Hedging forward positions: Basis risk versus liquidity costs

Thomas Kruse
Université d'Evry Val d'Essonne

Laboratoire de Mathématiques et Modélisation d'Évry

LaMME

Joint work with Stefan Ankirchner and Peter Kratz

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Intro: Risk management of a gas power plant

Companies operating a gas power plant have an immanent

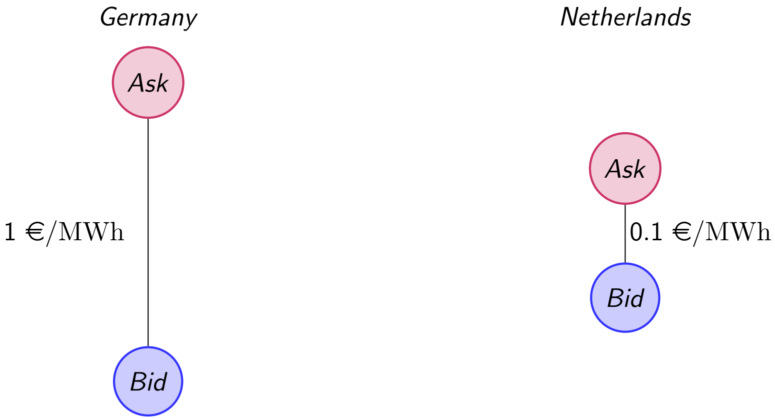
- short forward position of natural gas (NG),
- long forward position of power.

To reduce **price risk** they

- buy natural gas on forward markets,
- sell power on forward markets.

Suppose that a German energy company wants to buy today the NG it needs in 2016.

Problem: German gas **forward market** is very **illiquid**.



- Bid-ask-spread ↓ as time to delivery approaches
- Dutch and German gas prices are **highly** correlated

2 Ways of Hedging

- Hedge 1:

  Buy natural gas in G

- Hedge 2:

  Buy natural gas in NL.
  Shortly before delivery: sell in NL and buy in G.

Pros & Cons:

	Hedge 1	Hedge 2
Pro	No risk	Low liqu. costs
Con	High liqu. costs	Basis risk

Optimal trade-off via stochastic control

Trade-off: High liquidity costs versus basis risk

Question: What is the optimal position in German and Dutch NG at *any* time before 2016?

$\Longrightarrow$ **A singular (stochastic) control problem**

The model

- initial short position : $x_0 < 0$
- T = time horizon
- X_t = primary asset position (e.g. German NG);

$$\text{Constraints:} \qquad X_{0-} = x_0 \text{ and } X_T = 0$$

- Y_t = proxy position (e.g. Dutch NG)

$$\text{Constraints:} \qquad Y_{0-} = 0 \text{ and } Y_T = 0$$

Minimizing overall costs ⇔ minimizing execution costs

- P_t = forward price of the primary asset at time t (a continuous martingale)
- K_t = liquidity costs of primary asset at time t (a non-negative process with càdlàg paths)
- L = half bid-ask-spread of proxy

Expected costs in the primary asset:

$$C^1(X) = E\left[\int_{[0,T]} P_s dX_s + \int_{[0,T]} K_s |dX_s|\right] = -P_0 x_0 + E\left[\int_{[0,T]} K_s |dX_s|\right].$$

Expected costs in the proxy:

$$C^2(Y) = E\left[\int_{[0,T]} L |dY_s|\right]$$

Expected execution costs

$$C(X,Y) = E\left[\int_{[0,T]} K_s |dX_s| + \int_{[0,T]} L |dY_s|\right]$$

The model cont'd

Risk

- $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ covariance matrix
- Instantaneous risk at time t:

$$f(X_t, Y_t) = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}^T \Sigma \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$

- Overall risk:

$$R(X, Y) = \int_0^T \sqrt{f(X_t, Y_t)} dt$$

Target function and minimum variance hedge

Control problem:

$$C(X,Y) + \lambda R(X,Y) \longrightarrow \text{min!}$$

Lemma

Let X be a given primary position path and assume that $L = 0$. Then the optimal cross hedge is given by

$$Y_t^* = -\rho \frac{\sigma_1}{\sigma_2} X_t.$$

$h = \rho \frac{\sigma_1}{\sigma_2}$ = minimum variance hedge ratio

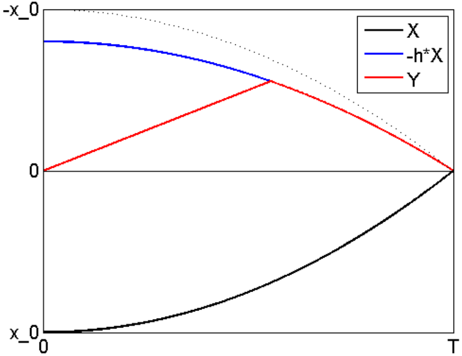
Optimal position paths are (piecewise) monotone

Proposition

Let (X^*, Y^*) be optimal. Then almost surely

a) *X_t^* is non-decreasing, and*

b) *there exists a càdlàg , adapted and non-decreasing process I such that $Y_t^* = I_t \wedge -hX_t^*$.*



Our method for getting explicit solutions

Assumption A: The optimal cross hedge $Y(X)$ associated to any X is non-increasing after 0, i.e. of the form

$$Y(X)_t = y \wedge -\rho \frac{\sigma_1}{\sigma_2} X_t.$$

Iterative Method:

1. For a given $y \geq 0$ determine the optimal primary position $X = X(y)$. To this end reformulate the problem as a stopping problem.
2. Determine optimal initial cross hedge position y^*.
3. The optimal positions are given by

$$X_t^* = X_t(y^*) \text{ and } Y_t^* = y^* \wedge -\rho \frac{\sigma_1}{\sigma_2} X_t^*.$$

The primary position via optimal stopping

For any $y \geq 0$ consider the problem

$$E\left[\int_{[0,T]} K_s dX_s + \int_0^T g(X_s)ds\right] \longrightarrow \min! \qquad (1)$$

where $g(x) = \lambda\sqrt{f(x, y \wedge -\rho\frac{\sigma_1}{\sigma_2}x)}$.

Proposition

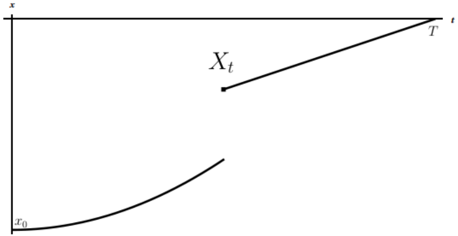
For all $x \in [x_0, 0]$ let $\tau(x)$ be the solution of the stopping problem

$$\inf_{\tau \in \mathcal{T}_{[0,T]}} E\left[K_\tau + \tau g'(x)\right].$$

Then an optimal primary position X for (1) is given by

$$X_t = \inf\{x \in [x_0, 0] | \tau(x) > t\}.$$

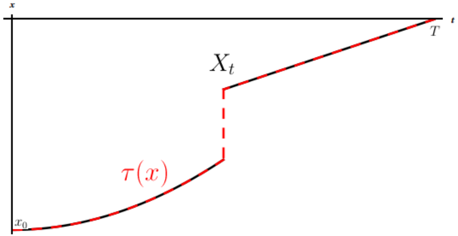
Proof



Right continuous inverse of a position path:

$$\tau(x) = \inf\{t \geq 0 | X_t > x\}$$

Proof



Right continuous inverse of a position path:

$$\tau(x) = \inf\{t \geq 0 | X_t > x\}$$

Apply a Change of Variables Formula to the cost term

$$\int_{[0,T]} K_s dX_s = \int_{x_0}^{0} K_{\tau(z)} dz$$

The primary position via optimal stopping

Proof

The risk term satisfies

$$\int_0^T g(X_s)ds = \int_0^T \left(\int_{x_0}^{X_s} g'(z)dz + g(x_0) \right) ds$$
$$= \int_{x_0}^{0} \tau(z)g'(z)dz + g(x_0)T$$

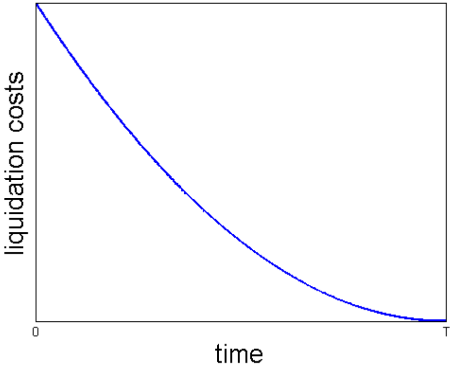
Hence

$$E\left[\int_{[0,T]} K_s dX_s + \int_0^T g(X_s)ds\right] = \int_{x_0}^{0} E\left[\underbrace{K_{\tau(z)}}_{\text{marginal cost}} + \underbrace{\tau(z)g'(z)}_{\text{marginal risk}}\right] dz + g(x_0)T$$

→ *Minimize marginal costs + marginal risk pointwise*

Example: Convex deterministic costs

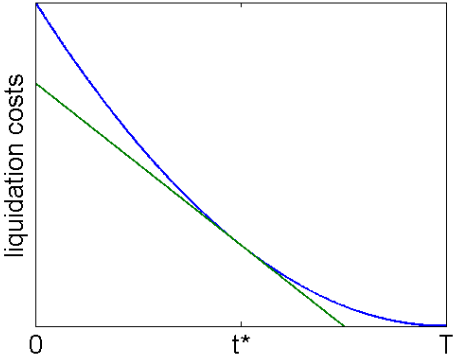
- Liquidity costs are *deterministic*, decreasing and convex in time
- $L = 0 \rightarrow$ marginal risk is constant in x
- marginal risk increases linearly in time



Example: Convex deterministic costs

- Liquidity costs are *deterministic* and convex in time
- $L = 0 \rightarrow$ marginal risk is constant in x
- marginal risk increases linearly in time

$\rightarrow \exists$ optimal turning point t^*



Example cont'd: Optimal buying time

Proposition

Suppose that $L = 0$ and that $K \in \mathcal{C}^1$ is decreasing and convex on $[0, T]$. If $\lambda\sigma_1\sqrt{1-\rho^2} \in [-\dot{K}(T), -\dot{K}(0)]$, then the optimal closing time is given by

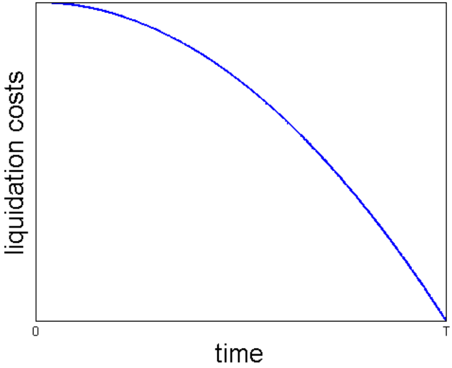
$$t^* = (\dot{K})^{-1}(-\lambda\sigma_1\sqrt{1-\rho^2}),$$

and $X^* = x_0 1_{[0,t^*)}$ and $Y^* = -\rho\frac{\sigma_1}{\sigma_2}x_0 1_{[0,t^*)}$ are the optimal position processes.

Example: Concave deterministic costs

- $L \geq 0$
- Liquidity costs in primary asset are *deterministic* and concave

$\rightarrow$ Optimal strategies are static



Example cont'd: Optimal strategies are static

Proposition

Suppose that K is decreasing and concave on $[0, T]$. Then the optimal position strategy is of the form

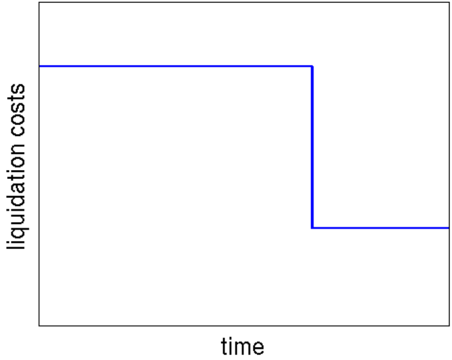
$$X_t^* = x^* 1_{[0,T)}(t) \text{ and } Y_t^* = y^* 1_{[0,T)}(t),$$

with $x_0 \leq x^* \leq 0$ and $y^* \geq 0$.
The optimal positions x^* and y^* can be calculated explicitly (tedious!).

Example: Active trading kicks in at a random time

- K jumps at a random time $\tilde{\tau}$ from a higher level K_+ to a lower level K_-.
- $\tilde{\tau}$ is the first jump time of an inhomogeneous Poisson process with non-decreasing jump intensity.

→ Close positions at time $\tilde{\tau}$: $X_s = Y_s = 0$ for all $s \geq \tilde{\tau}$.



Example cont'd: Optimal strategies are static

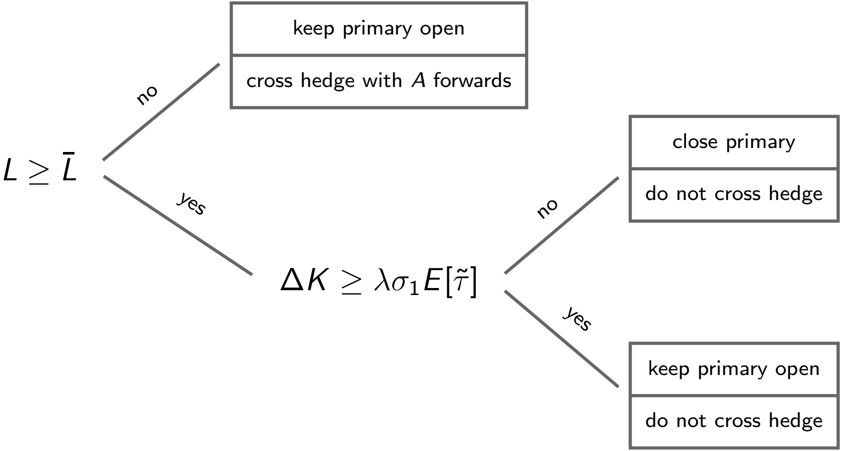
Proposition

Suppose that K jumps from K_+ to K_- at time $\tilde{\tau}$. Then the optimal position strategy is of the form

$$X_t^* = x^* 1_{[0,\tilde{\tau})}(t) \text{ and } Y_t^* = y^* 1_{[0,\tilde{\tau})}(t),$$

with $x^* \leq 0$ and $y^* \geq 0$.
The optimal positions x^* and y^* can be calculated explicitly.

Example cont'd: Decision tree



- $$\bar{L} = \frac{\sigma_2}{2\sigma_1}\left(\Delta K \rho - \sqrt{(1-\rho^2)(\lambda^2\sigma_1^2 E[\tilde{\tau}]^2 - \Delta K^2)^+}\right)$$
- $$A = -\frac{\sigma_1}{\sigma_2} \max\left(0, \rho - 2L\frac{\sqrt{1-\rho^2}}{\sqrt{(\lambda^2\sigma_2^2 E[\tilde{\tau}]^2 - 4L^2)^+}}\right) x_0$$

Conclusion

- When hedging on forward markets one frequently has to choose between liquidity costs and basis risk.
- We introduce a **singular control model** allowing to characterize optimal trade-offs.
- Optimal position paths can be obtained by solving families of stopping problems.
- For specific examples we present optimal position paths in closed form

Optimal closure of illiquid positions and applications to the management of a coal power plant

(based on joint work with S. Ankirchner, M. Jeanblanc and A. Popier)

Optimal position closure & Managing a coal power plant

- **short position: coal and emission rights**
- **long position: power**
- **close these positions on forward markets**

Here: Illiquidity is modeled by a volume-dependent price impact

Optimal position closure

	Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
−	ADS	A1EWWW	adidas AG	1	397	84,840	84,880	312	2	84,890	89	12:38:40	CO	85,920
		Bid/Ask Orders												
				1	876	84,870	84,900	281	2					
				3	455	84,860	84,910	392	3					
				5	494	84,850	84,920	275	2					
				9	1.187	84,840	84,930	1.040	9					
				9	1.408	84,830	84,940	889	5					
				7	602	84,820	84,950	994	7					
				7	760	84,810	84,960	358	4					
				3	400	84,800	84,970	631	6					
				5	929	84,790	84,980	922	6					
				3	639	84,780	84,990	974	7					

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9	1.408	84,830	84,940	889	5
7	602	84,820	84,950	994	7
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3	639	84,780	84,990	974	7

Bid/Ask Orders

Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz
4	276	84,850	84,900	484	5
2	275	84,840	84,910	631	5
7	843	84,830	84,920	808	8
9	829	84,820	84,930	976	9
9	1.696	84,810	84,940	937	6
4	522	84,800	84,950	1.171	7
6	921	84,790	84,960	358	4
4	717	84,780	84,970	471	5
2	134	84,770	84,980	438	3
4	274	84,760	84,990	723	3

The Almgren & Chriss framework

- $T < \infty$: time horizon
- $x \in \mathbb{R}$: initial position
- X_t: position size at time $t \in [0, T]$
- $\dot{X}_t$: trading rate ($\dot{X} \geq 0$: buying, $\dot{X} \leq 0$: selling)

$$X_t = x + \int_0^t \dot{X}_s ds$$

- **Constraint:** $X_T = 0$

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$$S_t^{\text{real}} = S_t^{\text{mid}} + \eta \text{sgn}(\dot{X}_t)|\dot{X}_t|^{p-1}$$

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$$X_t = x + \int_0^t \dot{X}_s ds$$

- **Constraint:** $X_T = 0$
- Trading at a rate $\dot{X}_t$ creates a temporary, stochastic price impact:

$$S_t^{\text{real}} = S_t^{\text{mid}} + \eta_t \text{sgn}(\dot{X}_t)|\dot{X}_t|^{p-1}$$

Stochastic liquidity

Liquidation problem:

$$E\left[\int_0^T \left(\underbrace{\eta_t|\dot{X}_t|^p}_{\text{execution costs}} + \underbrace{\gamma_t|X_t|^p}_{\text{“risk“}}\right) dt\right] \longrightarrow \min_{X_0=x, X_T=0} \tag{2}$$

- $p > 1$ (q its Hölder conjugate)
- (η_t): positive, progressively measurable
- (γ_t): nonnegative, progressively measurable
- stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ satisfying the usual conditions

Derivation of the BSDE

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s |\dot{X}_s|^p + \gamma_s |X_s|^p\right) ds \Big| \mathcal{F}_t\right]$$

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- The value function is explicit in the x variable:

$$v(t,x) = Y_t |x|^p$$

for some coefficient process Y.

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- By deriving a maximum principle we obtain:

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t\right) dt + dM_t \qquad (3)$$

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- Terminal constraint leads to singular terminal condition: $Y_T = \infty$

Integrability Assumptions and Approximation

- For the remainder of the talk we assume that η satisfies

$$E \int_0^T \frac{1}{\eta_t^{q-1}} dt < \infty, \quad E \int_0^T \eta_t^2 dt < \infty$$

and that γ satisfies

$$E \int_0^T \gamma_t^2 dt < \infty$$

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- Approximation

$$\begin{aligned} dY_t^L &= \left((p-1)\frac{(Y_t^L)^q}{\eta_t^{q-1}} - \gamma_t\right)dt + dM_t^L \\ Y_T^L &= L \end{aligned}$$

Existence and Minimality

Proposition

There exists a solution (Y^L, M^L). Y^L *is bounded from above*

$$Y_t^L \leq \frac{1}{(T-t)^p} E\left[\int_t^T (\eta_s + (T-s)^p \gamma_s) ds \middle| \mathcal{F}_t \right].$$

Existence in the Brownian case follows from Briand et al. 2003

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Theorem

There exists a process (Y, M) such that for every $t < T$ and as $L \nearrow \infty$

- $Y_t^L \nearrow Y_t$ *a.s.*
- $M^L \to M$ *in* $\mathcal{M}^2([0, t])$.

The pair (Y, M) is the minimal solution to (3) with singular terminal condition $Y_T = \infty$.

Optimal Controls

Theorem

The control

$$X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

belongs to $\mathcal{A}_0(0, x)$ *and is optimal in* (2). *Moreover,* $v(t, x) = Y_t |x|^p$.

Portfolio liquidation

- Liquidation of a portfolio consisting of $d \in \mathbb{N}$ assets
- Correlation structure leads to a multi-factor liquidation problem
- Value process:

$$v(t,x) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T (\dot{X}_r^T \eta_r \dot{X}_r + X_r^T \gamma_r X_r dr \middle| \mathcal{F}_t \right]$$

where η_t, γ_t are positive semidefinite matrices for every $t \in [0, T]$

Variable reduction

Assume that

- $\eta_t = diag(\eta^1, \dots, \eta^d)$ for every $t \in [0, T]$
- $\gamma_t = \lambda_t \Sigma$ where Σ is positive semidefinite matrix and (λ_t) a one-dimensional, nonnegative process

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- $\gamma_t = \lambda_t \Sigma$ where Σ is positive semidefinite matrix and (λ_t) a one-dimensional, nonnegative process

Then there exists a positive semidefinite matrix A such that

$$v(t,x) = x^T A^T \operatorname{diag}(Y_t^1, \ldots, Y_t^d) A x$$

where $Y_t^1, \ldots, Y_t^d$ denote the solutions of d independent versions of the BSDE (3).

Back to the power plant

- revenues from a coal power plant are essentially determined from the (clean) dark spread

$$CDS = S^P - \frac{1}{h} S^C - \rho S^{CO_2}$$

  - S^P: power price, S^C: coal price and S^{CO_2}: price of emission rights
  - h: heating rate, ρ: emission rate
- Here: Take relative dark spread

$$rDS = h \frac{S^P}{hS^C}$$

  and set $\lambda_t = c(a - rDS_t)^+$
- Solve the BSDEs with PDE methods

Price-sensitive position paths

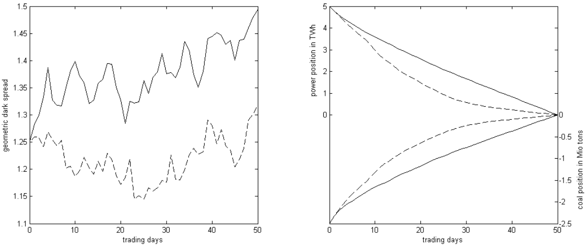

Figure : Position paths depending on rDS

Skewness in realized proceeds

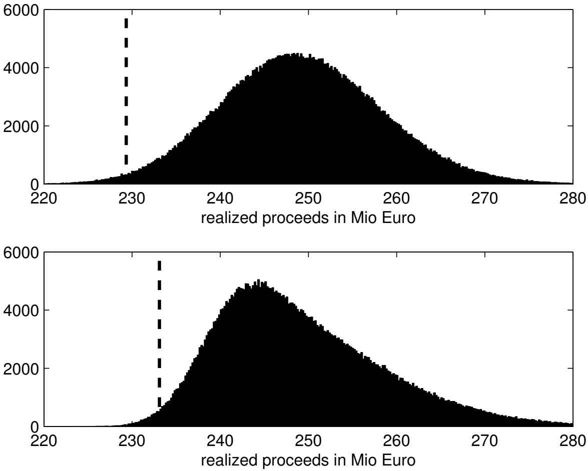

Figure : Histograms of realized proceeds

Literature

The talk is based on

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Thank you!