Optimal Production in the Carbon Emission Market

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Outline

1. Controlling Carbon emission without financial market
2. Introducing the Carbon Market
3. The Case of a Large Emission Firm

The Business-as-usual Optimal Production

- Consider a producer with **Profit rate function**

$\pi_t(q) : [0, T] \times \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad C^1, \quad$ increasing strictly convex in q

- The company's objective function is :

$$\sup_{q_\cdot \geq 0} \mathbb{E}\left[U\left(\int_0^T \pi_t(q_t)dt\right)\right]$$

Then the optimal production of the company is characterized by :

$$\frac{\partial \pi_t}{\partial q}(q_t^0) = 0$$

This induces a total quantity of carbon emissions

$E_T^q := \int_0^T e_t(q_t)dt$ which is not supported by the producer...

- The EU ETS aims at incurring a cost to the producer so as to obtain an overall reduction of the carbon emissions.

Taxing Carbon Emissions Witout Trading

• α : amount of tax to be paid per unit of carbon emission ($40 - -100$ Euros per Ton)

• Suppose emissions are taxed at the end of period which happens to coincides with the horizon of the company. Then the company's objective is :

$$\sup_{q_\cdot \geq 0} \mathbb{E}\left[U\left(\int_0^T \pi_t(q_t)dt - \alpha\left(E_T^q - E^{\max}\right)^+\right)\right]$$

• Direct calculation leads to

$$\frac{\partial \pi_t}{\partial q}\left(\bar{q}_t^0\right) \quad = \quad \alpha \frac{\partial e_t}{\partial q}\left(\bar{q}_t^0\right) \mathbb{E}_t^{\bar{\mathbb{Q}}^0}\left[\mathbb{1}_{\{E_T^{\bar{q}_t^0} > E^{\max}\}}\right]$$

where $\dfrac{d\bar{\mathbb{Q}}^0}{d\mathbb{P}} = \dfrac{U'\left(\int_0^T \pi_t(\bar{q}_t^0)dt - \alpha\left(E_T^{\bar{q}_t^0} - E^{\max}\right)^+\right)}{\mathbb{E}_t\left[U'\left(\int_0^T \pi_t(\bar{q}_t^0)dt - \alpha\left(E_T^{\bar{q}_t^0} - E^{\max}\right)^+\right)\right]}$

Comments on Optimal Production with End-of-period Taxation

- The production firm assigns an individual price to its emissions :

$$S_t \quad := \quad \mathbb{E}_t^{\bar{\mathbb{Q}}^0}\left[\mathbb{1}_{\{E_T^{\bar{q}_t^0} > E^{\max}\}}\right]$$

Then the profit rate function becomes $\pi_t(q) - e_t(q)S_t$ and the optimal production is dertermined by the zero marginal profit condition

- Non-trivial fixed point problem !
- Main difficulty for the manager : no market price for carbon emissions...
- No incentive to reduce emissions beyond $E^{\max}$
- No mutualization, incentive to merge...

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Market Price of Carbon Allowances

- To begin with, we assume

the market is organized in one single period

- Frictionless market, continuous trading, no constraints... Then the No-Arbitrage condition implies that the market price of unit carbon emission allowances is :

$$S_t = \mathbb{E}_t^{\mathbb{Q}}\left[\alpha \mathbb{1}_{\{Y_T>0\}}\right]$$

for some equivalent probability measure $\mathbb{Q}$ (which might be inferred from market data)

- Here, $Y_T = \text{Tot market emissions} - \text{Total allowances on } [0, T]$
- The carbon emission allowance is viewed as a

derivative security defined by
the payoff $\mathbb{1}_{\{Y_T>0\}}$ at time T

First Remarks

- Production firms have a clear incentive to reduce emissions as they have the possibility to sell their allowances on the EU ETS

- The taxation of carbon is mutualized, so no incentive to merge (we will see however that large producers can have a negative impact...)

- The price of carbon is available on the market at any time, so managers can better optimize their production scheme

- Since the market is frictionless, the initial holdings in (free) allowances can be expressed equivalently in terms of their value in cash

$$x := S_0 E^{\max}$$

Problem Formulation

Given a trading strategy θ and a production scheme q which induces the total emissions

$$E_T := \int_0^T e_t(q_t)dt$$

the firm's cumulated profit at time T is :

$$\begin{aligned} & x + \int_0^T \theta_t dS_t + \int_0^T \pi_t(q_t)dt - \alpha S_T E_T \\ = \quad & x + \int_0^T (\theta_t - E_t)dS_t + \int_0^T \left(\pi_t(q_t) - S_t e_t(q_t)\right) dt \end{aligned}$$

$\theta \in$ linear vector space $\Longrightarrow$ $\sup_{q \geq 0, \theta} \mathbb{E}\left[X_T^{x,\theta} + B_T^q\right]$

where $X_T^{x,\theta} := x + \int_0^T \theta_t dS_t$ and $B_T^q := \int_0^T \left(\pi_t(q_t) - S_t e_t(q_t)\right) dt$

Optimal Production Scheme

Want to solve

$$\sup_{q \geq 0, \theta} \mathbb{E}\left[U\left(X_T^{x,\theta} + B_T^q\right)\right]$$

where $dX_t^{x,\theta} = \theta_t dS_t$ and $dB_t^q = (\pi_t(q_t) - S_t e_t(q_t))\, dt$

- Partial maximization with respect to q :

$$\frac{\partial \pi_t}{\partial q}(q_t^1) \quad = \quad S_t \frac{\partial e_t}{\partial q}(q_t^1)$$

- Notice that $q_t^1 < q_t^0$

- Here, **optimal production scheme completely decoupled from the trading activity**

Optimal Trading Strategy

Let $B^1 := B_T^{q^1}$ and $E_t^1 := \int_0^t e_t(q_t^1)dt$, want to solve :

$$\sup_\theta \mathbb{E}\left[U\left(X_T^{x,\theta} + B^1\right)\right]$$

In the context of a complete market, the solution is given by :

$$x + \int_0^T \left(\theta_t^1 - E_t^1\right) dS_t + B^1 \quad = \quad (U')^{-1}\left(y^1 \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$$

where the Lagrange multiplier y^1 is defined by :

$$\mathbb{E}^{\mathbb{Q}}\left[(U')^{-1}\left(y^1 \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad = \quad x + \mathbb{E}^{\mathbb{Q}}\left[B^1\right]$$

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Large Emission Firm with No Impact on the Risk-Neutral Measure

• We now assume the total emission Y_T is the final value of the process

$$dY_t^q = (\mu_t + \beta e_t(q_t))\, dt + \gamma_t dW_t$$

$\longrightarrow$ Notice that the total emissions at $t < T$ is not observed. Only $\{Y_T > 0\}$ is observed at time T
$\longrightarrow$ Y_t is the market view of the total emissions at time t

• **Assume** the risk-neutral measure is not impacted by q, and the market price of carbon emission allowance is :

$$S_t^q = \alpha\, \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{\{Y_T^q > 0\}}\right]$$

Optimal Production Scheme

Want to solve

$$\sup_{q_\cdot \geq 0,\ \theta} \mathbb{E}\left[U\left(X_T^{x,\theta} + B_T^q\right)\right]$$

where

$$X_T^{x,\theta} := x + \int_0^T \theta_t dS_t^q \quad \text{and} \quad B_T^q := \int_0^T \pi_t(q_t)dt - S_T^q \int_0^T e_t(q_t)dt$$

Proposition Assume the market is complete. Then, the optimal production scheme q^1 is the solution of

$$\sup_{q_\cdot \geq 0} \mathbb{E}^{\mathbb{Q}}\left[B_T^q\right] = \mathbb{E}^{\mathbb{Q}}\left[B_T^{q^1}\right],$$

Proof of the Proposition

Since the market is complete, we perform the partial maximization with respect to θ by duality $\Longrightarrow$

$$\sup_{q_\cdot \geq 0} \mathbb{E}\left[U \circ (U')^{-1}\left(\hat{y}^q \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad (1)$$

where the Lagrange multiplier $\hat{y}^q$ is defined by

$$\mathbb{E}\left[(U')^{-1}\left(\hat{y}^q \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad = \quad x + \mathbb{E}^{\mathbb{Q}}\left[B_T^q\right] \quad (2)$$

Notice that $U \circ (U')^{-1}$ decreasing and $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$, then (1) reduces to

$$\inf\left\{\hat{y}^q : \; q_\cdot \geq 0\right\}$$

Since $(U')^{-1}$ is also decreasing, (2) converts the problem into

$$\sup\left\{\mathbb{E}^{\mathbb{Q}}\left[B_T^q\right] : \; q_\cdot \geq 0\right\}$$

Further Characterization in the Markov Framework

- Let Y be Markov, $\pi_t(q) = \pi(q)$, $e_t(q) = e(q)$, and define :

$$V(t,e,y) \quad := \quad \sup_{q_{\cdot} \geq 0} \mathbb{E}^{\mathbb{Q}}_{t,e,y}\left[\int_t^T \pi(q_t)dt - \alpha E_T \mathbb{1}_{\{Y_T>0\}}\right]$$

Then, V solves the Dynamic Programming Equation :

$$0 = \frac{\partial V}{\partial t} + (\mu - \lambda\gamma)V_y + \frac{1}{2}\gamma^2 V_{yy} + \max_{q\geq 0}\{\pi(q) + e(q)(V_e + \beta V_y)\}$$

together with the terminal condition $V(T,e,y) = -\alpha e \mathbb{1}_{\{y>0\}}$

- The optimal strategy is defined by

$$\pi'\left(q^{(2)}\right) \quad = \quad -e'\left(q^{(2)}\right)(V_e + \beta V_y)$$

Notice that $-V_e = S_t$ and $V_y > 0$, so $q^{(2)} < q^{(1)}$

Large Carbon Emission Impacting the Risk-Neutral Measure

Folowing the previous partial maximization with repect to θ, we reduce the production firm's problem to

$$\sup_{q_\cdot \geq 0} \mathbb{E}\left[U \circ (U')^{-1}\left(\hat{y}^q \frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right]$$

where $\hat{y}^q$ is defined by $\mathbb{E}^{\mathbb{Q}^q}\left[(U')^{-1}\left(\hat{y}^q \frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] = x + \mathbb{E}^{\mathbb{Q}^q}\left[B_T^q\right]$ and

$$\left.\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right|_T = \exp\left(-\int_0^T \lambda(q_t)dW_t - \frac{1}{2}\int_0^T \lambda(q_t)^2 dt\right)$$

Specialize the discussion to the exponential utility $U(x) = -e^{-\eta x}$, we obtain :

$$\sup_{q_\cdot \geq 0} \mathbb{E}^{\mathbb{Q}^q}\left[B_T^q + \frac{1}{\eta}\ln\left(\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] \qquad (!!)$$

DPE in the Markov Framework

Let Y be Markov, $\pi_t(q) = \pi(q)$ and $e_t(q) = e(q)$, and define

$$V(t,e,y) \;:=\; \sup_{q_\cdot \geq 0} \mathbb{E}^{\mathbb{Q}^q}_{t,e,y}\left[\int_t^T \pi(q_t)dt - \alpha E_T S_T + \frac{1}{\eta}\ln\left(\frac{(d\mathbb{Q}^q/d\mathbb{P})_T}{(d\mathbb{Q}^q/d\mathbb{P})_t}\right)\right]$$

$$= \sup_{q_\cdot \geq 0} \mathbb{E}^{\mathbb{Q}^q}_{t,e,y}\left[\int_t^T \left(\pi(q_t) + \frac{\lambda(q_t)^2}{2\eta}\right) dt - \alpha E_T \mathbb{1}_{\{Y_T>0\}}\right]$$

Then, V solves the Dynamic Programming Equation :

$$0 \;=\; \frac{\partial V}{\partial t} + \mu V_y + \frac{1}{2}\gamma^2 V_{yy} + \max_{q\geq 0}\left\{\pi(q) + \frac{1}{2\eta}\lambda(q)^2 + e(q)(V_e + \beta V_y) - \gamma\lambda(q)V_y\right\}$$

together with the terminal condition $V(T,e,y) = -\alpha e \mathbb{1}_{\{y>0\}}$

Comments on Large Emission Firm Impacting Risk-Neutral Measure

• For "nice" risk premium $\lambda(q)$, the optimal strategy is defined by

$$\pi'\left(q^{(3)}\right) = -V_e - e'\left(q^{(3)}\right)V_y + \lambda'\left(q^{(3)}\right)\left(\gamma V_y - \eta^{-1}\lambda\left(q^{(3)}\right)\right)$$

Notice that $-V_e = S_t$ again, but, it is not clear whether $q^{(3)} < q^{(1)}$! !

• The case of infinite risk-aversion...

• In fact, if $\lambda(q)$ is not "nice", the firm manager has a dominating position. The benefit from trading can dominate the carbon taxation, so that emissions can increase significantly compared to the Business-as-usual situation !

• Concentration of carbon emission firms may induce an increase of the total emissions

Numerical experiment

Goal ! : Understand the behavior of the difference term

$$\tau(e,y) = -\frac{\partial e}{\partial q}(t,\hat{q}^{(3)})\beta V_y(t,Y_t,E_t) + \frac{\partial \lambda}{\partial q}(t,\hat{q}^{(3)})\left(\gamma V_y^{(3)}(t,Y_t,E_t) - \frac{1}{\eta}\lambda(t,\hat{q}^{(3)})\right)$$

Let

$$\pi(q) = q(1-q), \;\; e(q) = \lambda(q) = q, \;\; \beta = 1$$

Then, the Dynamic Programming Equation is :

$$V_t + \mu V_y + \frac{1}{2}\gamma^2 V_{yy} + \frac{1}{4\rho}\left(1 + V_e + (1-\gamma)V_y\right)^2 = 0$$

and

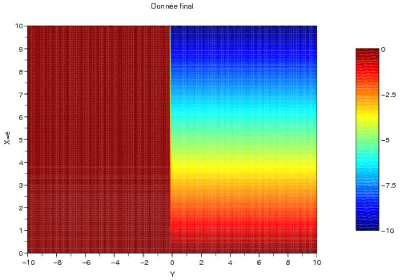
$$\tau(e,y) = \frac{2\eta(1-\gamma)}{2\eta-1}V_y + \frac{1}{2\eta-1}(1+V_e)$$

Numerical scheme

The main difficulty is the semi-linear terms.
$\Longrightarrow$ Time-splitting discretization :

- first part : solve the diffusion in a time step,
- second part : solve the coupling between the advection part with the non-linear effects (relaxation scheme).

The parameters are $\mu = 0.1$, $\gamma = 0.65$, $\eta = 5$ and the final time is $T = 10$



The value function

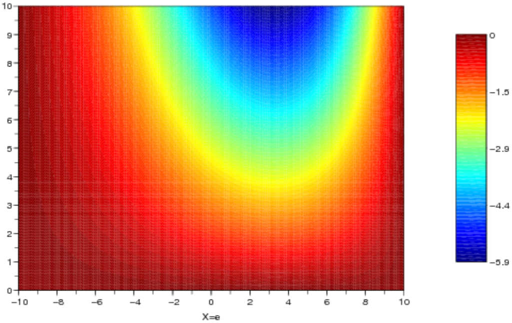


Fig.: The solution of the dynamic programming equation $V^3(e, y)$ at time $t = 0.2$

Market manipulation

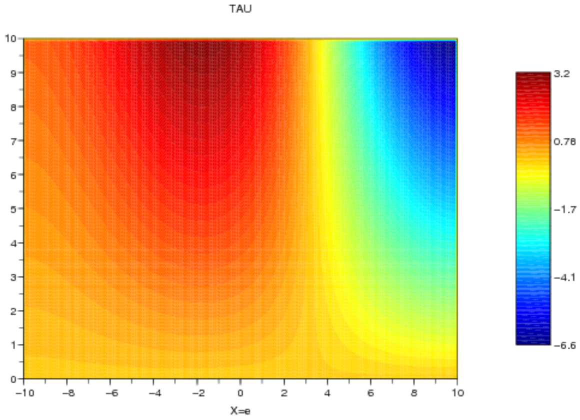


Fig.: The difference term $\tau(e, y)$ at time $t = 0.2$