On the control of the difference between two Brownian Motions: A dynamic copula approach

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Context

- Modeling dependence between two prices : one is electricity X_t, the other is a fuel Y_t used to produce electricity (coal, crude oil).
- This modeling is used to price and hedge spread options $(X_t - Y_t)^+$ (return of a plant).
- Today : Prices are modeled by diffusions and dependence by a constant correlation matrix.
- Limitations of this modeling :
    - Symmetry in the distribution of the difference between the prices,
    - $\mathbb{P}(X_t > Y_t) \simeq \frac{1}{2}$ for $t > 0$.

Motivation

- Is it possible to obtain higher values for the probability that the price of electricity is over the price of its fuel ?
- Use of **copulae** to find new structures of dependence in a **dynamic time framework.**
- Some bibliography on dynamic copulae :
  - Dynamic copulae in discrete time : Fermanian and Wegkamp (2004); Patton (2006).
  - Dynamic copulae for time-dependence of a process : Darsow et al. (1992).
  - Dynamic copulae for dependence between two processes : Jaworski and Krzywda (2013); Bosc (2012).

Contribution

- Find new asymmetric copulae admissible for Brownian motions.
- Control $\mathbb{P}(X_t - Y_t \geq \eta)$ with copulae between its infimum and its supremum for $t > 0$ and $\eta > 0$ when X_t and Y_t are Brownian motions .
- Derive a model allowing us to control $\mathbb{P}(X_t - Y_t \geq \eta)$, having value over $\frac{1}{2}$ for it and that can be applied to energy markets.

Outline

Copulae and Sklar's theorem

Definition (Copula)

A function $C : [0,1]^2 \mapsto [0,1]$ is a copula if :

(i) C is 2-increasing,

(ii) $C(u,0) = C(0,v) = 0,\ u,v \in [0,1]$,

(iii) $C(u,1) = u, C(1,u) = u,\ u \in [0,1]$.

Theorem (Sklar (1959))

Let X and Y be two random variables with marginal cdfs F^X and F^Y and bivariate cdf $F^{X,Y}$. There exists a copula C such that

$$F^{X,Y}(x,y) = C\left(F^X(x), F^Y(y)\right),\ x,y \in \mathbb{R}.$$

If F^X and F^Y are continuous, the copula is uniquely defined. Conversely, if C is a copula then the function defined by $F^{X,Y}(x,y) = C\left(F^X(x), F^Y(y)\right)$ for $x, y \in \mathbb{R}$ is a bivariate cdf.

Definition

Definition (Admissible copula for Markovian diffusions)

A collection of copula $C = (C_t)_{t \geq 0}$ is an admissible copula for the n real valued Markovian diffusions, $n \geq 2$, $\left(X^i\right)_{1 \leq i \leq n}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if there exists a $\mathbb{R}^m$ Markovian diffusion $Z = \left(Z^i\right)_{1 \leq i \leq m}$, $m \geq n$, defined on a probability extension of $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\begin{cases} \mathcal{L}\left(Z^i\right) = \mathcal{L}\left(X^i\right), 1 \leq i \leq n, \\ Z_0^i = X_0^i, 1 \leq i \leq n, \\ \text{for } t \geq 0, \text{ the copula of } \left(Z_t^i\right)_{1 \leq i \leq n} \text{ is } C_t. \end{cases}$$

Brownian motion case

- We denote by $\mathcal{C}_B$ the set of admissible copulae for Brownian motions in dimension 2.
- Gaussian copulae are well known for Brownian motions.
- Our definition includes several classical models :
    - Deterministic correlation models
    - Local correlation models
    - Stochastic correlation models

Closed formula

Theorem

Let $h > 0$. The copula

$$C_t^{ref,h}(u,v) = \begin{cases} v \text{ if } \Phi^{-1}(u) - \Phi^{-1}(v) \geq \frac{2h}{\sqrt{t}} \\ W(u,v) + \Phi\left(\Phi^{-1}(M(u,1-v)) - \frac{2h}{\sqrt{t}}\right) \\ \text{if } \Phi^{-1}(u) - \Phi^{-1}(v) < \frac{2h}{\sqrt{t}} \end{cases}$$

is in $\mathcal{C}_B$. It is the copula between a Brownian motion and its reflection according to the barrier h.

Φ is a cdf of a standard normal random variable, W and M are the lower and upper Frechet copulae.

Structure of the copula

- Two states of correlation : one of comonotonicity ($\rho = 1$) and one of countermonotonicity ($\rho = -1$).
- The copula is asymmetric ($C_t(u, v) \neq C_t(v, u)$).

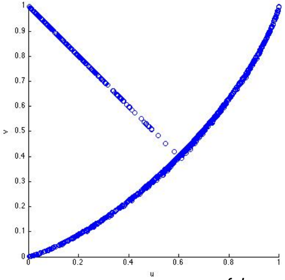


FIGURE: *Reflection Brownian Copula $C^{ref,h}$ at time $t = 1$ with $h = 2$.*

Extension : Correlated Reflection Brownian Motion

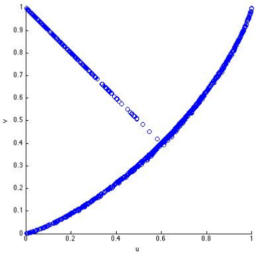
Let $h > 0$ and $\rho \in (0,1)$. The copula

$$\begin{cases} \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v) + \frac{2\rho h}{\sqrt{t}}\right) + v - \Phi\left(\Phi^{-1}(v) + \frac{2\rho h}{\sqrt{t}}\right) \text{ if } u \geq \Phi\left(\frac{h}{\sqrt{t}}\right) \\ \Phi_{-\rho}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) + \Phi_\rho\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}, \Phi^{-1}(1-v) - \frac{2\rho h}{\sqrt{t}}\right) + \\ \Phi_\rho\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}, \Phi^{-1}(v)\right) - \Phi\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}\right) \text{ if } u < \Phi\left(\frac{h}{\sqrt{t}}\right) \end{cases}$$

is in $\mathcal{C}^B$. Φ_ρ is the cdf of a couple of standard Gaussian random variables with correlation equal to ρ.

Structure of the Correlated Reflection Brownian Motion

- Two states of correlation ρ and $-\rho$.
- The copula is asymmetric.



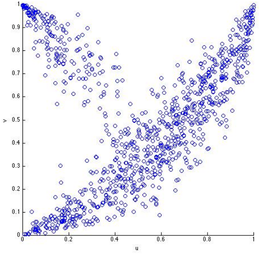


FIGURE: *Reflection Brownian Copula $C^{ref,h}$ at time $t = 1$ with $h = 2$ and extension with $\rho = 0.95$.*

Extension : Random Barrier

Let ξ be a positive r.v. with law having a density. The copula

$$v - \int_{-\infty}^{\Phi^{-1}(M(1-u,v))} \frac{e^{\frac{-w^2}{2}}}{\sqrt{2\pi}} \overline{F}^{\xi}\left(\frac{\sqrt{t}}{2}\left(\Phi^{-1}\left(M\left(u, 1-v\right)\right) - w\right)\right) dw$$

is in $\mathcal{C}^B$.

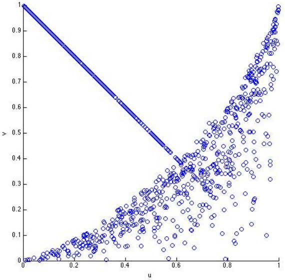


FIGURE: ξ *follows an exponential law with parameter* $\lambda = 2$.

1. New asymmetric dynamic copulae
2. Difference control between two Brownian motions
3. A two barrier correlation model
4. Application to Energy Markets

Static Result

If X and Y are two standard normal r.v. and $\mathbb{P}_C$ is the probability measure associated to the copula C of (X, Y), we consider :

$$\begin{array}{rcccc} \tilde{S}_\eta & : & \mathcal{C} & \to & [0,1] \\ & & C & \mapsto & \mathbb{P}_C(X - Y \geq \eta) \end{array}$$

Theorem

Let $\eta > 0$. We have :

(i) *$Ran\left(\tilde{S}_\eta|_{\mathcal{C}_G}\right) = \left[0, \Phi\left(\frac{-\eta}{2}\right)\right]$ with $\tilde{S}_\eta|_{\mathcal{C}_G}$ the restriction of $\tilde{S}_\eta$ to Gaussian copulae,*

(ii) $\sup_{C \in \mathcal{C}} \tilde{S}_\eta(C) = 2\Phi\left(\frac{-\eta}{2}\right)$,

(iii) *$Ran\left(\tilde{S}_\eta\right) = \left[0, 2\Phi\left(\frac{-\eta}{2}\right)\right]$.*

Sketch of the proof (1/2)

(ii) Let $r = 2\Phi\left(\frac{-\eta}{2}\right)$. We adapt the result of Frank et al. (1987) and the copula achieving the supremum is

$$C^r(u,v) = \begin{cases} M(u-1+r, v) & \text{if } (u,v) \in [1-r, 1] \times [0, r], \\ W(u,v) & \text{otherwise.} \end{cases}$$

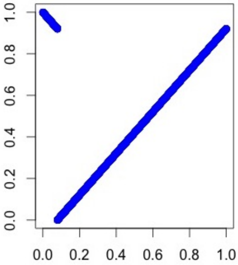


FIGURE: C^r *with* $\eta = 0.2$..

Sketch of the proof (2/2)

(iii) We use the concept of patchwork copula of Durante et al. (2013) : we consider the copula defined by $C_{G,\rho}$ in $[1-r,1]\times[0,r]$ and W otherwise, with ρ varying between -1 and 1.

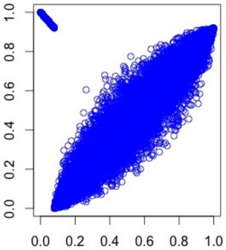


FIGURE: *Patchwork copula with* $\rho = 0.95$.

Dynamic Result

Given two Brownian motions B^1 and B^2, we consider :

$$\begin{array}{rcccl} S_{\eta,t} & : & \mathcal{C}_B & \to & [0,1] \\ & & C & \mapsto & \mathbb{P}_C\left(B_t^1 - B_t^2 \geq \eta\right). \end{array}$$

Theorem

Let $\eta > 0$ and $t > 0$. We have :

- (i) $Ran\left(S_{\eta,t}|_{\mathcal{C}_G^d}\right) = \left[0, \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$ *with* $S_{\eta,t}|_{\mathcal{C}_G^d}$ *the restriction of* $S_{\eta,t}$ *to dynamic Gaussian copulae,*
- (ii) $\sup_{C \in \mathcal{C}_B} S_{\eta,t}(C) = 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$,
- (iii) $Ran\left(S_{\eta,t}\right) = \left[0, 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$.

Sketch of the proof

(i) We use Gaussian copulae with constant correlations.

(ii) The Reflection Brownian Copula achieves the supremum.

(iii) We use the extension 2 of the Reflection Brownian Copula with a barrier following a translated exponential law $\frac{\eta}{2} + \mathcal{E}(\lambda)$, and λ varying between 0 and ∞.

Model

Let X and Y be two independent Brownian motions.

Let $\eta > 0$, $\nu < \eta$ and $\alpha_k = \begin{cases} 0 \text{ if } k = 0 \\ \eta \text{ if } k \text{ odd} \\ \nu \text{ if } k \text{ even, } k \neq 0 \end{cases}$.

Let $\left(\tilde{B}^k\right)_{k\geq 0}$, $\left(Y^k\right)_{k\geq 0}$ and $\left(\tau_k\right)_{k\geq 0}$ be defined by

$$\begin{cases} \tau_0 = 0 \\ \tilde{B}^0 = -X \end{cases},$$

$$\begin{cases} \tau_k = \inf\{t \geq \tau_{k-1} : X_t - Y_t^{k-1} = \alpha_k\} \quad k \geq 1 \\ \tilde{B}^k = \mathcal{R}(\tilde{B}^{k-1}, \tau_k) \quad k \geq 1 \\ Y^k = \rho \tilde{B}^k + \sqrt{1-\rho^2}\, Y \quad k \geq 0 \end{cases},$$

where $\mathcal{R}(B,\tau)_t = -B_t + 2(B_t - B_\tau)\mathbf{1}_{t\geq\tau}$.

Illustration

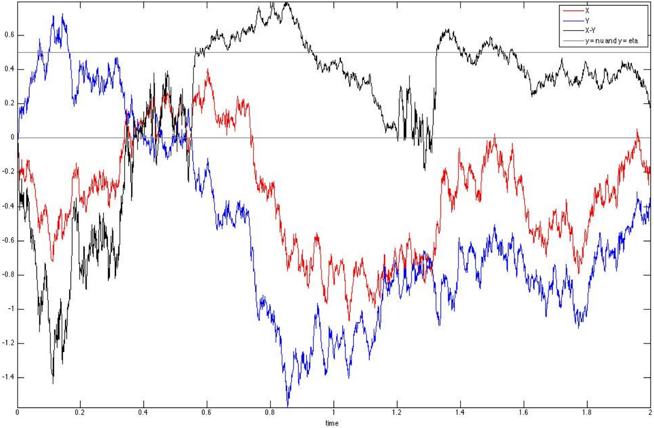


FIGURE: *One trajectory of X, Y^n, $X - Y^n$ in the multi-barrier correlation model with $\nu = 0$, $\eta = 0.5$, $\rho = 0.9$ and $n = \infty$.*

Main results

We can find a semi analytic formula for the CDF of $X_t - Y_t^n$ and we have the following results :

Theorem

Let $\rho > 0$. For $x \in [\nu, \eta]$, the sequence $\mathbb{P}\left(X_t - Y_t^n \geq x\right)$ is increasing with n. $X_t - Y_t^n$ converges in law towards $X_t - Y_t^{N_t}$ with $N_t < \infty$ a.s. a counting process.

Theorem

Let $t > 0$ and $\eta > z > 0$. $\forall x \in \left[0, \Phi\left(\frac{-z}{2\sqrt{t}}\right) + \Phi\left(\frac{z-2\eta}{2\sqrt{t}}\right)\right], \exists \rho \in [-1, 1] : \mathbb{P}\left(X_t(\rho) - Y_t^{N_t}(\rho) \geq z\right) = x$.

Survival function

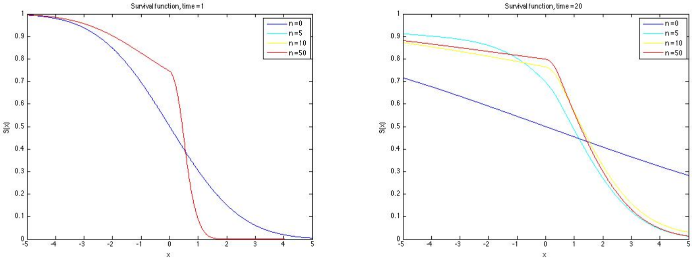


FIGURE: *Survival function of* $X_t - Y_t^n$ *with parameters* ν *= 0,* $\eta = 0.5$ *and* $\rho = 0.9$ *for* $t = 1$ *and* $t = 20$.

Trajectories of $X - Y^n$

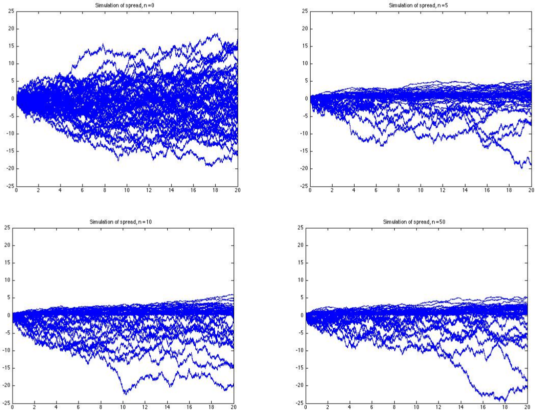


FIGURE: *50 simulations of $X - Y^n$ between time 0 and 20 with parameters ν = 0, $\eta = 0.5$ and $\rho = 0.9$ for $n = 0$ (upper left), $n = 5$ (upper right), $n = 10$ (bottom left), $n = 50$ (bottom right).*

Empirical copula

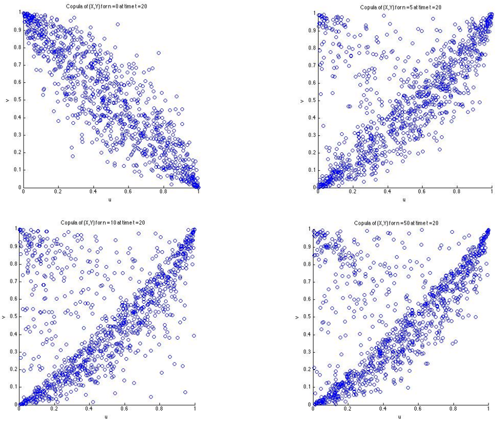


FIGURE: *Empirical copula of (X,Y) at time* $t = 20$ *with parameters* $\nu = 0$, $\eta = 0.5$ *and* $\rho = 0.9$ *for* $n = 0$ *(upper left),* $n = 5$ *(upper right),* $n = 10$ *(bottom left),* $n = 50$ *(bottom right).*

Equivalent Model

Let $\rho(x)$ be a continuous and monotone function such that $\rho(x) = \rho_1$ for $x \leq \nu$ et $\rho(x) = \rho_2$ for $x \geq \eta$. We consider :

$$\begin{cases} dX_t = dB_t^X \\ dY_t = \rho\left(X_t - Y_t\right) dB_t^X + \sqrt{1 - \rho\left(X_t - Y_t\right)^2} dB_t^Y \end{cases}$$

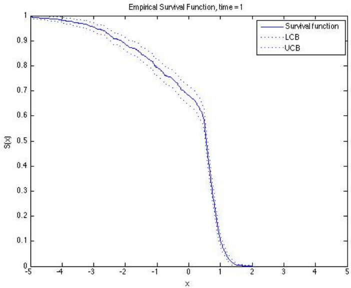


FIGURE: *Empirical survival function of $X_t - Y_t$ at $t = 1$.*

A two-factor model

Let $f^E(t,T)$ (resp. $f^C(t,T)$) the forward price of the electricity (resp. coal) at time t with maturity T. We consider the two-factor model (see Benth and Koekebakker (2008)) :

$$\begin{cases} df^E(t,T) = f^E(t,T)\left(\sigma_s^E e^{-\alpha_s^E(T-t)} dB_t^{E,s} + \sigma_l^E dB_t^{E,l}\right) \\ df^C(t,T) = f^C(t,T)\left(\sigma_s^C e^{-\alpha_s^C(T-t)} dB_t^{C,s} + \sigma_l^C dB_t^{C,l}\right) \end{cases}$$

We model the dependence as follow :

- $B^{E,s}$ and $B^{E,l}$ are independent,
- $B^{C,s}$ and $B^{C,l}$ are independent,
- $B^{E,s}$ and $B^{C,s}$ are independent,
- $B^{E,l}$ and $B^{C,l}$ are constructed following the multi-barrier correlation model.

The spot price for commodity i is equal to $f^i(t,t)$.

Data and parameters

We use the forward prices on electricity and on coal during 2014 in France to estimate these parameters. The method used for estimation is the first one of Féron and Daboussi (2015).

Parameters	Electricity	Coal
σ_l (year)	10.2555%	9.2602%
σ_s (year)	97.2925%	11.2134%
α_s (year)	17.0363	2.07832

TABLE: *Parameters of the two-factor model for electricity and coal*

We choose arbitrarly $\nu = 0$, $\eta = 0.5$, $\rho = 0.9$, $n = \infty$ for the multi-barrier correlation model. In the benchmark model (constant correlation), the correlation between $B^{E,l}$ and $B^{C,l}$ is equal to 0.275546.

Survival function

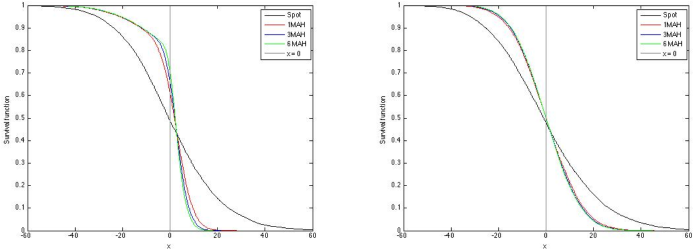


FIGURE: *Empirical survival function of the difference between the price of electricity and the price of coal at time $t = 335$ days for different products in the multi-barrier correlation model and in the benchmark model*

Example of trajectory

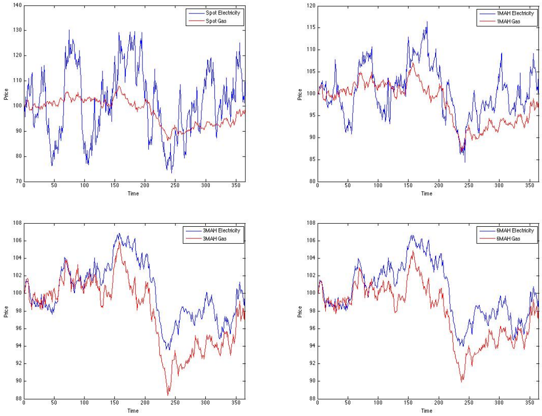


FIGURE: *One year trajectory of electricity and coal products (Spot, 1MAH, 3MAH, 6MAH)*

Limitations

- Sensitive to the difference of volatility between the two forward prices.
- Sensitive to the value of $f^E(0,T) - f^C(0,T)$. A solution is to change the value of η and ν. However, only one η and one ν for all maturities.

Perspectives and future works

- Generalize results on Brownian motions for more general Itô processes.
- Generalize results for $\mathbb{P}\left(B_t^1 - \min\left(B_t^2, .., B_t^n\right)\right) \geq \eta\right)$. Application : we want for the price of electricity to be greater than the minimal production cost.
- Is the local correlation model really equivalent to the multi-barrier correlation model ?
- Calibration of the model.

Thank you for your attention.

Do you have any questions ?

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