Emissions Derivatives and Singular Forward Backward SDEs

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Allowance prices dynamics

$(\Omega, \mathcal{F}, \mathbb{P})$
B d-dim Brownian motion under a risk-neutral measure $\mathbb{Q}$
$\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$: augmented filtration of the Brownian motion

Assume there is a liquid market of carbon emissions allowances, denote the price process

$$A = \{A_t;\, 0 \leq t \leq T\}$$

- A is a $\mathbb{F}$-martingale under $\mathbb{Q}$
- $dA_t = Z_t\, dB_t$ for some progressively measurable process Z
- $A_T = \lambda \mathbb{1}_{[\Lambda,\infty)}(E_T)$ where
  - λ is the penalty
  - $E_t = \sum_{i \in \mathcal{I}} E_t^i$ is the aggregate of the E_t^i representing the **cumulative emission** up to time t of firm i
  - Λ is the cap imposed by the regulator

Emissions Dynamics

Assume (dynamics **under** $\mathbb{P}$)

$$dE_t^i = (b_t^i - \xi_t^i)dt + \sigma_t^i dW_t, \qquad E_0^i = 0.$$

- If $\xi_t^i \equiv 0$, $\{E_t^i\}_{0\le t\le T}$ cumulative emissions of firm i in BAU
- $\{\xi_t^i\}_{0\le t\le T}$ abatement rate of firm i
- Assumptions on emission rates b_t^i and volatilities σ_t^i to be articulated later

One may view E_t^i as a prediction of the emissions of firm i

Individual Firm Optimization Problems

Abatement costs for firm i given by **cost function** $c_t^i : \mathbb{R} \to \mathbb{R}$

- c^i is C^1 and strictly convex
- c^i satisfies Inada-like conditions for each $t \in [0, T]$

$$(c^i)'(-\infty) = -\infty \qquad \text{and} \qquad (c^i)'(+\infty) = +\infty.$$

- $c^i(0) = \min c_t^i$ ($\xi^i \equiv 0$ corresponds to BAU)

(e.g. $c^i(x) = \beta|x|^{1+\alpha}$ for some $\beta > 0$ and $\alpha > 0$)

Each firm chooses its **abatement strategy** ξ^i and its **investment** θ^i in allowances. Its **terminal wealth** is given by

$$X_T^i = X_T^{i,\xi^i,\theta} = x^i + \int_0^T \theta_t^i dA_t - \int_0^T c^i(\xi_t^i)dt - E_T^{\xi^i} A_T.$$

Solving the Individual Firm Optimization Problems

Preferences of firm i given by a C^1, increasing, strictly concave **utility function** $U^i : \mathbb{R} \to \mathbb{R}$ satisfying Inada conditions :

$$(U^i)'(-\infty) = +\infty \qquad \text{and} \qquad (U^i)'(+\infty) = 0.$$

The optimization problem for firm i is :

$$V(x^i) := \sup_{(\xi^i,\theta^i)\in\mathcal{A}^i} \mathbb{E}^{\mathbb{P}}\{U^i(X_T^{i,\xi^i\theta^i})\}$$

If no non-standard restriction on $\mathcal{A}^i$ set of admissible strategies for firm i

Proposition

If an equilibrium allowance price $\{A_t\}_{0\le t\le T}$ exists, then the optimal abatement strategy $\hat{\xi}^i$ is given by

$$\hat{\xi}^i_t = [(c^i)']^{-1}(A_t).$$

NB : The optimal abatement strategy $\hat{\xi}^i$ is independent of the utility function U^i

Finding the Equilibrium Allowance Price

- Recall

$$dE_t^i = \left[\tilde{b}_t^i - [(c^i)']^{-1}(A_t)\right] dt + \sigma_t^i dB_t, \qquad E_0^i = 0, \text{ for each } i$$

- Assume
  - $\forall i, \tilde{b}_t^i = \tilde{b}^i(t)E_t^i$ or $\forall i, \tilde{b}_t^i = \tilde{b}^i(t)$
  - $\forall i, \sigma_t^i = \sigma^i(t)$.
- Set

$$b := \sum_{i \in \mathcal{I}} \tilde{b}^i, \ \sigma := \sum_{i \in \mathcal{I}} \sigma^i, \text{ and } f := \sum_{i \in \mathcal{I}} [(c^i)']^{-1}.$$

Therefore we have the following **FBSDE**

$$dE_t = \{b(t, E_t) - f_t(A_t)\}dt + \sigma(t)dB_t, \qquad E_0 = 0 \qquad (1)$$

$$dA_t = Z_t dB_t, \qquad A_T = \lambda \mathbb{1}_{[\Lambda, +\infty)}(E_T), \qquad (2)$$

with $b(t, E_t) = b(t)E_t^\beta$ with $\beta \in \{0, 1\}$ and f increasing.

Theorem

If $\sigma(t) \geq \underline{\sigma} > 0$ then for any $\lambda > 0$ and $\Lambda \in \mathbb{R}$, FBSDE (1)-(2) admits a unique solution $(E, A, Z) \in M^2$. Moreover, A_t is nondecreasing w.r.t λ and nonincreasing w.r.t Λ.

Proof

- Approximate the singular terminal condition $\lambda \mathbb{1}_{[\Lambda,+\infty)}(E_T)$ by an increasing sequence $\{\varphi^n(E_T)\}_n$ of smooth monotone functions of E_T
- Use
  - comparison results for BSDEs
  - Monotonicity of f and φ^n
  - the fact that E_T has a density (implying $\mathbb{P}\{E_T = \Lambda\} = 0$)

  to control the limits

PDE Characterization

Assume GBM for BAU emissions (**Chesney-Taschini**, **Seifert-Uhrig-Homburg-Wagner**, **Grüll-Kiesel**) i.e. $b(t,e) = be$ and $\sigma(t,e) = \sigma e$

$$\begin{cases} E_t = E_0 + \int_0^t (bE_s - f(A_s))ds + \int_0^t \sigma E_s d\tilde{W}_s \\ A_t = \lambda \, \mathbf{1}_{[\Lambda,\infty)}(E_T) - \int_t^T Z_t d\tilde{W}_t. \end{cases} \tag{3}$$

Then, allowance price is of the form $A_t = v(t, E_t)$ for a (deterministic function v which **MUST** solve (**Nonlinear PDE**)

$$\partial_t v(t,e) + \big(be - f(v(t,e))\big)\partial_e v(t,e) + \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} v(t,e) = 0$$

$$v(T,.) = \mathbf{1}_{[\Lambda,\infty)}$$

The price at time t of a **call option** with maturity τ and strike K on an allowance forward contract maturing at time $T > \tau$ is given by

$$V(t, E_t) = \mathbb{E}_t\{(A_\tau - K)^+\} = \mathbb{E}_t\{(v(\tau, E_\tau) - K)^+\}$$

V solves (**Linear PDE**)

$$\partial_t V(t,e) + \big(be - f(v(t,e))\big)\partial_e V(t,e) + \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} V(t,e) = 0$$
$$V(\tau, .) = (v(\tau, .) - K)^+$$

Explicit Solution (**R.C. - Hinz**)

$$v^0(t,e) = \lambda \mathbb{P}\left[E_T^0 \geq \Lambda | E_t^0 = e\right]$$
$$= \lambda \Phi\left(\frac{\ln(e/\Lambda e^{-b(T-t)})}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$
$$V^0(t,e) = \mathbb{E}\left[(v^0(\tau, E_\tau^0) - K)^+ | E_t^0 = e\right]$$

where E^0 is the geometric Brownian motion :

$$\frac{dE_t^0}{E_t^0} = bdt + \sigma dW_t$$

used as proxy estimation of the cumulative emissions.

Small Abatement Asymptotics

For $\epsilon \geq 0$ small, let v^ϵ and V^ϵ be the prices of the allowances and the option for $f = \epsilon f_0$, i.e.

$$-\partial_t v^\epsilon - \big(be - \epsilon f_0(v^\epsilon)\big)\partial_e v^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} v^\epsilon$$

$$v^\epsilon(T,.) = \lambda \mathbf{1}_{[\Lambda,\infty)}$$

$$-\partial_t V^\epsilon - \big(be - \epsilon f_0(v^\epsilon)\big)\partial_e V^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} V^\epsilon = 0$$

$$V^\epsilon(\tau,.) = (v^\epsilon(\tau,.) - K)^+$$

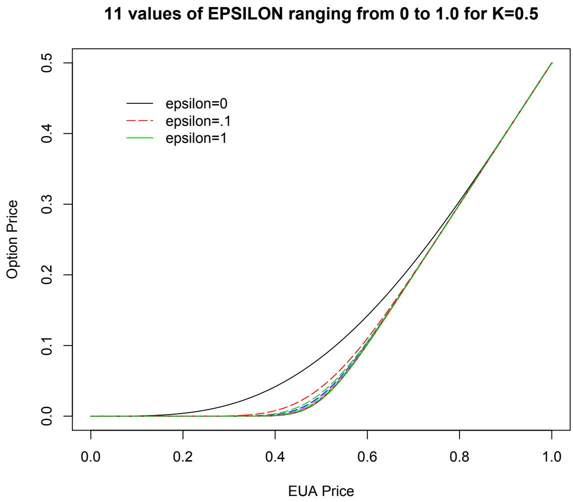
Proposition

$$V^\epsilon(t,s) = V^0(t,s)$$

$$+\epsilon\, \mathbb{E}_{t,e}\left[\mathbf{1}_{[\Lambda,\infty)}(v^0)(\tau, E^0_\tau) \int_t^T f_0(v^0)(s, E^0_s)\partial_e v^0(s\vee\tau, E^0_{s\vee\tau})\frac{E^0_{s\vee\tau}}{E^0_s}\,ds\right]$$

$$+ \circ(\epsilon),$$

Option Prices



A Slightly Different Model

Single good (e.g. **electricity**) regulated economy, with price dynamics given **exogenously** (e.g. regulated utilities) !

$$dP_t = \mu(t, P_t)dt + \sigma(t, P_t)dW_t$$

Firm i

- Controls its *instataneous rate of production* q_t^i
- **Production** over $[0, t]$

$$Q_t^i := \int_0^t q_t^i dt$$

- **Costs of production** given by $c_t^i : \mathbb{R}_+ \mapsto \mathbb{R}$ C^1 strictly convex satisfying Inada-like conditions

$$(c_t^i)'(0) = 0, \qquad (c_t^i)'(+\infty) = +\infty$$

- **Cumulative emissions** $E_t^i := e^i Q_t^i$

Individual Firm Optimization Problem

P&L (terminal wealth)

$$X_T^i = X_T^{i,q^i,\theta^i} = x^i + \int_0^T \theta_t^i dA_t + \int_0^T [P_t q_t^i - c_t^i(q_t^i)]dt - e^i Q_T^i A_T$$

Proposition

If such an equilibrium exits for the price of an allowance, the optimal production strategy $\hat{q}^i$ is given by :

$$\hat{q}_t^i = [(c^i)']^{-1}(P_t - e^i A_t).$$

NB : As before the optimal production schedule $\hat{q}^i$ does not depend on the utility function

- Set $E_t := \sum_{i \in \mathcal{I}} E_t^i$ for the total aggregate emissions up to time t
- Define $f(p,a) := \sum_{i \in \mathcal{I}} \varepsilon^i [(c^i)']^{-1}(p - \varepsilon^i a)$

Then the corresponding FBSDE under $\mathbb{Q}$ reads

$$\textbf{FBSDE} \begin{cases} dP_t &= \sigma(t, P_t)dB_t, \qquad P_0 = p \\ dE_t &= f(P_t, A_t)dt, \qquad E_0 = 0 \\ dA_t &= Z_t dB_t. \end{cases}$$

with terminal condition $A_T = \lambda \mathbf{1}_{[\Lambda, +\infty)}(E_T)$

NB : The volatility of the forward equation is **degenerate !**

Assume further **forward diffusion elliptic** $\delta^{-1} > \sigma(t,p) \geq \delta > 0$

Theorem

Assuming uniformly Lipschitz coefficients, there exists a unique progressively measurable quadruplet $(P_t, E_t, A_t, Z_t)_{0\leq t\leq T}$ satisfying **FBSDE** on $[0,T]$ and

$$\mathbf{1}_{(\Lambda,\infty)}(E_T) \leq A_T \leq \mathbf{1}_{[\Lambda,\infty)}(E_T).$$

The terminal condition $A_T = \mathbf{1}_{[\Lambda,\infty)}(E_T)$ may not be satisfied !

Singularity of the Terminal Value

Theorem

- E_t has a smooth density whenever $t < T$,
- The distribution of E_T has a (Dirac) point mass at Λ, i.e.

$$\mathbb{P}\{E_T = \Lambda\} > 0.$$

- The support of the conditional distribution of A_T given $\{E_T = \Lambda\}$ is the **WHOLE** interval $[0, 1]$!

Consequences

- **The terminal condition $A_T = 1_{[\Lambda,\infty)}(E_T)$ is not satisfied !**
- At time T, the **price A_T of one allowance is not determined** by the model **on the set $\{E_T = \Lambda\}$** of positive probability !