Hedging under multiple risk constraints

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Introduction

- Practical motivation: EDF is required by law to hold funds dedicated to the decommissionning of the nuclear power plants, as well as the treatment and storage of the radioactive waste.
- Management of an asset portfolio dedicated to cover the long-term future costs for the nuclear plants "with a high degree of confidence" - probabilistic risk constraints
- Related subjects: Asset Liability Management (ALM) problem for pension funds, banks and insurance companies; longevity risk; Basel or Solvency regulatory capital requirement etc.

Literature

- In the literature, the continuous-time setting is mostly considered, with a single liability. We mention for example :
  - Föllmer-Leukert (1999, 2000): quantile hedging
  - El Karoui-Jeanblanc-Lacoste (2001): portfolio with American guarantee
  - Boyle-Tian (2007): desired benchmark strategy problem
  - Bouchard-Elie-Touzi (2009), Bouchard-Moreau-Nutz (2012): stochastic target problem
- In our work, we consider a finite set of future liabilities, with risk constraints imposed at each payment date.

Outline

- Formulation of ALM problem with random liabilities.
- Three types of probabilistic risk constraints :
  - European-style constraint
  - Time-consistent constraint
  - Lookback constraint
- Solution of these problems by a dynamic programming approach
  - determine the relationship between the risk constraints at different dates
  - find the least expensive portfolio which outperforms the stochastic benchmark under different risk constraints.
- Examples and numerical illustrations

A discrete-time setting

- Market $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$
- Payment dates : $0 = t_0 < t_1 < \cdots < t_n \leq T$
- Let $\mathcal{F}_k := \mathcal{F}_{t_k}$
- Future liable payments : $P_1, \cdots, P_n$ at $t_1, \cdots, t_n$, which are $\mathcal{F}_1, \cdots, \mathcal{F}_n$ measurable random variables.
- Portfolio held by an agent with value $\widetilde{V}$ for the payment :

$$\widetilde{V}_{t_i} = \widetilde{V}_{t_i-} - P_i$$

- Let $\mathbb{Q}$ be an equivalent probability measure such that all admissible self-financing portfolios are $\mathbb{Q}$-supermartingales, and for any $\mathbb{Q}$-supermartingale $(M_t)_{0 \leq t \leq T}$, there exists an admissible portfolio $(V_t)_{0 \leq t \leq T}$, which satisfies $V_t = M_t$ for all $t \in [0, T]$.

The associated self-financing portfolio is :

$$V_t = \widetilde{V}_t + \sum_{i \geq 1, t_i < t} P_i$$

The benchmark process is :

$$S_t = \sum_{i \geq 1, t_i < t} P_i$$

The agent has certain risk tolerance and searches for

- the cheapest portfolio V which outperforms the benchmark process S
- the $\mathbb{Q}$-supermartingale M with the smallest initial value which dominates the benchmark S at all dates $t_1, \cdots, t_n$ under some risk constraint.

Risk constraints

Let $\ell : \mathbb{R} \to \mathbb{R}$ be a loss function which is convex, decreasing and bounded from below.

Example :

- $\ell(x) = (-x)^+$
- $\ell(x) = e^{-px} - 1,\ p > 0$

European-style constraint

Find the minimal value of M_0 s.t. there exists a $\mathbb{Q}$-supermartingale $(M_k)_{k=0}^n$ with

$$\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k)] \leq \alpha_k \quad \text{for} \quad k = 1, \ldots, n. \tag{1}$$

We denote the set of all such $\mathbb{Q}$-supermartingales by $\mathcal{M}_{EU}$.

American-style constraint

Time-consistent constraint

Find the minimal value of M_0 s.t. there exists a $\mathbb{Q}$-supermartingale $(M_k)_{k=0}^n$ with

$$\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k) \,|\, \mathcal{F}_{k-1}] \leq \alpha_k \quad \text{for} \quad k = 1, \ldots, n. \tag{2}$$

We denote the set of all such $\mathbb{Q}$-supermartingales by $\mathcal{M}_{TC}$.

The above constraint can be viewed as an American-style one:

- let $X_k = \sum_{i=1}^k l(M_i - S_i) - \alpha_i$
- condition (2) is equivalent to any of the following conditions :
  - $(X_k)_{k=0}^n$ is a $\mathbb{P}$-supermartingale
  - for all $\mathbb{F}$-stopping times τ and σ taking values in $\{0, \cdots, n\}$, such that $\tau \leq \sigma$,

$$\mathbb{E}^{\mathbb{P}}[\sum_{i=\tau+1}^{\sigma} \{\ell(M_i - S_i) - \alpha_i\}] \leq 0$$

Lookback-style constraint

Maximum constraint

Find the minimal value of M_0 s.t. there exists a $\mathbb{Q}$-supermartingale $(M_k)_{k=0}^n$ with

$$\mathbb{E}^{\mathbb{P}}[\max_{k=1,\ldots,n}\{\ell(M_k - S_k) - \alpha_k\}] \leq 0. \tag{3}$$

We denote the set of such $\mathbb{Q}$-supermartingales by $\mathcal{M}_{LB}$.

- So for a given threshold vector $(\alpha_1, \cdots, \alpha_n)$, the following relation holds:

$$\mathcal{M}_{LB} \subset \mathcal{M}_{TC} \subset \mathcal{M}_{EU}.$$

- The initial capital requirement for the three constraints satisfy

$$M_0^{EU} \leq M_0^{TC} \leq M_0^{LB}$$

Solving the three problems

- We apply a dynamic programming approach in each case.
- The dynamic programming structure depends on the nature of the constraint.
- At each time step, the constraint need to be verified for succeeding dates.
- We obtain recursive formulas for the three cases.

The time-consistent case

The (TC) problem :

Recall that $\mathcal{M}_{TC}$ denotes the set of all $\mathbb{Q}$-supermartingales $(M_k)_{k=0}^n$ such that

$$\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k) \,|\, \mathcal{F}_{k-1}] \leq \alpha_k \quad \text{for} \quad k = 1, \ldots, n.$$

Dynamic version :

For any $k \in \{0, \ldots, n\}$, let $\mathcal{M}_{TC,k}$ be the set of the $\mathbb{Q}$-supermartingales $(M_t)_{t=k}^n$ such that

$$\mathbb{E}^{\mathbb{P}}[\ell(M_t - S_t)|\mathcal{F}_{t-1}] \leq \alpha_t \text{ for } t = k+1, \ldots, n$$

Value process for the (TC) case

Define the value process in a backward manner :

- let

$$V_n = -\infty$$

- for any $k < n$,

$$V_k = \operatorname*{ess\,inf}_{M \in \mathcal{F}_{k+1}} \{\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_k] : M \geq V_{k+1} \text{ and } \mathbb{E}^{\mathbb{P}}[\ell(M - S_{k+1})|\mathcal{F}_k] \leq \alpha_{k+1}\}$$

Proposition

$$V_k = \operatorname*{ess\,inf}_{(M_t)_{t=k}^n \in \mathcal{M}_{TC,k}} M_k, \quad k = 0, \dots, n-1.$$

A more explicit result

Let ℓ be strictly convex, strictly decreasing and of class C^1. Assume $\alpha_k > \lim_{x\to+\infty} \ell(x)$ for all k and that the derivative $\ell'(x)$ satisfies Inada's conditions $\lim_{x\to-\infty} \ell'(x) = -\infty$ and $\lim_{x\to+\infty} \ell'(x) = 0$. Then

$$V_{n-1} = \mathbb{E}^{\mathbb{Q}}[S_n + I(\lambda_{n-1} Z_n / Z_{n-1}) | \mathcal{F}_{n-1}]$$

where I is the inverse of ℓ' and $\lambda_{n-1} \in \mathcal{F}_{n-1}$ is the solution of

$$\mathbb{E}^{\mathbb{P}}[\ell(I(\lambda_{n-1} Z_n / Z_{n-1})) | \mathcal{F}_{n-1}] = \alpha_n,$$

and Z is the Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$. For $k < n$,

$$\begin{aligned} V_{k-1} &= \mathbb{E}^{\mathbb{Q}}[V_k | \mathcal{F}_{k-1}] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[\{S_k - V_k + I(\lambda_{k-1} Z_k / Z_{k-1})\}^+ \middle| \mathcal{F}_{k-1}\right] \mathbf{1}_{\mathbb{E}^{\mathbb{P}}[l(V_k - S_k)|\mathcal{F}_{k-1}] > \alpha_k}, \end{aligned}$$

where $\lambda_{k-1} \in \mathcal{F}_{k-1}$ is the solution of

$$\mathbb{E}^{\mathbb{P}}\left[\ell\left(I(\lambda_{k-1} Z_k / Z_{k-1}) \vee (\widehat{V}_k - S_k)\right) \middle| \mathcal{F}_{k-1}\right] = \alpha_k.$$

The lookback-style case

The (LB) problem :

Recall that $\mathcal{M}_{LB}$ denotes the set of all $\mathbb{Q}$-supermartingales $(M_k)_{k=0}^n$ such that

$$\mathbb{E}^{\mathbb{P}}[\max_{k=1,\dots,n}\{\ell(M_k - S_k) - \alpha_k\}] \leq 0.$$

- In the dynamic programming of this problem, the maximum should be taken into account in the value process.

Value process of the (LB) problem

- For any $k = 0, \cdots, n-1$, let $V_k(Y_k, z)$ be the essential infimum of $M_k \in \mathcal{F}_k$ such that there exists a $\mathbb{Q}$-supermartingale $(M_t)_{t=k}^n$ and a $\mathbb{P}$-supermartingale $(Y_t)_{t=k}^n$ verifying

$$\max\left\{z, \max_{t \in \{k+1,\dots,n\}} \{\ell(M_t - S_t) - \alpha_t\}\right\} \leq Y_n.$$

Proposition

$$V_0(0, -\infty) = \inf_{(M_k)_{k=0}^n \in \mathcal{M}_{LB}} M_0.$$

Recursive formula for the (LB) problem

By convention, we define

$$V_n(y,z) = (+\infty)\mathbf{1}_{\{z>y\}} + (-\infty)\mathbf{1}_{\{z\le y\}}$$

Proposition

$V_k(Y_k, z)$ equals the essential infimum of $\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_k]$ where $M \in \mathcal{F}_{k+1}$ such that there exists $Y_{k+1} \in \mathcal{F}_{k+1}$ satisfying

$$\begin{cases} \mathbb{E}^{\mathbb{P}}[Y_{k+1}|\mathcal{F}_k] = Y_k, \\ M \ge V_{k+1}\big(Y_{k+1}, \max(z, \ell(M - S_{k+1}) - \alpha_{k+1})\big). \end{cases}$$

In particular,

$$V_{n-1}(Y_{n-1}, z) = \operatorname*{ess\,inf}_{M\in\mathcal{F}_n} \{\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_{n-1}] : Y_n \ge \max(z, \ell(M - S_n) - \alpha_n)\}$$

$$= \operatorname*{ess\,inf}_{M\in\mathcal{F}_n} \{\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_{n-1}] : Y_{n-1} \ge \mathbb{E}^{\mathbb{P}}[\max(z, \ell(M - S_n) - \alpha_n)|\mathcal{F}_{n-1}]\}$$

Solving the European-style case

The (EU) problem :

$\mathcal{M}_{EU}$ denotes the set of all $\mathbb{Q}$-supermartingales $(M_k)_{k=0}^n$ such that

$$\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k)] \leq \alpha_k \quad \text{for} \quad k = 1, \ldots, n.$$

- Let V_0 be the infimum value of M_0 such that there exist a $\mathbb{Q}$-supermatingale $(M_t)_{t=0}^n$ and a family of $\mathbb{P}$-supermartingales $(Y_t^k)_{t=0}^k$, $k = 1, \ldots, n$, satisfying

$$Y_0^k = \alpha_k \text{ and } \ell(M_k - S_k) \leq Y_k^k$$

Proposition

$$V_0 = \inf_{(M_k)_{k=0}^n \in \mathcal{M}_{EU}} M_0.$$

Dynamic version of the (EU) problem

- For any $k = 0, \dots, n-1$ and a family of $\mathcal{F}_k$-measurable random variables $Y^{k+1}, \dots, Y^n$, let $V_k(Y^{k+1}, \cdots, Y^n)$ be the essential infimum of all $M_k \in \mathcal{F}_k$ such that there exists a $\mathbb{Q}$-supermartingale $(M_t)_{t=k}^n$ and

$$\mathbb{E}^{\mathbb{P}}[\ell(M_t - S_t)|\mathcal{F}_k] \le Y^t, \quad t = k+1, \dots, n.$$

- By convention, $V_n = -\infty$.

Proposition

$V_k(Y^{k+1}, \dots, Y^n)$ equals the essential infimum of all $\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_k]$, $M \in \mathcal{F}_{k+1}$ such that there exist a family of $\mathbb{P}$-supermartingales $(Y_t^{k+1})_{t=k}^{k+1}, \cdots, (Y_t^n)_{t=k}^n$ which satisfy :

$$\begin{cases} Y_k^t = Y^t \text{ for } t = k+1, \cdots, n, \\ \ell(M - S_{k+1}) \le Y_{k+1}^{k+1}, \\ M \ge V_{k+1}(Y_{k+1}^{k+2}, \cdots, Y_{k+1}^n). \end{cases}$$

Corollary

$V_k(\alpha_{k+1}, \ldots, \alpha_n)$ equals the essential infimum of all $\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_k]$, $M \in \mathcal{F}_{k+1}$ such that there exist a family of $\mathcal{F}_{k+1}$-measurable random variables $Y^{k+2}, \cdots, Y^n$ which satisfy :

$$\begin{cases} \mathbb{E}^{\mathbb{P}}[Y^t|\mathcal{F}_k] = \alpha_t \text{ for } t = k+2, \cdots, n, \\ \mathbb{E}^{\mathbb{P}}[\ell(M - S_{k+1})|\mathcal{F}_k] \leq \alpha_{k+1}, \\ M \geq V_{k+1}(Y^{k+2}, \cdots, Y^n). \end{cases}$$

In particular

$$V_{n-1}(\alpha_n) = \operatorname*{ess\,inf}_{M \in \mathcal{F}_n} \{\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_{n-1}] \text{ s.t. } \mathbb{E}^{\mathbb{P}}[\ell(M - S_n)|\mathcal{F}_{n-1}] \leq \alpha_n\}$$

The risk-neutral case $\mathbb{P} = \mathbb{Q}$, $n = 2$

For the European and time-consistent constraints:

$$V_1(\alpha_2) = \mathbb{E}[S_2|\mathcal{F}_1] + \ell^{-1}(\alpha_2)$$

For the lookback constraint:

$$V_1(y,z) = +\infty \quad \text{if } y < z \text{ and } \quad \mathbb{E}[S_2|\mathcal{F}_1] + \ell^{-1}(y + \alpha_2) \quad \text{otherwise}$$

For the minimal initial value at $t = 0$:

- $$V_0^{EU} = \inf_{M \in \mathcal{F}_1} \{\mathbb{E}[M] : \mathbb{E}[\ell(M - S_1)] \leq \alpha_1, \mathbb{E}[\ell(M - \mathbb{E}[S_2|\mathcal{F}_1])] \leq \alpha_2\}$$
- $$V_0^{TC} = \inf_{M \in \mathcal{F}_1} \{\mathbb{E}[M] : \mathbb{E}[\ell(M - S_1)] \leq \alpha_1, \ell(M - \mathbb{E}[S_2|\mathcal{F}_1]) \leq \alpha_2\}$$
- $$V_0^{LB} = \inf_{M \in \mathcal{F}_1} \{\mathbb{E}[M] : \mathbb{E}[\max\{\ell(M - S_1) - \alpha_1, \ell(M - \mathbb{E}[S_2|\mathcal{F}_1]) - \alpha_2\}] \leq 0\}$$

Example with $\ell(x) = (-x)^+$

A technical lemma

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X, Y \in \mathcal{F}$ and $\alpha, \beta \geq 0$. Then,

$$
\begin{aligned}
V_0 = \inf_{M \in \mathcal{F}} \{\mathbb{E}[M] : \mathbb{E}[(X-M)^+] \leq \alpha, \mathbb{E}[(Y-M)^+] \leq \beta\} \\
= \max\{\mathbb{E}[X-\alpha], \mathbb{E}[Y-\beta], \mathbb{E}[X \vee Y - \alpha - \beta]\}.
\end{aligned}
$$

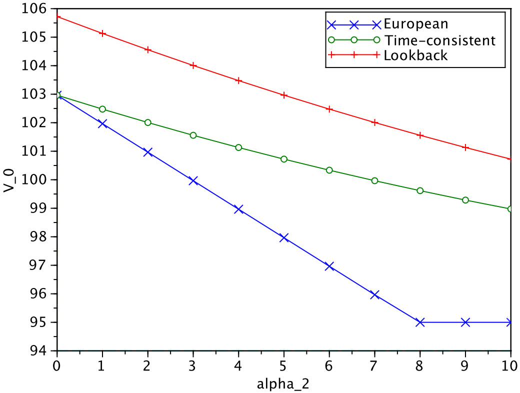
and

$$
\begin{aligned}
V_0' = \inf_{M \in \mathcal{F}} \{\mathbb{E}[M] : \mathbb{E}[\max((X-M)^+ - \alpha, (Y-M)^+ - \beta)] \leq 0\} \\
= \mathbb{E}[(X-\alpha) \vee (Y-\beta)].
\end{aligned}
$$

$$
\begin{aligned}
V_0^{EU} &= \max\left(\mathbb{E}[S_1] - \alpha_1, \mathbb{E}[S_2] - \alpha_2, \mathbb{E}[(S_1 \vee \mathbb{E}[S_2|\mathcal{F}_1])] - \alpha_1 - \alpha_2\right), \\
V_0^{TC} &= \max\left(\mathbb{E}[S_2] - \alpha_2, \mathbb{E}[S_1 \vee (\mathbb{E}[S_2|\mathcal{F}_1] - \alpha_2)] - \alpha_1\right), \\
V_0^{LB} &= \mathbb{E}[\max\left(S_1 - \alpha_1, (\mathbb{E}[S_2|\mathcal{F}_1] - \alpha_2)\right)].
\end{aligned}
$$

Numerical illustration : cost of hedging two objectives

$\ell(x) = (-x)_+$ and $\mathbb{P} = \mathbb{Q}$. The model : $S_1 = S_0 e^{\sigma Z_1 - \frac{\sigma^2}{2}}$ and $S_2 = S_0 e^{\sigma Z_2 - \frac{\sigma^2}{2}}$ where $S_0 = 100$, $\sigma = 0.2$ and $Z_1, Z_2 \sim N(0,1)$ with correlation $\rho = 50\%$. The first threshold $\alpha_1 = 5$.



Cost of hedging both objectives in an almost sure way: 107.966.

Explicit example for multi-objectives

- For arbitrary n, the explicit solution can only be obtained in some particular cases.

Example

Assume that $\ell(x) = (-x)^+$, $\mathbb{P} = \mathbb{Q}$, $\alpha_1, \ldots, \alpha_n \geq 0$ and the process $(S_t)_{0 \leq t \leq n}$ is non-decreasing. Then

$$M_0^{EU} = \max_{k \in \{1, \ldots, n\}} \{\mathbb{E}[S_k] - \alpha_k\}$$

Thanks for your attention !

TC case: Idea of the proof

Denote by $\widehat{V}_k$ the essential infimum of M_k with $(M_t)_{t=n}^{k} \in \mathcal{M}_{TC,k}$.

- The proof is by backward induction on k : assume $V_{k+1} = \widehat{V}_{k+1}$.
- "$V_k \leq \widehat{V}_k$" : If $(M_t)_{t=k}^{n} \in \mathcal{M}_{TC,k}$, then $(M_t)_{t=k+1}^{n} \in \mathcal{M}_{TC,k+1}$ and by induction hypothesis $M_{k+1} \geq M_{k+1}$. By supermartingale property, we have $V_k \leq \mathbb{E}^{\mathbb{Q}}[M_{k+1}|\mathcal{F}_k] \leq M_k$, so $V_k \leq \widehat{V}_k$.
- "$V_k \geq \widehat{V}_k$" : The opposite inequality is more delicate and relies on the following fact: if $(M_t)_{t=k+1}^{n}$ and $(M'_t)_{t=k+1}^{n}$ are supermartingales in $\mathcal{M}_{TC,k+1}$, then there exists $(M''_t)_{t=k+1}^{n} \in \mathcal{M}_{TC,k+1}$ such that $M''_{k+1} = \min(M_{k+1}, M'_{k+1})$. Thus we can realize the essential infimum defining $\widehat{V}_{k+1}$ as the limit of a decreasing sequence.