A probabilistic numerical method for optimal multiple switching problem and application to investments in electricity generation

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Motivation

Investments in power generation

- Capital intensive
- Very long-term returns
- Many technologies
- Many random factors :
  - Demand
  - Outages
  - Fuel prices
  - Inflows

Natural modelling : Stochastic control

- High-dimensional
- Infinite horizon

Motivation

More precisely : **Optimal multiple switching**

Outline

1. Numerical scheme
   - Convergence rate
   - Complexity analysis
2. Application
   - Detailed modelling
   - Numerical solution

Outline

1 Formulation

2 Numerical scheme

3 Complexity

4 Memory reduction

5 Application

6 Conclusion

Problem

$$v(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\left[\int_t^{\infty} f\left(s, X_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{\tau_n \geq t} k\left(\tau_n, \zeta_n\right)\right]$$

- $X^{t,x}$ Markovian diffusion in $\mathbb{R}^d$, $X_t = x$
- I^{α} piecewise constant in a finite set $\mathbb{I}_q = \{i_1, i_2, \ldots, i_q\}$ ($\Rightarrow$ **optimal switching**)
- $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$ impulse control strategy, τ_n increasing stopping times, ι_n random variables in $\mathbb{I}_q$, $\zeta_n = \iota_n - \iota_{n-1}$

$$I_s^{\alpha} = \iota_0 \mathbf{1}\{0 \leq s < \tau_0\} + \sum_{n \in \mathbb{N}} \iota_n \mathbf{1}\{\tau_n \leq s < \tau_{n+1}\}$$

- $\mathcal{A}_{t,i}$ set of admissible strategies s.t. $I_t^{\alpha} = i$

Assumptions

Diffusion coefficients

Lipschitz continuity & Linear growth

Reward function f

Lipschitz continuity & Linear growth

Exponential discount $e^{-\rho t}$

Cost function k

Lipschitz continuity & Linear growth

Exponential discount $e^{-\rho t}$

Minimum fixed cost

Triangular condition : $k(t;i,j) + k\left(t;j,k\right) > k\left(t;i,k\right)$

Properties

- $v_i = v(.,.,i)$ solutions of a system of Hamilton-Jacobi-Bellman Quasi-Variational Inequalities
- $v(t, X_t, i)$ solution of a Reflected Backward Stochastic Differential Equation
- **Dynamic Programming Principle** : $\forall \tau \geq t$ stopping time,

$$v(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\left[\int_t^{\tau} f\left(s, X_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{t \leq \tau_n \leq \tau} k(\tau_n, \zeta_n) + v\left(\tau, X_{\tau}^{t,x}, I_{\tau}^{\alpha}\right)\right]$$

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Finite time horizon

$$v_T(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}^T} \mathbb{E}\left[\int_t^T f(s,X_s^{t,x},I_s^\alpha)\,ds - \sum_{t \leq \tau_n \leq T} k(\tau_n,\zeta_n) + g\left(T,X_T^{t,x},I_T^\alpha\right)\right]$$

$$g\,(T,x,i) = \mathbb{E}\left[\int_T^\infty f\left(s,X_s^{T,x},i\right)ds\right]$$

OR any Lipschitz fct s.t. $|g(T,x,i)| \leq Ce^{-\rho T}\,(1+|x|)$

Approximation error

$$|v\,(t,x,i) - v_T\,(t,x,i)| \leq \mathbf{C}\,(\mathbf{1}+|\mathbf{x}|)\,\mathbf{e}^{-\bar{\rho}\mathbf{T}}$$

Time discretization

$$\bar{v}_{\Pi}(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}^{\Pi}} \mathbb{E}\left[\int_t^T f\left(\pi(s), \bar{X}_s^{t,x}, I_s^{\alpha}\right) ds - \sum_{t \leq \tau_n \leq T} k(\tau_n, \iota_{n-1}, \iota_n) + g\left(T, \bar{X}_T^{t,x}, I_T^{\alpha}\right)\right]$$

$\Pi = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$, step h

$d\bar{X}_s = b\left(\pi(s), \bar{X}_{\pi(s)}\right) ds + \sigma\left(\pi(s), \bar{X}_{\pi(s)}\right) dW_s, \ 0 \leq s \leq T, \ \bar{X}_0 = x_0$

Approximation error

$$\left|v_T(0,x,i) - \bar{v}_{\Pi}(0,x,i)\right| \leq \mathbf{C}\left(\mathbf{1} + |\mathbf{x}|^{\frac{3}{2}}\right)\sqrt{\mathbf{h}}$$

cf. [Gassiat et al., 2011]

Space localization

$$\bar{X}_t \in \mathbb{R}^d \to \mathcal{P}^\varepsilon\left(\bar{X}_t\right) \in \mathcal{D}^\varepsilon$$

$\mathcal{D}^\varepsilon$ s.t. $\mathbb{E}\left[\left|\bar{X}_t - \mathcal{P}^\varepsilon\left(\bar{X}_t\right)\right|\right] \leq \varepsilon \ \forall 0 \leq t \leq T$

$\mathcal{D}^\varepsilon$ bounded domain : $\forall x \in \mathcal{D}^\varepsilon$, $|x| \leq C\left(T, \varepsilon\right)$

Approximation error

$$\left|\bar{v}_\Pi\left(0, x, i\right) - \bar{v}_\Pi^\varepsilon\left(0, x, i\right)\right| \leq \mathbf{C}\varepsilon$$

Conditional expectation approximation (1/2)

Dynamic programming principle

$$\bar{v}_\Pi(T,x,i) = g(T,x,i)$$

$$\bar{v}_\Pi(t_n,x,i) = \max_{j\in\mathbb{I}_q}\left\{hf(t_n,x,j) - k(t_n,i,j) + \Phi_j^{t_n,x}(\bar{v}_\Pi)\right\}$$

$$\Phi_j^{t_n,x}(\varphi) = \mathbb{E}\left[\varphi\left(t_{n+1},\bar{X}_{t_{n+1}},j\right)\middle|\bar{X}_{t_n}=x\right]$$

Approximation : Least-squares regression

$$\Phi_j^{t_n,x}(\varphi) \rightsquigarrow \tilde{\Phi}_j^{t_n,x}(\varphi) = \min_{\lambda\in\mathbb{R}^K}\mathbb{E}\left[\left(\varphi\left(t_{n+1},\bar{X}_{t_{n+1}},i\right) - \sum_{k=1}^K\lambda_k e_k\left(\bar{X}_{t_n}\right)\right)^2\right]$$

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Conditional expectation approximation (1/2)

Local basis

$$e_k(x) = \mathbf{1}\{x \in B_k\}$$

$(B_k)_{1 \le k \le K}$ hypercubes $\subset \mathbb{R}^d$, partition of $\mathcal{D}^\varepsilon$, edges of length $\boldsymbol{\delta}$

cf. [Gobet et al., 2005], [Bouchard and Warin, 2011]

Approximation error

$$\left|\bar{v}_\Pi(0,x,i) - \tilde{v}_\Pi(0,x,i)\right| \le \mathbf{C}\frac{\boldsymbol{\delta}}{\mathbf{h}}$$

Conditional expectation approximation (2/2)

Sample approximation

$$\tilde{\Phi}_j^{t_n,x}(\varphi) \rightsquigarrow \hat{\Phi}_j^{t_n,x}(\varphi) = \min_{\lambda \in \mathbb{R}^K} \sum_{m=1}^{M} \left[\left(\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^m, i\right) - \sum_{k=1}^{K} \lambda_k e_k\left(\bar{X}_{t_n}^m\right) \right)^2 \right]$$

Lemma

$$\forall p \geq 1\,,\ \left\| \hat{\Phi}_j^{t_n,x}(\varphi) - \tilde{\Phi}_j^{t_n,x}(\varphi) \right\|_p \leq \frac{C_p}{\sqrt{M}} \frac{\Gamma^{t_n,x}(\varphi) + \bar{\varphi}^{t_n}}{\mathbb{P}\left(\bar{X}_{t_n} \in B(x)\right)^{1-\frac{1}{p \vee 2}}}$$

- Extension of [Tan, 2011]
- Makes use of $\left\| \frac{1}{M} \sum_{m=1}^{M} X_m \right\|_p \leq \frac{C_p}{\sqrt{M}} \|X\|_{p \vee 2}$ (Marcinkiewicz-Zygmund + Jensen)

Conditional expectation approximation (2/2)

Sample approximation

$$\tilde{\Phi}_j^{t_n,x}(\varphi) \rightsquigarrow \hat{\Phi}_j^{t_n,x}(\varphi) = \min_{\lambda\in\mathbb{R}^K} \sum_{m=1}^{M} \left[\left(\varphi\left(t_{n+1}, \bar{X}_{t_{n+1}}^m, i\right) - \sum_{k=1}^{K} \lambda_k e_k\left(\bar{X}_{t_n}^m\right) \right)^2 \right]$$

Lemma

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Rate of convergence

Approximation error

$$\left\| \tilde{v}_\Pi (0,x,i) - \hat{v}_\Pi (0,x,i) \right\|_p \leq \frac{\mathbf{C_p}}{\mathbf{h\sqrt{M}}} \frac{\mathbf{1+C\,(T,\varepsilon)+\sqrt{h}}}{\mathbf{p\,(T,\delta,\varepsilon)^{1-\frac{1}{p\vee 2}}}}$$

$$p\,(T,\delta,\varepsilon) = \min_{t_n\in\Pi,\, B_k\subset\mathcal{D}^\varepsilon} \mathbb{P}\left(\bar{X}_{t_n} \in B_k\right)$$

Full rate of convergence

$$\left\| v (0,x,i) - \hat{v}_\Pi (0,x,i) \right\|_p \leq$$

$$C_p\left\{(1+|x|)e^{-\bar{\rho}T}+\left(1+|x|^{\frac{3}{2}}\right)\sqrt{h}+\varepsilon+\frac{\delta}{h}+\frac{C_p}{h\sqrt{M}}\frac{1+C(T,\varepsilon)+\sqrt{h}}{p(T,\delta,\varepsilon)^{1-\frac{1}{p\vee 2}}}\right\}$$

Rate of convergence

Approximation error

$$\left\| \tilde{v}_\Pi(0,x,i) - \hat{v}_\Pi(0,x,i) \right\|_p \leq \frac{C_p}{h\sqrt{M}} \frac{1 + C(T,\varepsilon) + \sqrt{h}}{p(T,\delta,\varepsilon)^{1-\frac{1}{p\vee 2}}}$$

$p(T,\delta,\varepsilon) = \min_{t_n \in \Pi,\, B_k \subset \mathcal{D}^\varepsilon} \mathbb{P}\left(\bar{X}_{t_n} \in B_k\right)$

Full rate of convergence

$$\left\| v(0,x,i) - \hat{v}_\Pi(0,x,i) \right\|_p \leq$$

$$C_p \left\{ (1+|x|)e^{-\bar{\rho}T} + \left(1+|x|^{\frac{3}{2}}\right)\sqrt{h} + \varepsilon + \frac{\delta}{h} + \frac{C_p}{h\sqrt{M}} \frac{1 + C(T,\varepsilon) + \sqrt{h}}{p(T,\delta,\varepsilon)^{1-\frac{1}{p\vee 2}}} \right\}$$

Example

$X_t = W_t$ d-dim. B.M.

$$C(T,\varepsilon) < \sqrt{T \log\left(\frac{8T}{\pi\varepsilon^{\frac{2}{d}}}\right)}$$

$$p(T,\varepsilon,\delta) \gg \frac{\varepsilon}{(4T)^d}\delta^d$$

One-parameter rate of convergence

$$\left\| v(0,x,i) - \hat{v}_{\Pi}(0,x,i) \right\|_p \leq C_p \left(1 + |x|^{\frac{3}{2}}\right) \sqrt{h}$$

- $T = \frac{1}{2\bar{\rho}} \ln\left(\frac{1}{h}\right)$, $\varepsilon = \sqrt{h}$, $\delta = h^{\frac{3}{2}}$
- $M = \frac{1}{2\bar{\rho}} \left(\frac{2}{\bar{\rho}}\right)^{2d} \left(\ln\left(\frac{1}{h}\right)\right)^{2d+1} \ln\left(\frac{4}{\pi\bar{\rho}} \frac{\ln\left(\frac{1}{h}\right)}{h^{\frac{1}{d}}}\right) \frac{1}{h^{3(d+1)}}$ ...

Example

$X_t = W_t$ d-dim. B.M.

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- $M = \frac{1}{2\bar{\rho}}\left(\frac{2}{\bar{\rho}}\right)^{2d}\left(\ln\left(\frac{1}{h}\right)\right)^{2d+1} \ln\left(\frac{4}{\pi\bar{\rho}}\frac{\ln\left(\frac{1}{h}\right)}{h^{\frac{1}{d}}}\right)\frac{1}{h^{3(d+1)}}$ . . .

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Computational complexity

Dynamic programming

$$\hat{v}_\Pi\left(T, x_m, i\right) = g\left(T, x_m, i\right)$$
$$\hat{v}_\Pi\left(t_n, x_m, i\right) = \max_{j \in \mathbb{I}_q} \left\{ hf\left(t_n, x_m, j\right) - k\left(t_n, i, j\right) + \hat{\Phi}_j^{t_n, x_m}\left(\hat{v}_\Pi\right) \right\}$$

Complexity

$$\mathcal{O}\left(q^2 \times N \times M \log\left(M\right)\right)$$

- q =number of switches
- N =number of time steps
- M =number of Monte Carlo trajectories

or $\mathcal{O}\left(q \times N \times M \log\left(M\right)\right)$ under suitable conditions

Memory Complexity : Naive implementation

Euler scheme : **Forward**

$$
\begin{aligned}
x_{t_{n+1}} &= x_{t_n} + b\left(t_n, x_{t_n}\right) h + \sigma\left(t_n, x_{t_n}\right) \varepsilon_i \sqrt{h} \\
\varepsilon_i &\sim \mathcal{N}(0,1)
\end{aligned}
$$

Dynamic programming : **Backward**

$$
\hat{v}_\Pi(T, x_m, i) = g(T, x_m, i)
$$

$$
\hat{v}_\Pi(t_n, x_m, i) = \max_{j \in \mathbb{I}_q} \left\{ hf(t_n, x_m, j) - k(t_n, i, j) + \hat{\mathbb{E}}\left[\hat{v}_\Pi\left(t_{n+1}, \bar{X}_{t_{n+1}}^{t_n, x_m}, j\right)\right] \right\}
$$

$\Rightarrow$ storage of Monte Carlo sample $\Rightarrow$ $\mathcal{O}(N \times M)$

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How to avoid the sample storage ?

Memory reduction method

[Chan et al., 2004] 1-dim geometric B.M.

[Chan et al., 2006] d-dim geometric B.M.

[Chan and Wu, 2011] exponential Lévy

$\hookrightarrow$ limited to additive processes

Our contribution

Extension to any Markovian process

(Pseudo)Random number generators

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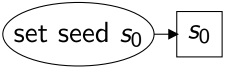
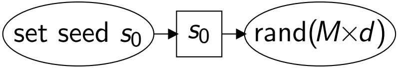


(Pseudo)Random number generators



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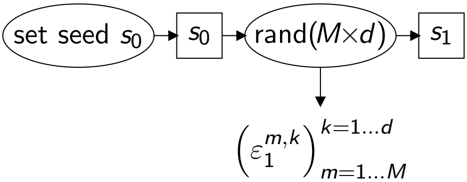

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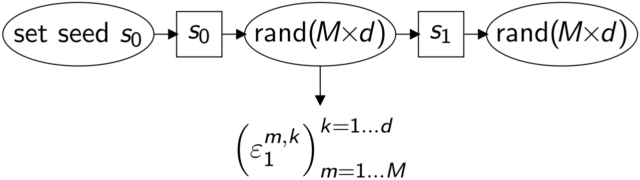

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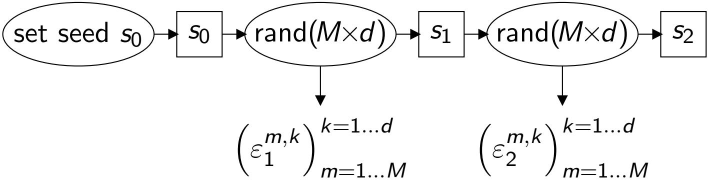

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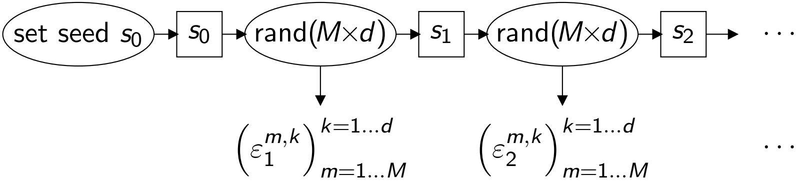

- The seeds s_n can be set and stored
- The i.i.d. sequence $\varepsilon_n^{m,k}$ can be used for the Euler scheme

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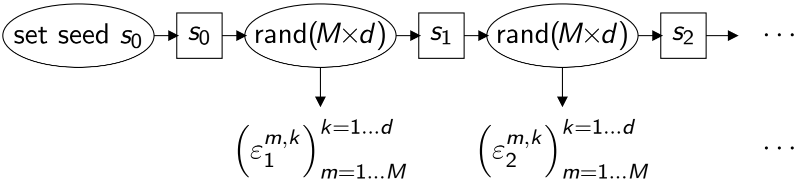

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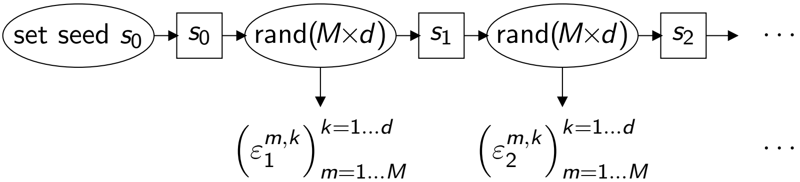

- The seeds s_n can be set and stored
- The i.i.d. sequence $\varepsilon_n^{m,k}$ can be used for the Euler scheme

Euler scheme

Case of a diffusion

$$
\begin{aligned}
x_{t_{n+1}} &= F\left(x_{t_n}, \varepsilon_i\right) \\
F\left(x, \varepsilon\right) &= x + b\left(t_n, x\right) h + \sigma\left(t_n, x\right) . \varepsilon \sqrt{h} \\
\varepsilon_i &\sim \mathcal{N}\left(0, 1\right)
\end{aligned}
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Euler scheme

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Suppose that $\exists F^{-1}(x, \varepsilon)$ s.t. $\forall \varepsilon$, $F\left(F^{-1}(x, \varepsilon), \varepsilon\right) = x$

General memory reduction method (v1)

Euler scheme

Initialize $X[m]\,,\ m = 1..M$
for $i = 1..N-1$
  $S[i] \leftarrow \mathrm{getseed}\,()$
  for $m = 1..M$
    $\varepsilon \leftarrow \mathrm{rand}\,(d)$
    $X[m] \leftarrow F\,(X[m]\,,\varepsilon)$
  end
end
$S[N] \leftarrow \mathrm{getseed}\,()$

Inverse Euler scheme

Final values in $X[m]\,,\ m=1..M$
for $i = N-1..1$
  $\mathrm{setseed}\,(S[i])$
  for $m = 1..M$
    $\varepsilon \leftarrow \mathrm{rand}\,(d)$
    $X[m] \leftarrow F^{-1}\,(X[m]\,,\varepsilon)$
  end
end
$\mathrm{setseed}\,(S[N])$

- Doubles the number of calls to $\mathrm{rand}\,()$
- But memory complexity down to $\mathcal{O}\,(M+N)$

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  for $m = 1..M$
    $\varepsilon \leftarrow \text{rand}(d)$
    $X[m] \leftarrow F^{-1}(X[m], \varepsilon)$
  end
end
$\text{setseed}(S[N])$

- Doubles the number of calls to $\text{rand}()$
- But memory complexity down to $\mathcal{O}(M+N)$

General memory reduction method (v2)

Euler scheme

for $i = 1..N-1$
  $S[i] \leftarrow \mathrm{getseed}\,()$
  for $m = 1..M$
    $\varepsilon \leftarrow \mathrm{rand}\,(d)$
  end
end
Initialize $X[m]\,,\ m = 1..M$
for $i = 1..N-1$
  $\mathrm{setseed}\,(\mathrm{S[N-i+1]})$
  for $m = 1..M$
    $\varepsilon \leftarrow \mathrm{rand}\,(d)$
    $X[m] \leftarrow F\,(X[m]\,,\varepsilon)$
  end
end

Final values in $X[m]\,,\ m = 1..M$
$\mathrm{setseed}\,(S[0])$
$\mathrm{free}\,(S)$
for $i = N-1..1$
  for $m = 1..M$
    $\varepsilon \leftarrow \mathrm{rand}\,(d)$
    $X[m] \leftarrow F^{-1}\,(X[m]\,,\varepsilon)$
  end
end

- No seed change nor storage

General memory reduction method (v2)

Euler scheme

for $i = 1..N-1$
 $S[i] \leftarrow \text{getseed}()$
 for $m = 1..M$
  $\varepsilon \leftarrow \text{rand}(d)$
 end
end
Initialize $X[m]\,,\ m = 1..M$
for $i = 1..N-1$
 $\text{setseed}(\text{S}[\text{N}-\text{i}+1])$
 for $m = 1..M$
  $\varepsilon \leftarrow \text{rand}(d)$
  $X[m] \leftarrow F(X[m], \varepsilon)$
 end
end

Inverse Euler scheme

Final values in $X[m]\,,\ m=1..M$
$\text{setseed}(S[0])$
$\text{free}(S)$
for $i = N-1..1$
 for $m = 1..M$
  $\varepsilon \leftarrow \text{rand}(d)$
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Euler scheme

for $i = 1..N-1$
  $S[i] \leftarrow \text{getseed}\,()$
  for $m = 1..M$
    $\varepsilon \leftarrow \text{rand}\,(d)$
  end
end
Initialize $X[m]\,,\; m = 1..M$
for $i = 1..N-1$
  $\text{setseed}\,(\mathrm{S[N-i+1]})$
  for $m = 1..M$
    $\varepsilon \leftarrow \text{rand}\,(d)$
    $X[m] \leftarrow F\,(X[m]\,,\varepsilon)$
  end
end

Inverse Euler scheme

Final values in $X[m]\,,\, m{=}1..M$
$\text{setseed}\,(S[0])$
$\text{free}\,(S)$
for $i = N-1..1$
  for $m = 1..M$
    $\varepsilon \leftarrow \text{rand}\,(d)$
    $X[m] \leftarrow F^{-1}\,(X[m]\,,\varepsilon)$
  end
end

- No seed change nor storage

General memory reduction method (v2)

Euler scheme

for $i = 1..N-1$
    $S[i] \leftarrow \text{getseed}()$
    for $m = 1..M$
        $\varepsilon \leftarrow \text{rand}(d)$
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Initialize $X[m]\,,\ m = 1..M$
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Stability ?

$\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{x})\right) = \mathbf{x}$?

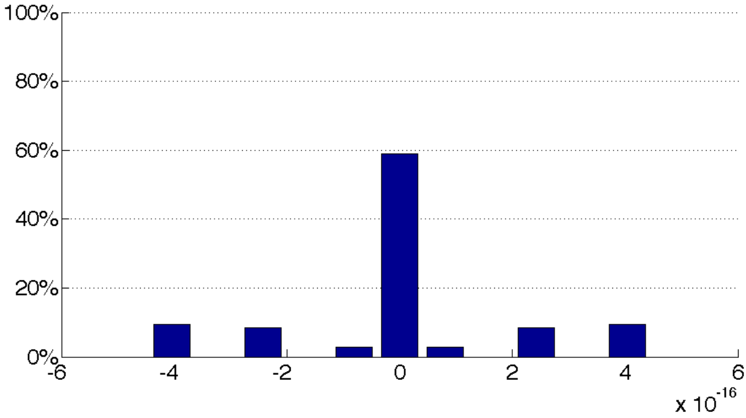


Figure: $f\left(f^{-1}(x)\right) - x$ for $f(x) = 2x + 3$ on a sample of 10^7 pts in $[0, 1]$

Stability ?

$$f\left(f^{-1}(x)\right) = x + \epsilon Z\,,\ \epsilon > 0\,,\ Z \text{ r.v.}$$

Examples

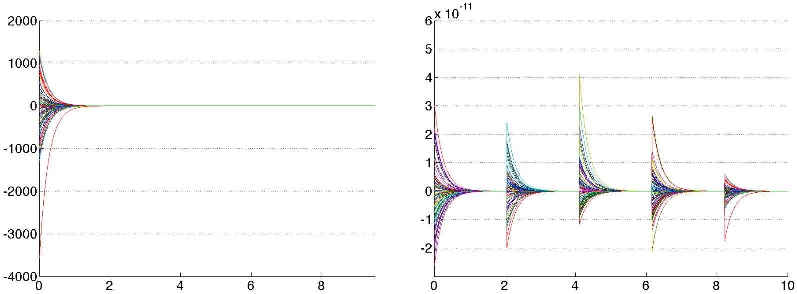
Arithmetic B.M. $\quad \tilde{x}_{t_0} - x_{t_0} = \epsilon \sum_{i=0}^{N-1} z_i$

$\hookrightarrow$ tame

Ornstein-Uhlenbeck $\quad \tilde{x}_{t_0} - x_{t_0} = \epsilon \sum_{i=0}^{N-1} \frac{1}{(1-\alpha h)^i} z_i$

$\hookrightarrow$ problem if $T > \frac{1}{\alpha} \ln\left(\frac{1}{\epsilon}\right)$

Stabilizing correction



(a) Without intermediate saves (b) With intermediate saves

Figure: Compound rounding error, Ornstein-Uhlenbeck

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2 Numerical scheme

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Application to investments in power generation

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Not included

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- Multizone / Network
- Dynamic constraints / Time to build

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Factors modelling

Demand D_t

deterministic seasonalities + O.U. process

Capacities C_t

Controlled installed capacities : $I_t = I_{0-} + \sum_{\tau_n \leq t} \zeta^n$

Available capacities : $C_t = I_t \times A_t$

A_t = availability rate (outages)

$A_t = \mathcal{T}\,(\mathrm{O.U.} + \mathrm{seasonalities})$

$\mathcal{T} : \mathbb{R} \mapsto]0,1[$ (cf. [Wagner, 2012])

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Fuels and CO_2 prices

$S_t^0 = \mathrm{CO}_2$ price
S_t^k =fuel price for tech. $k \geq 1$

$$dS_t = \Xi S_t dt + \Sigma S_t dW_t^S$$

$\Rightarrow$ **Cointegrated geometric B.M.**

- Non-stationary individual prices
- Stationary linear combinations

(provided $\mathrm{rank}(\Xi)$ not full, cf. [Benmenzer et al., 2007])

Full fuel price in €/MWh

$$\tilde{S}_t^k = h_k^0 S_t^0 + h_k S_t^k$$

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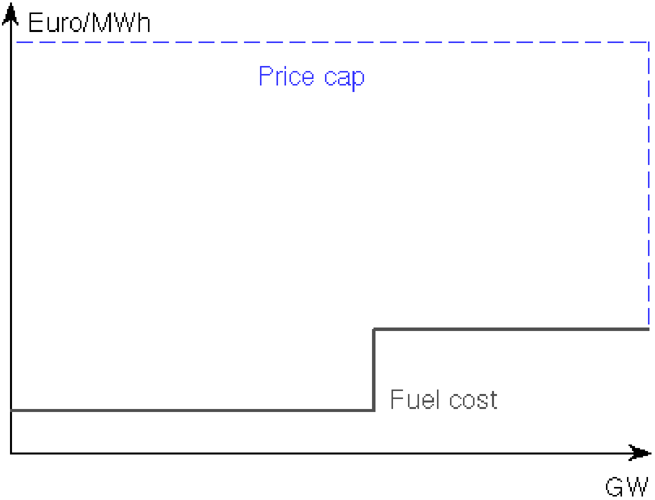
Power price



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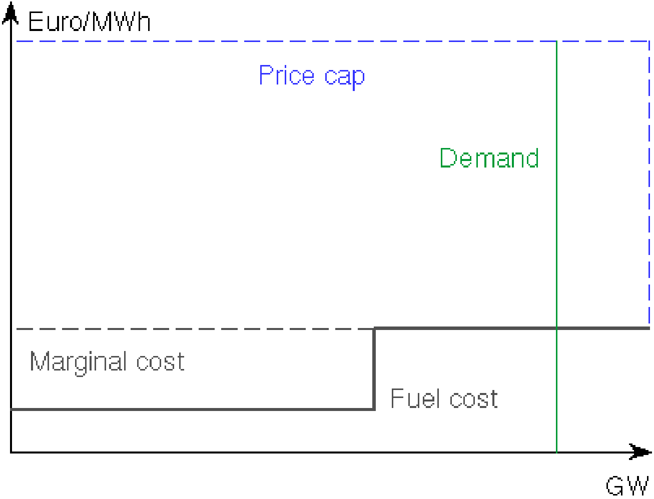

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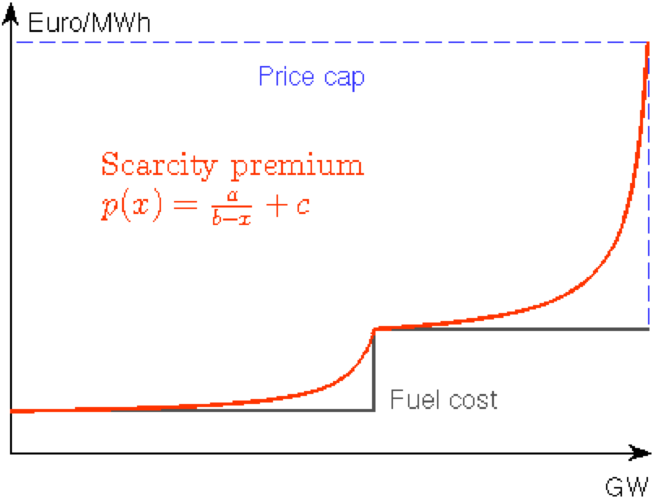
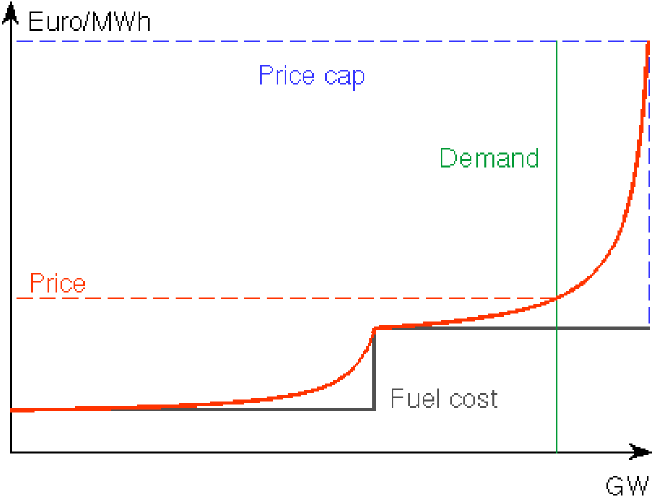

Euro/MWh
Price cap
Scarcity premium
$p(x) = \frac{a}{b-x} + c$
Fuel cost
GW

Power price



Objective function

New plant ζ^j of type j

Cost $\kappa^j\left(\zeta^j\right) = \kappa_f^j + \zeta^j \times \kappa_p^j$

Output $O_t^j = \min\left(\zeta^j \times A_t^j, \left(D_t - \overline{C}_t^{(j-1)}\right)^+\right)$

Revenue $\left(P_t - \tilde{S}_t^j\right)^+ - \kappa_m^j$

Total Gain $\int_0^\infty e^{-\rho s}\left(O_s^j\left(P_s - \tilde{S}_s^j\right)^+ - \kappa_m^j\right) ds - \kappa^j\left(\zeta^j\right)$

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Numerical example

2 technologies

"Base" tech. $\tilde{S}_0^1 = 40€/\text{MWh}$, vol.5%,
$I_0^1 = 67\text{GW}$, $\kappa_p^1 = 2.00\,10^9€/\text{GW}$

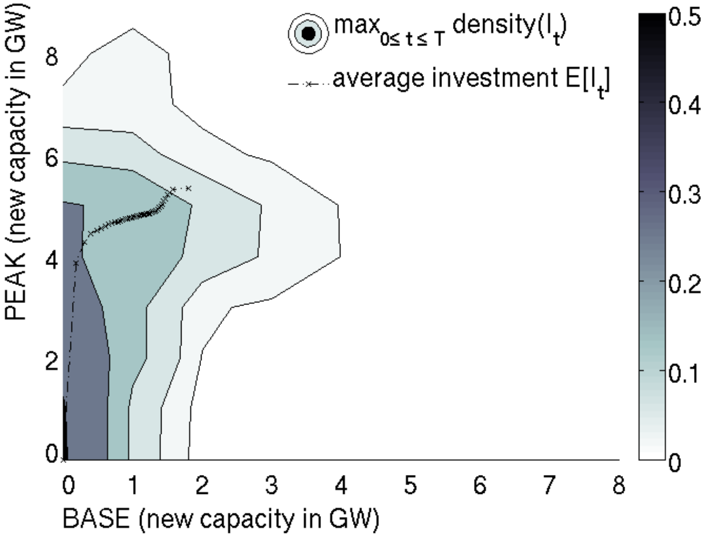
"Peak" tech. $\tilde{S}_0^2 = 80€/\text{MWh}$, vol.15%,
$I_0^1 = 33\text{GW}$, $\kappa_p^2 = 0.24\,10^9€/\text{GW}$

$\tilde{S}_t^2 - 2\tilde{S}_t^1$ stationary, $D_0 = 70\text{GW}$

Numerical parameters

$T = 40$ years (+20 years for terminal values)
$h = 1/730$ (2 steps per day), but investments only once a year
$b = 2^6 = 64$ base functions, piecewise linear
$M = 5000$ M.C. trajectories

Optimal investments



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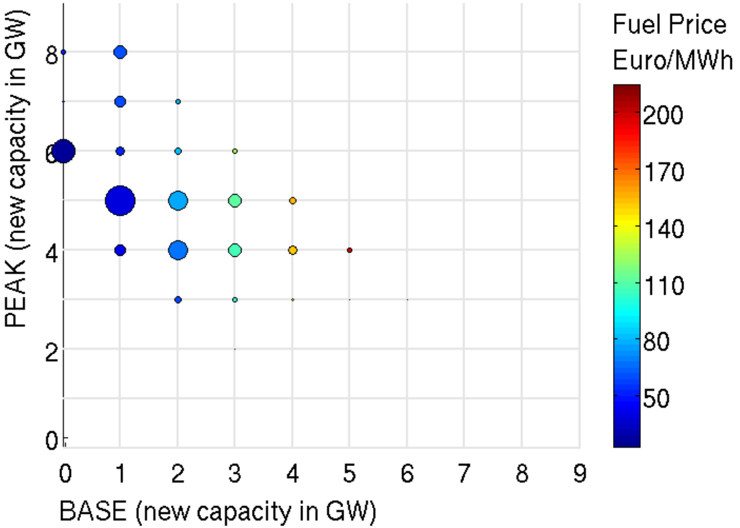
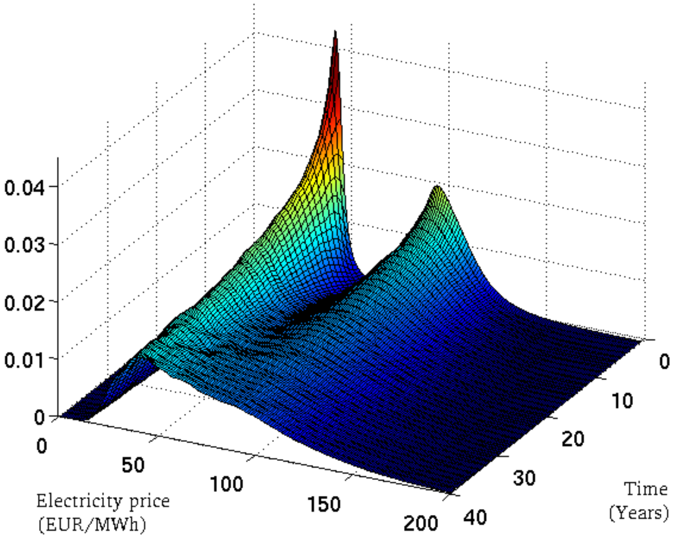
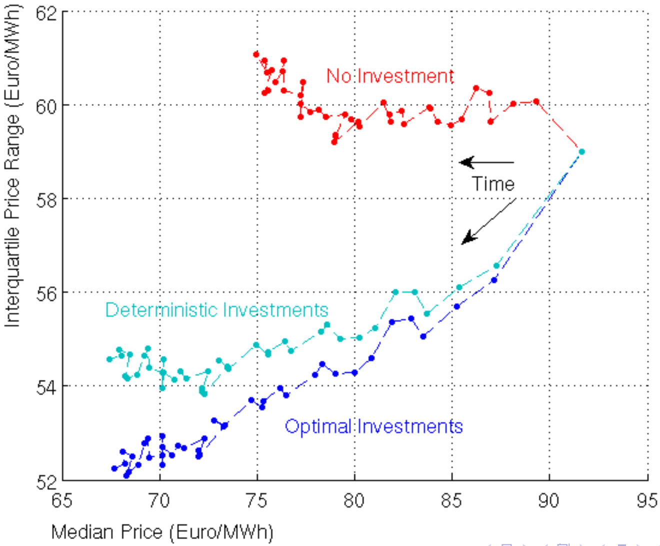


Price densities



Strategies comparison



Extensions

- More technologies ⇒ more dimensions
- Asymptotic confidence intervals ⇒ second algorithm
- Multiple agents

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Questions

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