COUPLING POISSON PROCESSES BY SELF-DECOMPOSABILITY: AN APPLICATION TO ENERGY FACILITIES[1]

Piergiacomo Sabino

UGC - RQPR

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[1] Joint work with N. Cufaro Petroni, University of Bari, Italy.
The technique here discussed does not reflect Uniper view.

Contents

- It is commonly accepted that the spot dynamics of commodity markets shows mean reversion, seasonality and jumps (see for example Cartea and Figueroa [1]).
- In addition, some methodologies have been proposed to take dependency into account based on correlation and co-integration (see Döttling and Heider. [2])
- However, these approaches can become mathematically complex or non-treatable when leaving the Gaussian-Itō world (see Kallsen and Tankov [4]).
- In this work we address the problem of dependency in the 2-dimensional case and start considering 2-dimensional jump diffusion processes with a 2-dimensional compound Poisson component.
- We then introduce an intuitive approach to model the dependency of 2-dimensional Poisson processes based on the self-decomposability (see Cufaro Petroni [7], Cufaro Petroni and Sabino [9], Sato [11]) of the exponential random variables used for its construction.

A law with *pdf* $f(x)$ and *chf* $\varphi(u)$ is said to be *self-decomposable* (*sd*) (see Sato [11] and Cufaro Petroni [7]) when for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u)$$

We will also say that a *rv* X is *sd* when its law is *sd*: for every $0 < a < 1$ we can always find two *independent rv*'s Y (with the same law of X), and Z_a with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that *in distribution*

$$X \stackrel{d}{=} aY + Z_a$$

We can look at this, however, also from a different point of view: to the extent that for $0 < a < 1$ the law of Z_a is known, we can define the *rv*

$$X = aY + Z_a$$

which by self-decomposability will now have the same law of Y.
It is easy to show that

$$r_{XY} = a$$

Self-decomposability. Part2

Based on the definition of self-decomposability we can derive the following results (see Cufaro Petroni and Sabino [9])

- The general form of $\kappa(x,y,z)$, the joint *pdf* of the triplet (X, Y, Z_a) is:

$$\kappa(x,y,z) = f(y)\, g_a(x-ay)\,\delta[z-(x-ay)]$$

  where $g_a(x)$ is the *pdf* of Z_a.

- The marginal, joint *pdf* of (X, Y) and (X, Z_a) will respectively be

$$h(x,y) = \int \kappa(x,y,z)\,dz = f(y)\,g_a(x-ay) \quad (1)$$

$$\ell(x,z) = \int \kappa(x,y,z)\,dy = \frac{1}{a} f\left(\frac{x-z}{a}\right) g_a(z) \quad (2)$$

- The joint cumulative distribution function (*cdf*) of (X, Y) is

$$H(x,y) = \int_{-\infty}^{y} f(y') G_a(x-ay')\,dy' \quad (3)$$

  where

$$G_a(z) = \int_{-\infty}^{z} g_a(z')\,dz'$$

  is the *cdf* of Z_a. The form of $H(x,y)$ will be useful in finding the copula functions pairing X and Y

It is well known that the exponential laws $\mathfrak{E}_1(\lambda)$ with *pdf* and *chf*

$$f_1(x) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} \qquad\qquad \varphi_1(u) = \frac{\lambda}{\lambda - iu}$$

are *sd* laws (see Sato [11]).
Remark that if $Y' \sim \mathfrak{E}_1(\mu)$, then $\alpha Y' \sim \mathfrak{E}_1\left(\frac{\mu}{\alpha}\right)$ for every $\alpha > 0$, and hence in particular

$$Y = \frac{\mu}{\lambda} Y' \sim \mathfrak{E}_1(\lambda)$$

As a consequence we could also state the self-decomposability by means of *exponential rv's with different parameters* $X \sim \mathfrak{E}_1(\lambda)$ and $Y' \sim \mathfrak{E}_1(\mu)$ because of course we have

$$X = aY + Z_a = \frac{a\mu}{\lambda} Y' + Z_a = \gamma Y' + Z_a.$$

provided that $0 < a < 1$, $\gamma > 0$. Hereafter, however, we will stick to the original formulation with $\mu = \lambda$.
It is possible to show now (Cufaro Petroni and Sabino [9])

$$Z_a = B(1)\, Z\,, \quad B(1) \sim \mathfrak{B}(1, 1-a) \text{ with } a = \mathbf{P}\left\{B(1) = 0\right\}$$

X is nothing else than the exponential Y *down a-rescaled*, plus another independent, but *intermittent* with frequency $1 - a$, exponential Z.

Equation (3) is instrumental to derive the joint *cdf*

$$H(x,y) = \mathbb{1}_{(y\wedge\frac{x}{a})}\left[\left(1-e^{-\lambda\left(y\wedge\frac{x}{a}\right)}\right) - e^{-\lambda x}\left(1-e^{-\lambda(1-a)\left(y\wedge\frac{x}{a}\right)}\right)\right]$$

with the marginals

$$F(x) = \mathbb{1}_{x\geq 0}\left(1-e^{-\lambda x}\right) \qquad G(y) = \mathbb{1}_{y\geq 0}\left(1-e^{-\lambda y}\right).$$

The joint *cdf* can also conveniently written as

$$H(x,y) = F(x) - [1-G(y)]\left(1-\frac{1-F(x)}{[1-G(y)]^a}\right)^+ .$$

As a consequence we get a family of copula functions

$$C_a(u,v) = u - (1-v)\left[1-\frac{1-u}{(1-v)^a}\right]^+ = u - \frac{[(1-v)^a-(1-u)]^+}{(1-v)^{a-1}}$$

which runs between the endpoints of the interval $0 \leq a \leq 1$ to give the extremal copulas

$C_0(u,v)$	$=$	uv	independent marginals
$C_1(u,v)$	$=$	$u\wedge v$	fully positively correlated marginals

It is easy to see that $C_1(u,v)$ also coincides with the Fréchet-Höffding upper bound $\overline{C}(u,v)$ for copulas (see (Cufaro Petroni and Sabino [9]))

Financial Applications

Considering then X and Y' as two random times with a positive random delay Z_a, self-decomposability can help describing their co-movement and can answer some common questions in the financial context:

- Once a financial institution defaults how long should one wait for a dependent institution to default too?
- A market receives a news interpreted as a shock: how long should one wait to see the propagation of that shock onto a dependent market?
- If different companies are interlinked, what is the impact on insurance risk?

Questions like the ones above are covered by the special case $\gamma > 1$. Our model is then rich enough to describe cases where the second random time event does not only occur after the first one.

- Similar results based on linear structure of exponential *rv*'s can be found in Iyer et al. [14] whose purpose was to model a multi-component reliability system.
- It is worthwhile noticing that this bivariate exponential model implies a copula function, the copula function does not define the model, but rather the opposite.
- Hereafter our goal is to derive (semi)-close formulas for spread options where the underlying are driven by dependent jump diffusion processes.

As initially suggested in Iyer et al. [14], we take now a sequence of *iid rv*'s

$$X_k = aY_k + B_k(1)Z_k \qquad k = 1, 2, \dots$$

in such a way that for every k: X_k, Y_k, Z_k are $\mathfrak{E}_1(\lambda)$, $B_k(1)$ is $\mathfrak{B}(1, 1-a)$, and $Y_k, Z_k, B_k(1)$ are mutually independent. Add moreover $X_0 = Y_0 = Z_0 = 0$, $\mathbb{P}$-*a.s.* to the list, and then define the point processes

$$T_n = \sum_{k=0}^{n} X_k \sim \mathfrak{E}_n(\lambda) \qquad n = 0, 1, 2, \dots$$

$$S_n = \frac{\mu}{\lambda}\sum_{k=0}^{n} Y'_k \sim \mathfrak{E}_n(\mu) \qquad n = 0, 1, 2, \dots$$

where $\mathfrak{E}_n(\lambda)$ are Erlang (gamma) laws with *pdf*'s and *chf*'s

$$f_n(x) = \lambda\frac{(\lambda x)^{n-1}}{(n-1)!}e^{-\lambda x}\mathbb{1}_{x\geq 0} \qquad \varphi_k(u) = \left(\frac{\lambda}{\lambda - iu}\right)^n \qquad n = 0, 1, 2, \dots$$

where it is understood that $\mathfrak{E}_0 = \boldsymbol{\delta_0}$. We will finally denote with $N(t) \sim \mathfrak{P}(\lambda t)$ and $M(t) \sim \mathfrak{P}(\mu t)$ the *dependent* Poisson processes associated respectively to T_n and S_n, and for our purposes we are interested in finding an explicit form of

$$p_{m,n}(t) = \mathbb{P}\{M(t) = m,\ N(t) = n\} \qquad n, m = 0, 1, 2, \dots \qquad t \geq 0$$

It is easy to see now that the *rv*'s

$$\zeta_n = \sum_{k=0}^{n} B_k(1) Z_k$$

are the sum of a random, binomial number $B(n) \sim \mathfrak{B}(n, 1-a)$ of *iid* exponentials $\mathfrak{E}_1(\lambda)$, and hence is nothing else than an Erlang $\mathfrak{E}_{B(n)}(\lambda)$ with a random index $B(n)$ (here $B(0) = 0$). As a consequence we can also write

$$\zeta_n = \sum_{k=0}^{n} B_k(1) Z_k = \sum_{k=0}^{B(n)} Z_k$$

and then from

$$\sum_{k=0}^{n} X_k = a \sum_{k=0}^{n} Y_k + \sum_{k=0}^{n} B_k(1) Z_k$$

We will also have

$$T_n = \frac{a\mu}{\lambda} S_n + \zeta_n = \frac{a\mu}{\lambda} S_n + \sum_{k=0}^{B(n)} Z_k = \frac{a\mu}{\lambda} S_n + R_{B(n)}$$

where $R_{B(n)}$ is the point process R_n with a random index $B(n)$. The equation above is nothing else than the self-decomposability of Erlang *rv*'s.

Based on the results above, we can explicitly calculate

$$p_{m,n} = \boldsymbol{P}\left\{M(t) = m,\, N(t) = n\right\} \qquad n, m = 0, 1, 2, \ldots$$

Proposition

Denote $p_{m,n} = \mathbb{P}\left[M(t) = m, N(t) = n\right]$*, when* $\boldsymbol{\gamma} \geq \mathbf{1}$*, we have* $p_{m,n} = 0$ *for* $\boldsymbol{m} < \boldsymbol{n}$*, while for* $\boldsymbol{m} \geq \boldsymbol{n}$

$$p_{m,n}(t) = \begin{cases} 0 & n > m \geq 0 \\ Q_{n,n}(t) & m = n \geq 0 \\ Q_{m,n}(t) - Q_{m,n+1}(t) & m > n \geq 0 \end{cases}$$

$$Q_{m,n}(s,t) = \sum_{k=n}^{m} (-1)^k \sum_{j=k}^{m} \binom{j}{k} \frac{\pi_{m-j}(\mu t)}{(-a)^j} \sum_{\ell=0}^{n} \beta_\ell(n) \pi_{j+\ell}(\lambda t) \Phi(j+1; j+\ell+1; \lambda t)$$

When $\gamma \leq 1$*, namely* $a\mu \leq \lambda$*, we have*

$$p_{m,n}(t) = \begin{cases} A_{m,n}(t) - A_{m,n+1}(t) + B_{m,n}(t) - B_{m,n-1}(t) & n > m \geq 0 \\ A_{n,n}(t) - A_{n,n+1}(t) + B_{n,n}(t) + C_{n,n}(t) & m = n \geq 0 \\ A_{m,n}(t) - A_{m,n+1}(t) + C_{m,n}(t) - C_{m,n+1}(t) & m > n \geq 0 \end{cases}$$

where $\pi_n(\cdot)$ and $\beta_\ell(n)$ denote respectively for the distributions of a Poisson and a binomial $\mathfrak{B}(n, 1-a)$ (it is understood that $\beta_0(0) = 1$). We then define for every $n, m \geq 0$

$$A_{m,n}(t) = \pi_m(\mu t) \sum_{k=0}^{n} \beta_k(n) \left[1 + \pi_k(\lambda t - a\mu t) - \sum_{j=0}^{k} \pi_j(\lambda t - a\mu t) \right]$$

while for $n \geq m \geq 0$, and $\lambda t - a\mu t = w$ for short, it is

$$B_{m,n}(t) = \pi_m(\mu t) \sum_{k=0}^{n-m} \pi_k \left(\frac{w}{a} \right) \sum_{\ell=0}^{n+1} \beta_\ell(n+1) \frac{w^\ell k!}{(k+\ell)!} \Phi \left(\ell, k+\ell+1, \frac{1-a}{a} w \right)$$

and for $m \geq n \geq 1$ it is (for $n = 0$ we have $C_{m,0}(t) = 0$)

$$C_{m,n}(t) = \frac{e^{-(1-a)\mu t}}{a^m} \sum_{\ell=1}^{n} \beta_\ell(n) \sum_{k=n}^{m} \sum_{j=0}^{\ell-1} \binom{k+\ell-j-1}{k} (-1)^{\ell-1-j} \pi_j(\lambda t) \pi_{m+\ell-j}(a\mu t) \Phi(k+\ell-j, m+\ell-j+1, a\mu t)$$

and $\Phi(j+1; j+\ell+1; \lambda t)$ for $0 \leq \ell \leq n \leq j \leq m$ are the confluent hypergeometric functions that are in fact elementary functions as proved Cufaro Petroni and Sabino [9] Remark that the law of the positively correlated pair of Poisson above differs from the one obtained for fatal shock models.

Geometric Brownian Motions with Jumps

Consider a Black-Scholes market with two risky underlying assets whose dynamics are driven by SDE's with the following solution (Merton model):

$$S_i(T) = \exp\left[\log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_i(T) + \sum_{n_i=1}^{N_i(T)} \log J_i^{n_i}\right], \quad i = 1, 2, \tag{4}$$

with $dW_1(t)dW_2(t) = \rho^{(W)}dt$ and log-normal jumps:

$$J_i = M_i \exp\left(-\frac{\nu_i^2}{2} + \nu_i Z_i\right), \quad i = 1, 2. \tag{5}$$

where $Z_i \sim N(0,1)$ and $Corr(Z_1 Z_2) = \rho^{(D)}$.
We assume that the compound Poisson processes and BM are independent. We now concentrate on the logarithm:

$$\log S_i(T) \stackrel{d}{=} \log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + N_i(T)\log M_i - \frac{\nu_i^2}{2}N_i(T) + \sqrt{\sigma_i^2 T + N_i(T)\nu_i^2}\,H_i, \quad i = 1, 2$$

For simplicity we denote

$$v_i^{(J,n)}(T) = \left(\sigma_i^{(J,n)}\right)^2 = \sigma_i^2 T + n\nu_i^2 = v_i^{(C)}(T) + v_i^{(D,n)}, \tag{6}$$

where $v_i^{(C)}$ and $v_i^{(D)}$ denote the terminal variances of the continuous and discontinuous parts.
No-arbitrage conditions imply (see Joshi [5] pag 344):

$$\mu_i - r = -\lambda_i \mathbb{E}[J_i - 1] \quad i = 1, 2. \tag{7}$$

Geometric Ornstein-Uhlenbeck with Jumps.

Energy markets often display mean-reversion and jumps. Consider a market driven by a stochastic process whose solution is:

$$S_i(t) = F_i(0,t)\exp\left\{U_i(t) + h(t)\right\}, \quad i = 1, 2, \tag{8}$$

where h(t) is a pure deterministic function and $U_i(t)$ is

$$U_i(t) = U_i(0)e^{-k_i t} + \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t} \sum_{n_i=1}^{N_i(t)} Y_i^{n_i} = U_i^C(t) + U_i^D(t) \tag{9}$$

whose SDE is:

$$dU_i(t) = -k_i U_i(t)dt + \sigma_i dW_i(t) + e^{-k_i t} Y_i dN_i(t). \tag{10}$$

$Y_i^{n_i}$ are copies of $Y_i \sim N(M_i, \nu_i^2)$ and $Corr(Y_1, Y_2) = \rho^{(D)}$. Remark that compared to the GBM case the rv's $Y_i^{n_i}$ are not in terms of logarithms.
No-arbitrage conditions are fulfilled if, $h_i(t) = -a_i(t) - b_i(t)$, with (see Cufaro Petroni and Sabino [8]):

$$a_i(t) = \frac{\sigma_i^2}{4k_i}\left(1 - e^{-2k_i t}\right) \quad b_i(t) = \lambda_i t \left(e^{e^{-k_i t}\left(M_i + \frac{1}{2}e^{-k_i t}\nu_i^2\right)} - 1\right)$$

Schwarts-Smith with Jumps

Consider the two factor Schwartz-Smith model (see Schwartz Smith [12]):

$$\begin{aligned} U_1(t) &= U_1(0)e^{-kt} + \sigma_1 \int_0^t e^{-k(t-s)} dW_1(s) + e^{-kt} \sum_{n_1=1}^{N_1(t)} Y_1^{n_1} \\ U_2(t) &= U_2(0) + \mu t + \sigma_2 W_2(t) + \sum_{n_2=1}^{N_2(t)} Y_2^{n_2} \\ U(t) &= U_1(t) + U_2(t). \end{aligned} \tag{11}$$

where $S(t) = F(0,t)e^{h(t)+U(t)}$ and we assume that the jumps of both process share the same distribution $Y_1, Y_2 \sim N(M, \nu)$. Simply taking the differential and some algebra:

$$dU(t) = -k\left(\mu + U_2(t) - U(t)\right)dt + \sigma dW + Y\left(e^{-kt}dN_1 + dN_2\right) \tag{12}$$

No arbitrage conditions can be found using the same procedure used for the GOU case (see Cufaro Petroni and Sabino [8].

We represent the price of a call option at time zero $c(0)$ in terms of an abstract BS formula

$$c(0) = BS\left(P_0, K, r, T, v, q\right). \tag{13}$$

Where P_0, K, r, T, v, q denote the usual arguments for the Black-Scholes formula.

- **GBM Case** The price of a call (put) option is :

$$c(0) = \sum_{n=0}^{\infty} \pi_{n_1}(\lambda_1 T) BS(S_1^{(n)}(0), K, r, T, v_1^{(J,n)}(T), 0). \tag{14}$$

  where

$$S_1^{(n)}(0) = S_1(0) M_1^n \exp\left[\lambda_1 T(1 - M_1)\right]. \tag{15}$$

  and $v_1^{(J,n)}(T)$ is defined in Equation (6).

- **GOU Case**. In the abstract BS formula one needs to feed:

$$S_1^{(n)}(0) = F_1(0, T) e^{p_i^n(T)}, \tag{16}$$

  $v_1^{(J,n)}(T) = \mathbb{V}ar\left[U_1^{(C)}(T)\right] + ne^{-2k_1 T}\nu_1^2$ and

$$p_1^n(t) = -b_1(t) + ne^{-k_1 t}\left(\frac{1}{2}e^{-k_1 t}\nu_1^2 + M_1\right). \tag{17}$$

- **Schwartz-Smith Case**. Assuming Equation (11) a semi-closed form formula can be found following the procedure outlined in the GBM and GOU cases.

$$c(0) = \sum_{n_1, n_2 = 0}^{\infty} \mathbb{P}\left(N_1(T) = n_1, N_2(T) = n_2\right) BS(S^{(n_1,n_2)}(0), K, r, T, v^{(J,n_1,n_2)}(T), 0). \quad (18)$$

where

$$S^{(n_1,n_2)}(0) = F(0,t)e^{-b(t)+\mu t + n_1(t)e^{-kt}\left(M + e^{-kt}\frac{\nu^2}{2}\right) + n_2(t)\left(M + \frac{\nu^2}{2}\right)}, \quad (19)$$

and

$$v^{(J,n_1,n_2)}(T) = \mathbb{V}ar\left[U^C(T)\right] + \left(e^{-2k_i T} n_1 + n_2\right)\nu^2 \quad (20)$$

The application to spread options is the native framework to compare our approach with cointegrated jumps compared to other jump-diffusion cases. We consider spread options with zero-strike (Margrabe formula [6]).

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \mathbb{P}\left(N_1(T) = n_1; N_2(T) = n_2\right) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0), \quad (21)$$

where $v^{(M,n_1,n_2)}(T) = v_1^{(J,n_1)}(T) + v_2^{(J,n_2)}(T) - 2\rho^{(J,n_1,n_2)}\sqrt{v_1^{(J,n_1)} v_2^{(J,n_2)}}$ is the spread terminal variance. In the following, we compare three different Poisson models:

- **Independent Jumps**. $N_1(t)$ and $N_2(t)$ are independent Poisson processes:

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \pi_{n_1}(\lambda_1 T)\pi_{n_2}(\lambda_2 T) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0). \quad (22)$$

- **One Common Jump**. $N_i(t) = N(t) + N_i^X$, $i = 1, 2$, where $N(t)$ and N_i^X are all mutually independent Poisson processes:

$$s(0) = \sum_{n=0,n_1,n_2 \geq n}^{\infty} \pi_{n_1-n}(\lambda_1^X T)\pi_{n_2-n}(\lambda_2^X T)\pi_n(\lambda T) \times BS(S_1^{(n_1-n)}(0), S_2^{(n_2-n)}(0), 0, T, v^{(M,n_1-n,n_2-n)}, 0) \quad (23)$$

- **Cointegrated Jumps**. $p_{n_1,n_2} = \mathbb{P}\left(N_1(T) = n_1; N_2(T) = n_2\right)$ are defined in Proposition 2.1:

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \mathbb{P}\left(N_1(T) = n_1; N_2(T) = n_2\right) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0) \quad (24)$$

- The approximation formulas can be used when the strike is different from zero and for more than two legs (Deng and Lee [10] and Pellegrino and Sabino [16] or Pellegrino and Sabino [15]).
- Equations (22)-(24) depends on the values of the probabilities p_{n_1,n_2} and the values of the BS formulas separately. The former quantities do not depend on the distribution of the jumps while the latter ones are independent on the structure of dependence between the Poisson processes.
- The payoff of the spread options above considers the values of the two underlying at the same time T. Other types of spread options instead look at the two underlying at different times, e.g. the payoff may be $(S_1(T_1) - S_2(T_2))^+$, $T_2 < T_1$.
- In this case one needs to readapt the formulas and consider the probabilities $p_{n_1 n_2} = \mathbb{P}(N_1(T_1) = n_1; N_2(T_2) = n_2)$ and they can be found in Cufaro Petroni and Sabino [9].

- We presents the numerical experiments assuming the GBM and GOU dynamics plus jumps.
- The case with GBM considers realistic parameters and is meant to study the spread option values with different types of bivariate Poisson processes.
- In contrast the GOU case is based on real data of TTF and NCG day-ahead prices.

GBM. Application to Spread Options. Parameters

Table: Parameters of the GBM and Compound Poisson processes

(a) Continuous Part.

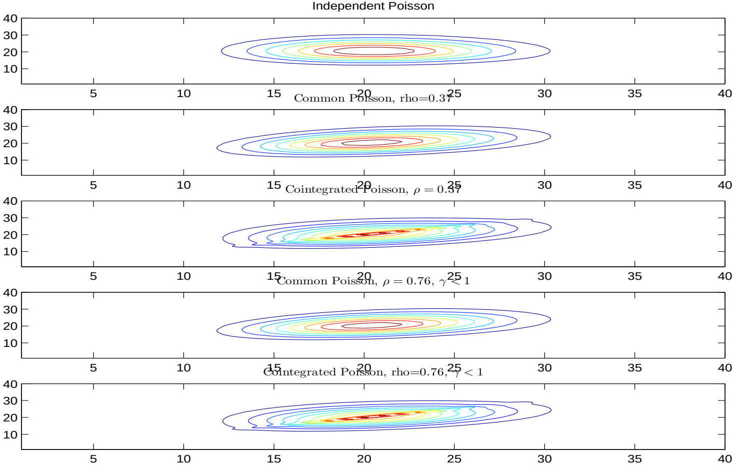
	No Jump		With Jumps
	Case A	Case B	
$S_1(0)$	100	100	100
$S_2(0)$	100	100	100
σ_1	0.49	0.37	0.2
σ_2	0.35	0.23	0.15
$\rho^{(W)}(\%)$	96	60	80

(b) Discontinuous Part

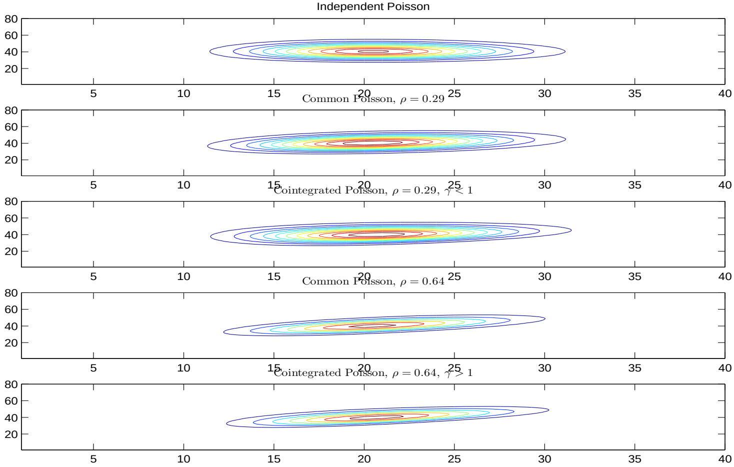
	Case A	Case B
$\rho^{(D)}(\%)$	99	50
λ_1	20	40
λ_2	20	20
ν_1	0.10	0.05
ν_2	0.07	0.04
M_1	1.1	1.05
M_2	1.1	1.05

GBM. Application to Spread Options. Probabilities $\lambda_1 = \lambda_2 = 20$

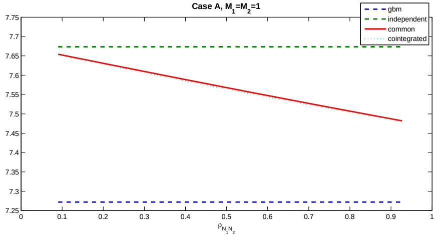
Figures below show the difference among the joint probabilities of the Poisson processes.



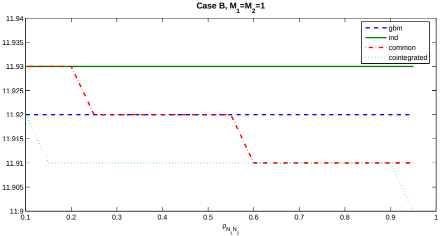
(a) $\lambda_1 = \lambda_2 = 20$



(b) $\lambda_1 = 40$, $\lambda_2 = 20$



(c) Case A



(d) Case B

Figure: Spread Option Values in the Cases A and B when $M_1 = M_2 = 1$

Table: Spread Option Values without jumps and independent Compound Poisson processes.

	No Jump		Independent Jump	
	Case A	Case B	Case A	Case B
Option Value	7.27	11.92	25.23	19.27

Table: Spread Option Values with Common and Cointegrated Compound Poisson.

	Case A		Option Value Case A		Case B		Option Value Case B	
a	$\rho_{N_1N_2}$ (%)	λ	Common	Cointegrated	$\rho_{N_1N_2}$ (%)	λ	Common	Cointegrated
0.1	9	1.80	24.30	24.22	7	2.09	18.87	18.87
0.15	14	2.71	23.76	23.64	11	3.13	18.66	18.67
0.2	18	3.63	23.20	23.05	15	4.16	18.45	18.46
0.25	23	4.55	22.63	22.44	18	5.19	18.25	18.26
0.3	27	5.47	22.04	21.81	22	6.21	18.04	18.05
0.35	32	6.40	21.42	21.16	26	7.23	17.83	17.83
0.4	37	7.34	20.78	20.48	29	8.24	17.61	17.62
0.45	41	8.29	20.11	19.78	33	9.25	17.40	17.40
0.5	46	9.24	19.41	19.05	36	10.25	17.18	17.18
0.55	51	10.20	18.68	18.29	40	11.25	16.97	16.96
0.6	56	11.17	17.90	17.49	43	12.24	16.75	16.74
0.65	61	12.16	17.08	16.64	47	13.23	16.53	16.51
0.7	66	13.15	16.20	15.74	50	14.21	16.30	16.29
0.75	71	14.17	15.25	14.78	54	15.19	16.08	16.06
0.8	76	15.20	14.21	13.75	57	16.16	15.85	15.83
0.85	81	16.26	13.06	12.61	61	17.13	15.62	15.60
0.9	87	17.36	11.74	11.33	64	18.09	15.38	15.37
0.95	93	18.53	10.14	9.82	67	19.05	15.15	15.14

- The effect of the correlation between the Poisson processes is noticeable. The value of the spread option is decreasing when $\rho_{N_1N_2}$ is increasing that is in line with the intuition because the spread terminal variance decreases.
- In the case A, the jump sizes are perfectly correlated and the spread option values using the common Poisson setting is always higher than the values obtained with our methodology. This is somehow reflected by the concentration of the isoline of the probabilities.
- Using a common Poisson reduces the spectrum of jump events, for instance in an extreme setting where $\lambda = \lambda_1 = \lambda_2$, $N_1(t)$ cannot jump more that $N_2(t)$, while this is not the case for the cointegrated Poisson process. In contrast, the choice of the Poisson model has no remarkable effect on the price of the spread option in the configuration B.
- This in our opinion does not diminish the value of our methodology because in any case the probabilities $p_{n_1n_2}$ differ between the two different Poisson examples.
- With the same $\rho_{N_1N_2}$ the price of the spread option seems to highly depend on the number of the jumps of both processes rather than when they occurred and that explains the small differences.
- Furthermore, assuming for instance $\lambda_2^X = 0$ implies $\lambda_2 = \lambda$ and $N_2(t)$ cannot have more jumps than $N_2(t)$ that coincides with the properties of our model only if $\gamma > 1$, that means that our model gives a richer set of combinations.

GOU. Application to Transportation between TTF and NBP. Results

- The calculation date is end of December 2013 with a historical time window of 2 years for the estimation period. The transportation cost is set to zero.
- We concentrate then on the transportation value for the first and second quarters, Q1, Q2 [2].
- The expected jump sizes and their correlation are very small and negligible.
- Comparing the values of λ and a or γ the correlation between the two Poisson processes is also small.
- Based on the results on the GBM case, we can expect that the selection of a specific Poisson model will not bring a remarkable difference.

Table: Market parameters for NCG and TTF

(a) Parameters of the Single Underlyings.

Market	k	σ_i	μ_i	ν_i	λ_i
NCG	9.75	0.09	0.001	0.07	35.37
TTF	26.38	0.15	0.003	0.02	26.70

(b) Common Parameters

Method	$\rho^{(W)}(\%)$	$\rho^{(D)}(\%)$	λ	a
Independent	43	-1.0	NA	NA
Common	40	-1.0	8.93	NA
Cointegrated	35	-1.0	NA	0.27

[2] The technique here discussed does not reflect UGC view.

- The prices obtained with the different configurations meet the expectations after having a look at the estimated parameters.[3].
- Remark that in this case the prices with cointegrated jumps are higher than those with common jumps; this is explained by the fact that the correlation parameters are different, the latter configuration has higher values both for ρ^W and $\rho_{N_1N_2}$.
- Once more, although our methodology is parameterized by γ and a, the correlation between the exponential *rv*'s that construct the Poisson process, it implies a structure that goes beyond the linear correlation.

Table: TTF-NCG Gas Transport Prices

	Transportation Value		
	Independent	Common	Cointegrated
Q1	49.14	49.35	49.79
Q2	34.99	35.48	36.27

[3] The technique here discussed does not reflect UGC view.

Conclusions and Future Studies

- We have analyzed a method to produce pairs of non independent Poisson processes $(M(t), N(t))$ from positively correlated, self-decomposable, exponential renewals. In particular we have also provided the family of copulas pairing the renewals, along with the closed form for the joint distribution.
- This second result turns out to be instrumental to model energy derivatives and in general to price spread options. Due to the particular relationships among inter arrival times, we can see this dependence as a form of coitegration among jumps that differs from the fatal shock models.
- Comparing our methodology and different types of Poisson processes. We have shown that our methodology can cope with a wide range of possibilities that go beyond the pure correlation between marginal Poisson.
- Further straightforward applications are in credit and insurance risk where our approach can answer questions regarding the time of contagion or time of propagation of certain information.
- Self-decomposability and subordination technique can be promising tools to study dependency beyond the Gaussian-Itō world: Erlang (Gamma) *rv*'s that can be used to create and simulate dependent variance gamma processes.
- Extension to two sided Exponential-Polynomial-Trigonometric (EPT) density functions (see Sexton and Hanzon [13] and Hanzon et al. [3]) have studied the use of to option pricing where EPT are distributions with a strictly proper rational characteristic function.

THANK YOU

and

QUESTIONS?

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