High order schemes for HJB equations
FIME

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Schedule

Problem

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x) &- \inf_{a \in A}\Big(\frac{1}{2} tr(\sigma_a(t,x)\sigma_a(t,x)^T D^2 v(t,x)) + b_a(t,x) Dv(t,x) \\
&\quad + c_a(t,x) v(t,x) + f_a(t,x)\Big) = 0 \text{ in } Q \\
v(0,x) &= g(x) \text{ in } \mathbf{R}^d
\end{aligned} \tag{1}
$$

where

- $Q := (0,T] \times \mathbf{R}^d$, A complete metric space,
- $\sigma_a(t,x)$ is a $d \times q$ matrix,
- b_a and f_a coefficients functions defined on Q in $\mathbf{R}^d$ and R.

HJB associated to a controled process with W_s a q D brownian motion:

$$
dX_s^{t,x} = b_a(s, X_s^{t,x})dt + \sigma_a(t, X_s^{t,x})dW_s \tag{2}
$$

Notation and hypothesis

Classical spaces :

$$|w|_0 = \sup_{(t,x)\in Q} |w(t,x)|, \qquad [w]_1 = \sup_{(s,x)\neq(t,y)} \frac{|w(s,x)-w(t,y)|}{|x-y|+|t-s|^{\frac{1}{2}}}$$

$$|w|_1 = |w|_0 + [w]_1$$

$C_1(Q)$ space space of functions with a finite $|\quad|_1$ norm.
Classical assumption :

$$\sup_a |g|_1 + |\sigma_a|_1 + |b_a|_1 + |f_a|_1 + |c_a|_1 \leq \hat{K} \tag{3}$$

Proposition

If the coefficients of the equation 1 satisfy 3, there exists an unique viscosity solution of the equation 1 belonging to $C_1(Q)$. If u_1 and u_2 are respectively sub and supersolution of equation 1 satisfying $u_1(0,.) \leq u_2(0,.)$ then $u_1 \leq u2$.

Monotone scheme consideration

- Monotone Scheme (Barles Souganidis framework [1]) converge to viscosity solution.
- Non monotone scheme may not converge [3] or converge to bad solution
- If $D_a = \sigma_a(t,x)\sigma_a(t,x)^T$ not diagonally dominant no classical Finite Difference Method is monotone,

Barles Souganidis for PDE

Parabolic equation

$$\frac{\partial v}{\partial t}(t,x) + F(t,x,v(t,x),Dv(t,x),D^2v(t,x)) = 0 \text{ in } Q$$
$$v(0,x) = g(x) \text{ in } \mathbf{R}^d \qquad (4)$$

where F elliptic

$$F(t,x,u,p,A) \leq F(t,x,u,p,B) \text{ if } A \geq B \qquad (5)$$

In our problem

$$F(t,x,u,p,A) = -\inf_{a \in A}\left(\frac{1}{2}tr(D_a A) + b_a(t,x)p + \right. \qquad (6)$$
$$\left. c_a(t,x)u(t,x) + f_a(t,x)\right) \qquad (7)$$

Barles Souganidis for PDE

The Scheme

$$S(h,t,x,v_h(t,x),[v_h]_{t,x}) = 0, (t,x) \in X_h \tag{8}$$
$$v_h(0,x) = g(x), x \in X_h \cup \{t=0\} \tag{9}$$

- A grid $h = (\Delta t, \Delta x)$ and $X_h = \Delta t \{0, 1..., N_T\} \times \Delta x Z^d$,
- v_h approximation of v, $[v_h]_{t,x}$ values at other points than (t,x)

Barles Souganidis for PDE

- Suppose the scheme monotone : if $u \leq v$

$$S(h, t, x, r, u) \geq S(h, t, x, r, v) \tag{10}$$

- Suppose it is consistent : ϕ smooth, when h goes to 0 ,

$$|S(h, t, x, \phi(t, x), [\phi]_{t,x}) - \phi_t - F(t, x, \phi, D\phi, D^2\phi)| \longrightarrow 0$$

- Stable v_h uniformly bounded in h

Then v_h converge to viscosity solution of 4

An example of monotone scheme : the implicit scheme for heat equation

$$v_t - D^2 v = 0 \text{ in }]0,T] \times \mathbf{R} \tag{11}$$

$$v(0,x) = g(x) \tag{12}$$

The implicit scheme

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \frac{v_{i+1}^{n+1} + v_{i-1}^{n+1} - 2v_i^{n+1}}{\Delta x^2} \tag{13}$$

$$S(\Delta t, \Delta x, (n+1)\Delta t, i\Delta x, u_i^{n+1}, [u_{i-1}^{n+1}, u_i^n, u_{i+1}^{n+1}]) = \tag{14}$$

$$u_i^{n+1}\left(\frac{1}{\Delta t} + \frac{2}{\Delta x^2}\right) - \frac{u_i^n}{\Delta t} - \frac{u_{i-1}^{n+1}}{\Delta x^2} - \frac{u_{i+1}^{n+1}}{\Delta x^2} \tag{15}$$

Decreasing in u_{i-1}^{n+1}, u_{i+1}^{n+1} and u_i^n

Numerical resolution method

Currently used monotone scheme relying on Barles Souganidis:

- Generalized finite differences by Bonnans Zidani [2] : take non adjacent point to approximate finite difference such that the approximation is monotone. Not too costly in 2D.
- Monte Carlo techniques and the resolution of a Second Order Backward Stochastic Differential Equation. See Fahim, Touzi, Warin [4], Tan [5] for non linear problem,
- Semi Lagrangian methods developed by Camilli Falcone [6], generalized by Munos Zidani [7], Debrabant Jakobsen [8] with linear interpolation. Easy to implement.

1D example of Semi Lagrangian scheme

Using developpment of ϕ

$$\phi(x + b_a h + \sigma_a \sqrt{h}) = \phi(x) + b_a D\phi(x) + \sqrt{h}\sigma_a D\phi + \frac{\sigma_a^2 h}{2} D^2\phi + \frac{\sigma_a^2 h^{3/2}}{6} D^3\phi + O(h^2)$$

$$\phi(x + b_a h - \sigma_a \sqrt{h}) = \phi(x) + b_a D\phi(x) - \sqrt{h}\sigma_a D\phi + \frac{\sigma_a^2 h}{2} D^2\phi - \frac{\sigma_a^2 h^{3/2}}{6} D^3\phi + O(h^2)$$

So

$$(\phi(x + b_a h + \sigma_a \sqrt{h}) + \phi(x + b_a h - \sigma_a \sqrt{h}) - 2\phi(x)) \simeq 2b_a D\phi(x) + \sigma_a^2 D^2\phi + O(h^2)$$

And use explicit scheme

$$\begin{aligned}
v(t+h,x) &= v(t,x) + \inf_{a \in A} \frac{1}{2}(v(t, \phi^+_{a,h}(t,x)) + v(t, \phi^-_{a,h}(t,x)) - 2v(t,x)) + \\
&\quad hc_a(t,x)v(t,x) + hf_a(t,x) \\
\phi^+_{a,h}(t,x) &= x + b_a(t,x)h + \sigma_a(t,x)\sqrt{h} \\
\phi^-_{a,h}(t,x) &= x + b_a(t,x)h - \sigma_a(t,x)\sqrt{h}
\end{aligned}$$

Remark on the scheme

- Either a finite difference scheme,
- A characteristic method with an explicit Euler Scheme discretization where brownian discretized with binomial :

$$X_{t+h}^{t,x,\pm} = x + b_a(s,x)h \pm \sigma_a(t,x)\sqrt{h}$$

So

$$\begin{aligned}\phi(t+h,x) &= \mathbb{E}(\phi(X_{t+h}^{t,x,+})) + hc_a(t,x)\phi(t,x) + hf_a(t,x)\\ &\simeq \mathbb{E}(e^{c_a(t,x)h}\phi(X_{t+h}^{t,x})) + hf_a(t,x)\end{aligned}$$

D dimensional version

$$v(t+h,x) = v(t,x) + \inf_{a \in A} L_{a,h}(v)(t,x)$$

with

$$\begin{aligned} L_{a,h}(v)(t,x) &= \sum_{i=1}^{q} \frac{1}{2q}(v(t,\phi^{+}_{a,h,i}(t,x)) + v(t,\phi^{-}_{a,h,i}(t,x)) - 2v(t,x)) + hc_a(t,x)v(t,x) \\ &\quad + hf_a(t,x) \\ \phi^{+}_{a,h,i}(t,x) &= x + b_a(t,x)h + (\sigma_a)_i(t,x)\sqrt{hq} \\ \phi^{-}_{a,h,i}(t,x) &= x + b_a(t,x)h - (\sigma_a)_i(t,x)\sqrt{hq} \end{aligned}$$

where $(\sigma_a)_i$ column i of σ_a

And initialize

$$v(t,x) = (1-\frac{t}{h})g(x) + \frac{t}{h}v(h,x), \forall t \in [0,h].$$

Convergence for time discretization (Debrabant-Jakobsen 2013), Camilli Falcone

Proposition

The solution v_h of previous equations is uniquely defined and belongs to $C^1(Q)$. We check that if $h \leq (16 \sup_a \{|\sigma_a|_1^2 + |b_a|_1^2 + 1\} \wedge 2 \sup_a |c_a^+|_0)^{-1}$, there exists C such that

$$|v - v_h|_0 \leq C h^{\frac{1}{4}} \tag{16}$$

Moreover, there exists C independent of h such that

$$|v_h|_0 \leq C \tag{17}$$

$$|v_h(t,x) - v_h(t,y)| \leq C|x-y|, \forall (x,y) \in Q^2 \tag{18}$$

Spatial discretization

- Meshe $(i_1\Delta x, .., i_d\Delta x)$, $\bar{i} = (i_1, ..., i_d) \in \mathbf{Z}^d$ coordinate of $M_{\bar{i}}$
- For a grid $(\xi_i)_{i=0,..N} \in [-1, 1]^N$, $y_{\bar{i},\tilde{j}}$ point of $M_{\bar{i}}$ with coordinate $(\Delta x(i_1 + 0.5(1 + \xi_{j_1})), .., \Delta x(i_d + 0.5(1 + \xi_{j_d}))$ and $\tilde{j} = (j_1, .., j_d) \in [0, N]^d$

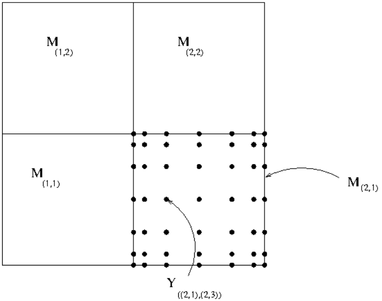


Figure : Discretization grids in 2D

Approximation operator

For v in $C(\mathbf{R}^d)$, $T_{h,\Delta x,N}((v^{\bar{i},\tilde{j}})_{\bar{i},\tilde{j}})$ function $C(\mathbf{R}^d)$ constructed from the set $(v^{\bar{i},\tilde{j}})_{\bar{i},\tilde{j}}$ and defining an approximation of v in $C(\mathbf{R}^d)$.

Remark

Example : $N = 1$, $\xi_0 = -1$, $\xi_1 = 1$, take $T_{h,\Delta x,1}$ as the linear interpolator

General Scheme

Approximation of the function needed to evaluate

$$v(y_{\bar{i},\tilde{j}} + b_a h \pm \sigma_a \sqrt{h}) \tag{19}$$

Use $T_{h,\Delta x,N}$

$$v^{\bar{i},\tilde{j}}_{h,\Delta x,N}(t+h) = v^{\bar{i},\tilde{j}}_{h,\Delta x,N}(t) + \inf_{a\in A}\left[\hat{L}_{a,h}(t, y_{\bar{i},\tilde{j}})\right]$$

$$\begin{aligned}\hat{L}_{a,h}(t, y_{\bar{i},\tilde{j}}) = & \sum_{i=1}^{q} \frac{1}{2q}(T_{h,\Delta x,N}((v^{\bar{k},\tilde{l}}_{h,\Delta x,N}(t))_{\bar{k},\tilde{l}}))(\phi^{+}_{a,h,i}(t,x)) + \\ & T_{h,\Delta x,N}((v^{\bar{k},\tilde{l}}_{h,\Delta x,N}(t))_{\bar{k},\tilde{l}}))(\phi^{-}_{a,h,i}(t,x)) - \\ & 2v(t, y_{\bar{i},\tilde{j}})) + hc_a(t, y_{\bar{i},\tilde{j}})v(t, y_{\bar{i},\tilde{j}}) + hf_a(t, y_{\bar{i},\tilde{j}})\end{aligned}$$

Linear interpolation between 0 and h once $v^{\bar{i},\tilde{j}}_{h,\Delta x,N}(h)$ calculated

How to choose $T_{h,\Delta x,N}$?

- $N = 1$, linear interpolator, defines a monotone scheme,
- $N > 1$ want to use a high degree approximation on each mesh (Spline , Lagrange)

Assumption

Suppose $T_{h,\Delta x,N}$ is from $C_0(Q)$ to $C_0(Q)$, that there exists a function of h $\tilde{K}_h \xrightarrow{h \longrightarrow 0} 0$ s.t. for $x \in M_{\bar{i}}$

$$(T_{h,\Delta x,N} f)(x) = \sum_{\tilde{j}} (w^h_{\bar{i},\tilde{j}}(f))(x) f(y_{\bar{i},\tilde{j}}) \tag{20}$$

$$0 \leq (1 - \tilde{K}_h h) \leq \sum_{\tilde{j}} (w^h_{\bar{i},\tilde{j}}(f))(x) \leq 1 + \tilde{K}_h h \tag{21}$$

and functions $w^h_{\bar{i},\tilde{j}}(f)$ positive weights functions depending on f, h and the support $M_{\bar{i}}$.

Convergence result

Theorem

Suppose $T_{h,\Delta x,N}$ satisfies previous assumptions, $(h_p, \Delta x_p) \longrightarrow (0,0)$ such that $\frac{\Delta x_p}{h_p} \longrightarrow 0$, $h_p \leq \min(\frac{1}{\theta \hat{K}}, 1)$. $\tilde{v}_{h_p,\Delta x_p,N}(t) = T_{h,\Delta x,N}((v^{\bar{i},\tilde{j}}_{h,\Delta x,N}(t))_{(\bar{i},\tilde{j})})$ converges to the viscosity solution of HJB. There exists C independent on h_p, N, Δx_p s.t. for h_p small enough

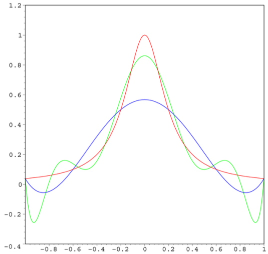
$$|\tilde{v}_{h_p,\Delta x,N} - v|_0 \quad \leq \quad C(h_p^{\frac{1}{4}} + \frac{\Delta x_p}{h_p} + \tilde{K}_{h_p}) \tag{22}$$

When linear interpolation $w^{h}_{\bar{i},\tilde{j}}(f)$ independant of f.

Can we build other approximations ?

Truncated Lagrange interpolators

- 1D Lagrange interpolator $I^X_{\Delta x,N}$ with quadrature points $X = (\xi_i)_{i=0,N} \in [-1,1]^{N+1}$ given.
- $|I^X_{\Delta x,N}(v)|_\infty \leq (1+\lambda_N(X))|v|_\infty$ with $\lambda_N(X)$ Lebesgue constant associated to X.
  - Runge effect associated to equidistant points :



  - So use polynomial interpolator associated to Gauss Lobatto Legendre or Gauss Lobatto Chebyshev grid.

Truncated Lagrange interpolators (nD)

Define

$$\underline{v}_{\bar{i}} = \min_{\tilde{j}} v(y_{\bar{i},\tilde{j}})$$
$$\bar{v}_{\bar{i}} = \max_{\tilde{j}} v(y_{\bar{i},\tilde{j}})$$

$$\hat{I}^X_{h,\tilde{K}_h,\Delta x,N}(v) = (\underline{v}_{\bar{i}} - \tilde{K}_h h|\underline{v}_{\bar{i}}|) \vee I^X_{\Delta x,N}(v) \wedge (\bar{v}_{\bar{i}} + \tilde{K}_h h|\bar{v}_{\bar{i}}|)$$

Proposition

The interpolator $\hat{I}^X_{h,\tilde{K}_h,\Delta x,N}$ satisfies the previous assumptions s.t. $\tilde{v}_{h,\Delta x,N}$ converges to the viscosity solution.

$$||v - \tilde{v}_{h,\Delta x,N}||_\infty \leq O(h^{\frac{1}{4}}) + O(\frac{\Delta x}{h}) + O(K_h)$$

Truncated Lagrange interpolators (nD)

Proposition

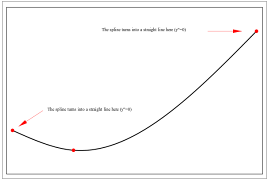
Consistency error is at worst $O(h + \frac{\Delta x^2}{h})$ *and at best in* $O(h + \frac{\Delta x^{N+1}}{h})$

Remark

- *Convergence result not better than linear (should take* $\Delta x = h^{5/4}$*)*
- *We expect truncature to be limited to non smooth area as we refine*
  - *Consistency equal to consistency of linear interpolator when truncature,*
  - *Consistency of high order schemes when no troncature*

Truncated cubic spline

- In 1D $y_{\bar{i},\tilde{j}}$ with $\bar{i}$ the mesh number and $\tilde{j} = 0$ or 1 corresponding to the left or right part of the mesh. $y_{\bar{i},1} = y_{\bar{i}+1,0}$. Use three successive points



- in ND, spline in first dimension, then spline on the coefficients in second dimension etc....

Truncated cubic spline

- $I^{X_c}_{\Delta x,2}$ cubic spline interpolator,
- Truncated interpolator:

$$\hat{I}^{X_c}_{h,\tilde{k}_h,\Delta x}(v) \quad = \quad (\underline{v}_{\bar{i}} - \tilde{K}_h h|\underline{v}_{\bar{i}}|) \vee I^{X_c}_{\Delta x,2}(v) \wedge (\bar{v}_{\bar{i}} + \tilde{K}_h h|\bar{v}_{\bar{i}}|)$$

Proposition

The consistency error (Cubic Spline) is at worst in $O(h + \frac{\Delta x^2}{h})$ and at best in $O(h + \frac{\Delta x^4}{h})$.

Proposition

$\tilde{v}_{h,\Delta x,1}$ obtained by the interpolator $\hat{I}^{X_c}_{h,\tilde{k}_h,\Delta x}$ converge to the viscosity solution v

$$||v - \tilde{v}_{h,\Delta x,1}||_\infty \leq O(h^{\frac{1}{4}}) + O(\frac{\Delta x}{h}) + O(K_h)$$

Approximation with Bernstein polynomials

- Not an interpolation
- Definition in 1D

$$B_N(f)(x) = \sum_{i=0}^{N} f(\frac{i}{N}) P_{N,i}(x)$$

$$P_{N,i} = \left(\begin{array}{c} N \\ i \end{array} \right) x^i (1-x)^{N-1}$$

- Positive weights, independant of the function so monotone approximation
- Define by tensorization in ND

Convergence property of Bernstein approximation

$$|f(x_1,..,x_d) - B_{N,...,N}(f)(x_1,..,x_d)| \quad \leq \quad \frac{C}{N}\sum_{i}^{d}|\frac{\partial^2 f}{\partial x_i^2}(x_1,..,x_d)|$$

s

Result for Bernstein

Proposition

$\tilde{v}_{h,\Delta x,N}$ converges to the viscosity solution

$$||v - \tilde{v}_{h,\Delta x,N}||_\infty \leq O(h^{\frac{1}{4}}) + O(\frac{\Delta x}{h})$$

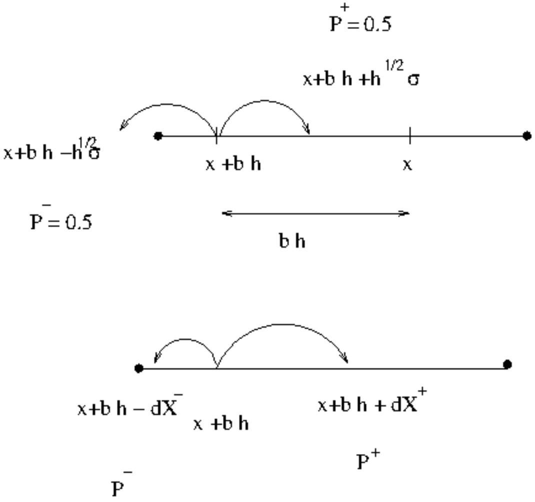
consistency error of order $O(h + \frac{\Delta x^2}{Nh})$.

Boundary conditions

- $x + bh \pm \sqrt{h}\sigma$ can lead us outside of domain,
- avoid extrapolation when possible
- modify the scheme if possible to respect 'mean' and 'variance'
  - initial scheme $dX^+ = dX^- = \sigma\sqrt{h}$ with $P^- = P^+ = \frac{1}{2}$.
  - modified scheme : try to find dX^+, dX^-, P^+, P^- s.t.

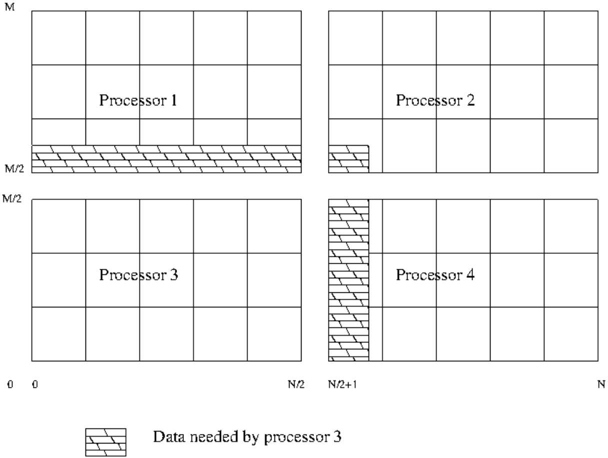
$$
\begin{aligned}
dX^+ dX^- &= \sigma^2 h \\
P^+ &= \frac{\sigma^2 h}{(dX^+)^2 + \sigma^2 h} \\
P^- &= \frac{(dX^+)^2}{(dX^+)^2 + \sigma^2 h}
\end{aligned}
$$

Modification of the scheme for boundary conditions



Parallelization techniques

- each processor computes at time t^n a part of the discretization grid;
- communication needed at date t^{n+1}.



A first case without control

$$
\begin{aligned}
f_a(t,x) &= sinx_1 sinx_2((1+2\beta)(2-t)-1) \\
& \quad -2(2-t)cosx_1 cosx_2 sin(x_1+x_2)cos(x_1+x_2) \\
c_a(t,x) &= 0, \quad b_a(t,x)=0 \quad \sigma_a(t,x) = \sqrt{2}\begin{pmatrix} sin(x_1+x_2) & \beta & 0 \\ cos(x_1+x_2) & 0 & \beta \end{pmatrix}
\end{aligned}
$$

- $\beta = 0.1$, $Q = (0,1] \times [-2\pi, 2\pi]^2$.
- Number of time steps 2000.
- Solution $u(t,x) = (2-t)sinx_1 sinx_2$

LINEAR				CUBIC				MPCSL				TCHEB 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
240	0.310		112	20	0.461		2	20	0.815		3	10	0.888		6
480	0.119	1.37	448	40	0.037	3.61	10	40	0.166	2.28	10	20	0.165	2.42	31
960	0.040	1.57	1815	80	0.005	2.88	41	80	0.007	4.55	42	40	0.0086	4.25	133
1920	0.0075	2.40	7334	160	0.0005	3.16	165	160	0.0005	3.61	170	80	0.00108	3.004	552

LEGEND 2				LEGEND 3				BERN 2				BERN 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
10	0.86		1	10	0.877		5	120	0.643		718	120	0.5528		2811
20	0.059	3.85	6	20	0.165	2.40	21	240	0.227	1.50	2878	240	0.1784	1.631	11292
40	0.0069	3.0	25	40	0.0085	4.27	92	480	0.077	1.55	11551	480	0.0557	1.678	45446
80	0.0010	2.70	104	80	0.00107	2.99	380	980	0.021	1.84	46467	960	0.01532	1.86239	181897

Non regular case without control

$$u(t,x) = (1+t)sin(\frac{x_2}{2}) \begin{cases} sin\frac{x_1}{2} & \text{pour } -2\pi < x_1 < 0 \\ sin\frac{x_1}{4} & \text{pour } 0 < x_1 < 2\pi \end{cases}$$

with

$$\begin{aligned} f_a(t,x) &= sin\frac{x_2}{2} \begin{cases} sin\frac{x_1}{2}(1+\frac{1+t}{4})(sin^2 x_1 + sin^2 x_2) & \text{for } -2\pi < x_1 < 0 \\ sin\frac{x_1}{4}(1+\frac{1+t}{16})(sin^2 x_1 + 4sin^2 x_2) & \text{for } 0 < x_1 < 2\pi \end{cases} \\ & -sinx_1 sinx_2 cos\frac{x_2}{2} \begin{cases} \frac{1+t}{2} cos\frac{x_1}{2} & \text{for } -2\pi < x_1 < 0 \\ \frac{1+t}{4} cos\frac{x_1}{4} & \text{for } 0 < x_1 < 2\pi \end{cases} \\ c_a(t,x) &= 0, \quad b_a(t,x) = 0 \quad \sigma_a(t,x) = \sqrt{2} \begin{pmatrix} sinx_1 \\ sinx_2 \end{pmatrix} \end{aligned}$$

On take $Q = (0,1] \times [-2\pi, 2\pi]^2$, the number of time step is equal to 2000.

Table : Test case 2

LINEAR				CUBIC				MPCSL				TCHEB 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
640	0.038	1.858	557	80	0.00875		17	80	0.00875		18	10	0.290		3
1280	0.013	1.481	2240	160	0.00439	0.994	70	160	0.00439	0.994	72	20	0.0136	4.41	12
2560	0.0070	0.89	8662	320	0.00220	0.998	285	320	0.00220	0.998	288	40	0.00398	1.77	51
5120	0.0035	0.998	34820	640	0.00110	0.999	1177	640	0.00110	0.999	1221	80	0.00128	1.63	212
LEGEND 2				LEGEND 3				BERN 2				BERN 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
10	0.0593		1	10	0.2859		2	80	0.3774		112	80	0.3144		429
20	0.01422	2.06	2	20	0.0137	4.378	9	160	0.1889	0.998	448	160	0.135	1.21	1734
40	0.00411	1.788	11	40	0.0040	1.780	39	320	0.066	1.513	1794	320	0.0460	1.557	6937
80	0.00132	1.63	46	80	0.00129	1.629	160	640	0.0186	1.82	7197	640	0.0128	1.837	27892

Control problem

Solution

$$u(t,x_1,x_2) = (\frac{3}{2} - t) sin x_1 sin x_2$$

Coefficients are given by

$$\begin{aligned}
f_a(t,x) &= (\frac{1}{2} - t) sin x_1 sin x_2 + (\frac{3}{2} - t) \Big[\sqrt{cos^2 x_1 sin^2 x_2 + sin^2 x_1 cos^2 x_2} \\
& \quad -2 sin(x_1 + x_2) cos(x_1 + x_2) cos x_1 cos x_2 \Big] \\
c_a(t,x) &= 0, \quad b_a(t,x) = a \quad \sigma_a(t,x) = \sqrt{2} \begin{pmatrix} sin(x_1+x_2) \\ cos(x_1+x_2) \end{pmatrix}, \\
A &= \{ a \in \mathbf{R}^2 : a_1^2 + a_2^2 = 1 \}
\end{aligned}$$

$Q = (0,1] \times [-\pi, \pi]^2$, number of time step 1000, number of control 4000. CPU time for 192 cores.

Table : Test case 3

LINEAR				CUBIC				MPCSL				TCHEB 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
80	0.59		237	10	0.312		19	10	0.688		30	4	0.671		27
160	0.147	2.00	850	20	0.0499	2.64	30	20	0.050	3.773	29	8	0.0986	2.767	47
320	0.044	1.738	3334	40	0.0072	2.79	96	40	0.0064	2.97	98	16	0.0119	3.046	184
640	0.014	1.659	13259	80	0.001	2.85	384	80	0.001	2.57	387	32	0.00122	3.28	735
LEGEND 2				LEGEND 3				BERN 2				BERN 3			
NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time	NbM	Err	Rate	Time
4	0.632		6	4	0.677		20	20	0.7479		181	20	0.769		758
8	0.0710	3.152	14	8	0.0988	2.77	31	40	0.706	0.0832	789	40	0.5898	0.383	2362
16	0.0094	2.913	49	16	0.0117	3.07	116	80	0.3210	1.136	2533	80	0.2334	1.3369	9436
32	0.0023	2.01	149	32	0.0011	3.339	465	160	0.0801	2.003	10111	160	0.0563	2.050	37750

Problem to solve

$$F(x, u, \mathbf{D}u, \mathcal{D}^2 u) = 0 \text{ in } \mathbf{R}^N,$$

with

$$\begin{aligned} F(x,t,p,X) &= \sup_{\alpha \in \mathcal{A}} \mathcal{L}^\alpha(x,t,p,X), \\ \mathcal{L}^\alpha(x,t,p,X) &= -tr[a^\alpha(x)X] - b^\alpha(x)p + c^\alpha(x)t - f^\alpha(x), \end{aligned}$$

a, b, c, f are at least continuous on $\mathbf{R}^N \times \mathcal{A}$ with values in $S(N)$ matrices,$\mathbf{R}^N$, $\mathbf{R}$ and $\mathbf{R}$ $\mathcal{A}$ is a compact metric space.

Assumptions

Assumption

For any $\alpha \in \mathcal{A}$, $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha t}$ *for some* $N \times P$ *matrix* σ^α*. Furthermore there exists* λ, K *independent of* α *such that:*

$$c^\alpha \geq \lambda > 0, \text{ and } |\sigma^\alpha|_1 + |b^\alpha|_1 + |f_1^\alpha| \leq K \tag{23}$$

Assumption

The constant λ *in 2 satisfies* $\lambda > \sup_\alpha [\sigma^\alpha]_1^2 + [b^\alpha]_1$

Assumption

For every $\delta > 0$, *there are* $M \in \mathbf{N}$ *and* $\{\alpha_i\}_{i=1}^M \subset \mathcal{A}$, *such that for any* $\alpha \in \mathcal{A}$

$$\inf_{1 \leq i \leq M} (|\sigma^\alpha - \sigma^{\alpha_i}|_0 + |b^\alpha - b^{\alpha_i}|_0 + |c^\alpha - c^{\alpha_i}|_0 + |f^\alpha - f^{\alpha_i}|_0) < \delta$$

Previous results (Barles Jakobsen [9], [10])

Proposition

Assume 2. There exists an unique viscosity u solution of 23 in $C_b(\mathbf{R}^N)$. If w_1 and w_2 are in $C_b(\mathbf{R}^N)$ and are sub- and supersolution of 23 respectively, then $w_1 \leq w_2$ in $\mathbf{R}^N$. Assume 3, then u is in $C^{0,1}(\mathbf{R}^N)$.

Theorem

Let S be a monotone, consistent and uniformly continuous and when a discrete bounded solution u_h can be found for

$$S(h, x, u_h(x), [u_h]_x) = 0, x \in \mathbf{R}^N \tag{24}$$

then u_h converges to the viscosity solution of the problem (and convergence rates)

Definitions

Definition

An $\epsilon(p, K)$ **monotone scheme** S *is s.t. there exists* $\bar{\lambda}$ *satisfying:*

- *for every* $h > 0$, $x \in \mathbf{R}^N$, $r \in \mathbf{R}$, *for every functions* $w \in C^{0,1}(\mathbf{R}^N)$, $v \in C_b(\mathbf{R}^N)$ *such that* $v \geq w$:

$$S(h, x, r, [v]_x) \leq S(h, x, r, [w]_x) + K|w|_{0,1}h^p \tag{25}$$

- *for every* $h > 0$, $x \in \mathbf{R}^N$, $r \in \mathbf{R}$, *for every functions* $w \in C_b(\mathbf{R}^N)$, $v \in C^{0,1}(\mathbf{R}^N)$ *such that* $v \geq w$:

$$S(h, x, r, [v]_x) \leq S(h, x, r, [w]_x) + K|v|_{0,1}h^p \tag{26}$$

- *for every* $h > 0$, $x \in \mathbf{R}^N$, $r \in \mathbf{R}$, $m \geq 0$, *for every function* $u \in C_b(\mathbf{R}^N)$:

$$S(h, x, r + m, [u + m]_x) \geq S(h, x, r, [u]_x) + \bar{\lambda}m \tag{27}$$

Definitions

Definition

An $\epsilon(c)$ **solution** u **of scheme** S *is a continuous function which satisfies:*

$$|S(h, x, u(x), [u]_x)| < c$$

Definition

An $\epsilon(c)$ **subsolution (supersolution)** u **of scheme** S *is a continuous function which satisfies:*

$$S(h, x, u(x), [u]_x) < c \qquad (S(h, x, u(x), [u]_x) > -c)$$

Assumptions for the scheme

Assumption

The scheme S is $\epsilon(p, K)$ monotone.

Assumption

- *(Regularity of S scheme) For every $h > 0$ and $\phi \in C_b(\mathbf{R}^N)$, $x \longrightarrow S(h, x, \phi(x), [\phi]_x)$ is bounded and continuous in $\mathbf{R}^N$ and the function $r \longrightarrow S(h, x, r, [\phi]_x)$ is uniformly continuous for bounded r, uniformly in $x \in \mathbf{R}^N$.*
- *(Consistency) There exists integers m , k_i $i = 1, m$, and a constant K such that for every $h \geq 0$, $x \in \mathbf{R}^N$ and a smooth function ϕ*

$$|F(x, \phi(x), \mathcal{D}\phi(x), \mathcal{D}^2\phi) - S(h, x, \phi(x), [\phi]_x)| \leq K \sum_{i=1}^{m} h^{k_i} |\mathcal{D}^i \phi|_0$$

- *We suppose that there exists C and r independent of h such that for each h we can find a $\epsilon(Ch^r)$ solution u_h of scheme S.*

Discrete comparison result

Lemma

Assume 5. Let $v \in C_b(\mathbf{R}^N)$ and $u \in C^{0,1}(\mathbf{R}^N)$. If u is a subsolution of scheme S and v is an $\epsilon(C)$ supersolution of scheme S then

$$u \leq v + \frac{1}{\lambda}(Kh^p|u|_{0,1} + C)$$

Convergence result

Theorem

Assume 2, 3, 5, 6. We have the following bounds for u_h a $\epsilon(\tilde{C}h^r)$ solution:

$$u - u_h \quad \leq \quad \hat{C}(h^{\min(p,r)} + h^{\min_{i=1,m} \frac{k_i}{i}})$$

where $\hat{C}$ depends on $\tilde{C}$.
Besides assume 4 then there exists $\hat{C}$ depending on $\tilde{C}$ such that :

$$u_h - u \quad \leq \quad \hat{C}(h^{\min(p,r)} + h^{\min_{i=1,m} \frac{k_i}{3i-2}})$$

Remark

Same estimation as Barles Jakobsen with some other terms

- h^r *error associated to discrete resolution*
- h^p *error associated to monotony perturbation in comparison result*

A Camilli Falcone style scheme

"Time" monotone scheme setting $[\phi]_x(z) = \phi(x+z)$

$$\hat{S}(h, y, t, [\phi]_x) = \sup_{\alpha \in \mathcal{A}} \left\{ -\frac{1}{h}(G(h, \alpha, y, [\phi]_x) - t) + c^\alpha(y)t - f^\alpha(y) \right\}$$

$$G(h, \alpha, y, [\phi]_x) = \frac{1 - hc^\alpha(y)}{2P} \sum_{i=1}^{P} \left([\phi]_x(hb^\alpha(y) + \sqrt{hP}\sigma_i^\alpha(y)) + [\phi]_x(hb^\alpha(y) - \sqrt{hP}\sigma_i^\alpha(y)) \right)$$

Non monotone scheme (interpolation error)

$$S(h, y, t, [\phi]_x) = \hat{S}(h, y, t, [\hat{I}_{\Delta x, M}\phi]_x)$$

where $\hat{I}_{\Delta x, M}$ a truncated interpolator (Lagrange, Spline ...)
Try to find ϵ solution :

$$|S(h, y, U(y), [I_{\Delta x, M}U]_y)| \leq \epsilon(h, \Delta x), \text{ for } y \in X_{\Delta X, M} := (y_{\tilde{i}, \tilde{j}})_{\tilde{i}, \tilde{j}} \quad (28)$$

Convergence result

Proposition

Assume 2 hold and that $\Delta x = h^q$. Then the scheme 28 satisfies assumptions 5, 6 with $k_2 = 2q-1$, $k_4 = 1$, $p = q-1$

T defined for U a function on $X_{\Delta x, M}$:

$$(T_{h,\Delta x}U)(x) = \inf_{\alpha \in \mathcal{A}} \{(1 - hc^{\alpha}(x))(\Pi_{\Delta x,h}(U))(x) + hf^{\alpha}(x)\} \text{ for } x \in X_{\Delta x,M}$$

where the operator $\Pi_{\Delta x,h}$ is

$$(\Pi_{\Delta x,h}U)(x) = \frac{1}{2P}\sum_{i=1}^{2P}\Big((I_{\Delta x,M}U)(x + hb^{\alpha}(x) + \sqrt{h}\sigma_i^{\alpha}(x)) + (I_{\Delta x,M}U)(x + hb^{\alpha}(x) - \sqrt{h}\sigma_i^{\alpha}(x))\Big)$$

Proposition

Assume 2, 3 hold. Suppose that $\Delta x = h^q$ with $q > 2$. There exists s depending on h and C independent of h such that $u_h = T^s_{h,\Delta x}0$ is a $\epsilon(Ch^{q-2})$ solution of scheme S.

Convergence results

Proposition

Suppose u_h has been constructed as in proposition 10, framework gives us that we can find q such that

$$|u - u_h| < Ch^{\frac{1}{10}}$$

Previous result non optimal :

Proposition

Taking $q \geq \frac{5}{4}$, u_h been constructed as in proposition 10, we have

$$|u - u_h| < Ch^{\frac{1}{4}}$$

Regular problem

$$u(x,y) \quad = \quad sin(\pi x) sin(\pi y)$$

$$c_a(t,x) \quad = \quad C, \quad \sigma_a(t,x) = \sigma a \mathbb{1}, \quad b^\alpha = b 1$$

$$f^\alpha(x,y) \quad = \quad (C + \pi^2 \sigma^2 1_{u(x,y)>0}) u(x,u) - b\pi(cos(\pi x) sin(\pi y) + sin(\pi x) cos(\pi y))$$

$$Q = [0,2]^2, \quad b = 0.3, c = 0.55, \sigma = 1, A = [0,1]$$

and the boundary condition is a Dirichlet one with 0 value, $h = 0.0002$

Table : Test case 1

NbM	LINEAR			QUADRATIC			CUBIC		
	Err	ItN	Time	Err	ItN	Time	Err	ItN	Time
10	1.169	25900	138	0.051	29483	919	0.183	45108	5182
20	1.028	26555	468	0.0065	29398	3619	0.0083	29486	10770
40	0.758	27211	1879	0.0012	29485	12633	0.0011	29706	43371
80	0.243	28076	6923	0.0003	29621	50988	0.0003	29788	175163
160	0.103	28620	27762	0.0003	29748	19517			
320	0.018	29282	108000						

Second regular problem

solution

$$u(x,y) = sin(\pi x) sin(\pi y)$$

$$c_a(t,x) = C, \quad \sigma_a(t,x) = \sigma a \mathbb{1}, \quad b^\alpha = b(a, \sqrt{1-a^2}), a \in [\underline{a}, \bar{a}]$$

Noting

$$\tilde{a} = \frac{sin(\pi y)cos(\pi x)}{\sqrt{sin(\pi x)^2 cos(\pi y)^2 + cos(\pi x)^2 sin(\pi y)^2}},$$

$$\phi(a) = a sin(\pi y)cos(\pi x) + \sqrt{1-a^2} sin(\pi x)cos(\pi y),$$

for $b \leq 0$, the function f^α is here given by

$$f^\alpha(x,y) = (C + \pi^2\sigma^2)u(x,u) - b\pi K$$

where K is the maximum of $\phi(\underline{a})$, $\phi(\bar{a})$ and $\phi(\tilde{a})$ conditionally to $\underline{a} \leq \tilde{a} \leq \bar{a}$. We take $\underline{a} = -1$, $\bar{a} = 1$, $\sigma = 1$, $C = 0.6$, $b = -1$, $Q = [0, \frac{1}{2}]$. As boundary Dirichlet.

Table : Test case 2

NbM	LINEAR			QUADRATIC			CUBIC		
	Err	ItN	Time	Err	ItN	Time	Err	ItN	Time
8	0.1195	493	2	0.0115	1304	33	0.012	1571	99
16	0.0632	903	13	0.0022	1318	100	0.0022	1589	390
32	0.0191	1197	50	0.0003	1319	405	0.0003	1590	1574
64	0.0062	1278	207						

Available on...

- Time dependant :

  http://arxiv.org/abs/1310.6121

- Stationnary :

  http://arxiv.org/abs/1312.5052

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