Risk-parameter estimation in volatility models

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Model risk/Estimation risk

Risk assessment framework defined by "Pillar II" directives: panel of risks including market risk.

In July 2009, the Basel Committee issued a directive requiring that financial institutions quantify "model risk":

"Banks must explicitly assess the need for valuation adjustments to reflect two forms of model risk: the model risk associated with using a possibly incorrect valuation methodology; and the risk associated with using unobservable (and possibly incorrect) calibration parameters in the valuation model."

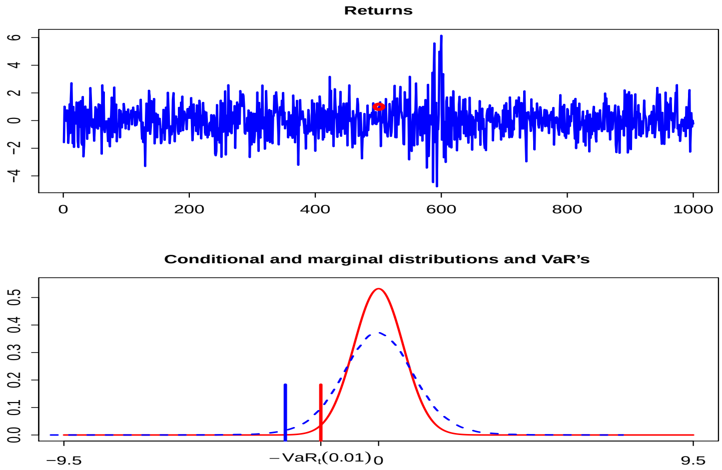
This talk is about quantifying the estimation risk in some dynamic models.

Conditional risk

Modern financial risk management focuses on risk measures based on distributional information.

- Standard approaches:
  - marginal distributions of loss (and profit) variables
  - risk $=$ a number (parameter)
- More sophisticated approaches:
  - conditional distributions of loss (and profit) variables
  - risk $=$ a stochastic process

Conditional VaR for a simulated process



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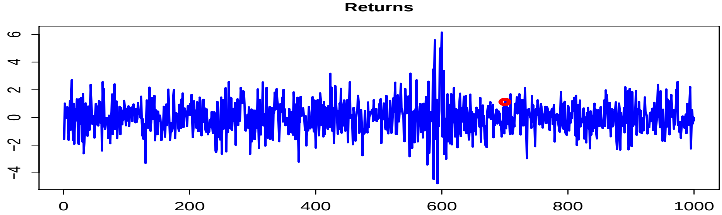
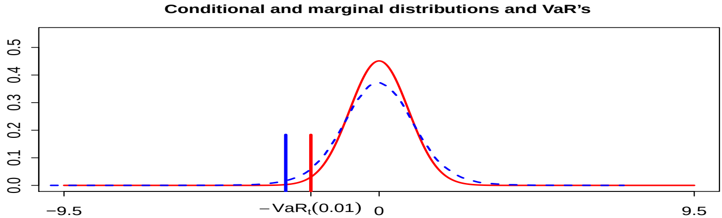
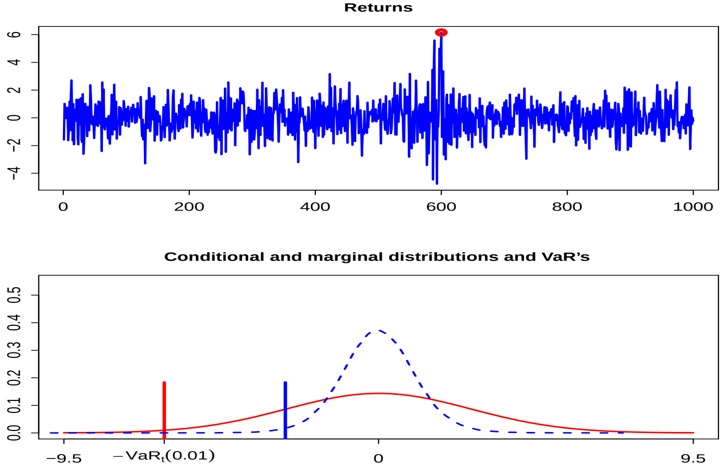
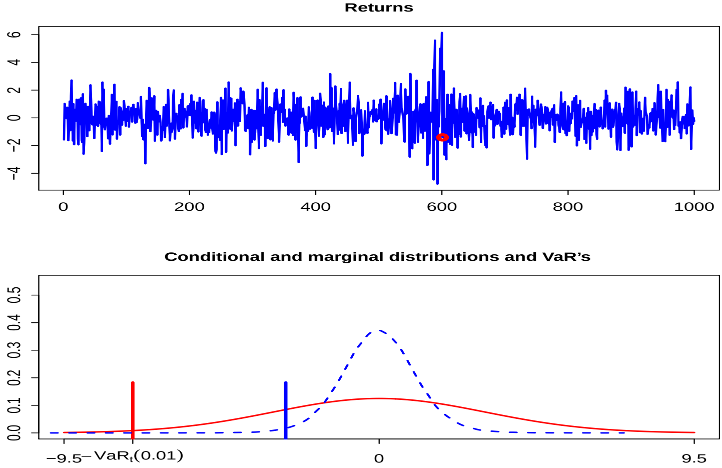




Conditional VaR for a simulated process



Conditional VaR for a simulated process



Aims

- Summarize the conditional risk by a parameter.
- Estimate the risk parameter (in particular the VaR parameter).
- Compare of the performance of different estimators.
- Derive confidence intervals for the VaR by accounting for the estimation risk.

Outline

A general conditional volatility model

$$\begin{cases} \epsilon_t = \sigma_t(\theta_0)\eta_t, \\ \sigma_t(\theta_0) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \ldots; \theta_0) > 0, \end{cases}$$

- $\theta_0 \in \mathbb{R}^m$ is a parameter and $\sigma_t(\theta_0)$ is the volatility;
- (η_t) is a sequence of iid r.v. with $\eta_t \perp \epsilon_{t-j}, j > 0$.

Examples

No specification of the distribution of η_t
(semi-parametric model)

For the (statistical) identification of θ_0, an assumption is needed.

On the role of an identifiability assumption on η_t

For any constant $K > 0$,

$$\epsilon_t = \underbrace{K\sigma_t(\theta_0)}_{\sigma_t(\theta_0^*)} \times \underbrace{K^{-1}\eta_t}_{\eta_t^*}$$

Multiplication by a constant must be precluded.

$\rightarrow$ a moment, a quantile, or another characteristic of the distribution of η_t must be fixed.

Standard identifiability assumption: $E\eta_1^2 = 1$.

Under this condition and $E\eta_1 = 0$, the volatility $\sigma_t^2(\theta_0)$ is the conditional variance of ϵ_t.

On the role of $E\eta_t^2 = 1$ for quasi-maximum likelihood (QML) estimation

Under $E\eta_t^2 = 1$, the Gaussian QML criterion (to be minimized)

$$\frac{1}{n}\sum_{t=1}^{n} \log \sigma_t^2(\theta) + \frac{\epsilon_t^2}{\sigma_t^2(\theta)}$$

gives a consistent estimator because the limit criterion

$$E\left(\log \sigma_t^2(\theta) + \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)}\eta_t^2\right) = E\left(\log \sigma_t^2(\theta) + \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)}\right)$$

is uniquely minimized at θ_0
(assuming that $\sigma_t^2(\theta) = \sigma_t^2(\theta_0) \quad \Rightarrow \quad \theta = \theta_0$).

Conditional risk

Consider a risk measure, r, that is, a mapping from the set of the real random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$.

Assume that r is nonnegative, and

- Positively homogenous: $r(\lambda X) = \lambda r(X)$ for any variable X and any $\lambda \geq 0$,
- Law invariant: $r(X) = r(Y)$ if X and Y have the same distribution.

The conditional risk of $\epsilon_t = \sigma_t(\theta_0)\eta_t$ is given by

$$r_{t-1}(\epsilon_t) = \sigma_t(\theta_0) r(\eta_t),$$

where $r(\eta_t)$ is a constant.

Example: conditional VaR

The conditional VaR of the process (ϵ_t) at risk level $\alpha \in (0,1)$, denoted by $\text{VaR}_t(\alpha)$, is defined, in the continuous case, by

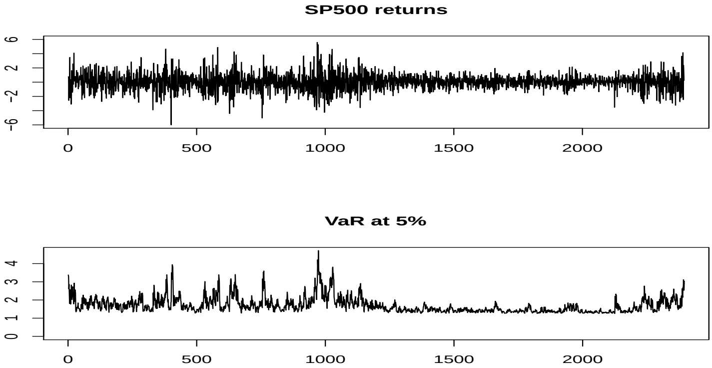
$$P_{t-1}[\epsilon_t < -\text{VaR}_t(\alpha)] = \alpha,$$

where P_{t-1} denotes the historical distribution conditional on $\{\epsilon_u, u < t\}$.

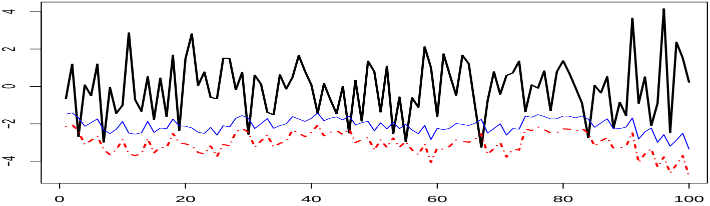
For the conditional volatility model, the conditional VaR is

$$\text{VaR}_t(\alpha) = -\sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \theta_0) F_\eta^{-1}(\alpha)$$

where F_η is the c.d.f. of η_t.



Log-returns of the SP500 and estimated VaR's at the 5% levels, from September 16, 1998 to March 25, 2008.



Log-returns of the SP500 and estimated $-$VaR's at the 1% and 5% levels, from October 30, 2007 to March 25, 2008.

Assumption on the volatility function

The goal is to define a risk parameter (for a given risk r), similar to the volatility parameter.

The following assumption is made on the volatility function.

A0: There exists a function H such that for any $\theta \in \mathbb{R}^m$, for any $K > 0$, and any sequence $(x_i)_i$

$$K\sigma(x_1, x_2, \dots; \theta) = \sigma(x_1, x_2, \dots; \theta^*), \quad \text{where } \theta^* = H(\theta, K).$$

Volatility parameter

Most conditional volatility models are such that for $K \geq 1$, $\theta^* \geq \theta$ componentwise.

For instance, in the GARCH(1,1) case $\theta^* = (K^2\omega, K^2\alpha, \beta)'$.

The parameter θ_0 can be interpreted as a *volatility parameter* in the sense that the larger θ_0 the larger the volatility.

$\rightarrow$ allows to compare the volatilities of two series (under $E\eta_1^2 = 1$).

Conditional risk parameter

We have $r_{t-1}(\epsilon_t) = \sigma_t(\theta_0) r(\eta_t)$.

If $r(\eta_t) > 0$, let $\eta_t^* = \eta_t / r(\eta_t)$ and let $\theta_0^* = H(\theta_0, r(\eta_t))$.

Under **A0**, the model can be reparameterized as

$$\begin{cases} \epsilon_t = \sigma_t^* \eta_t^*, & r(\eta_t^*) = 1, \\ \sigma_t^* = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \theta_0^*). \end{cases}$$

Because the conditional risk of ϵ_t is now simply

$$r_{t-1}(\epsilon_t) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \theta_0^*),$$

θ_0^* will be called the risk parameter.

Example: Conditional VaR for a GARCH(1,1)

GARCH(1,1) Model:

$$\begin{cases} \epsilon_t = \sigma_t(\theta_0)\eta_t, \\ \sigma_t^2(\theta_0) = \omega_0 + \alpha_0\epsilon_{t-1}^2 + \beta_0\sigma_{t-1}^2(\theta_0) \end{cases}$$

with $\theta_0 = (\omega_0, \alpha_0, \beta_0) \in (0, \infty) \times (\mathbb{R}^+)^2$.

$\text{VaR}_t(\alpha) = -\sigma_t(\theta_0)F_\eta^{-1}(\alpha)$.

Risk parameter at level α (with $F_\eta^{-1}(\alpha) < 0$):

$$\theta_0^* = (K^2\omega_0, K^2\alpha_0, \beta_0)'$$

with $K = -F_\eta^{-1}(\alpha)$.

This coefficient takes into account the dynamics of the GARCH process, but also the lower tail of the innovations distribution.

Example: Conditional VaR for a GARCH(1,1)

Numerical illustration:

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, \quad \eta_t \sim \mathcal{N}(0,1) \\ \sigma_t^2 = 1 + 0.05\epsilon_{t-1}^2 + 0.9\sigma_{t-1}^2 \end{cases} \quad \text{and} \quad \begin{cases} \epsilon_t = \sigma_t \eta_t, \quad \eta_t \sim \frac{1}{\sqrt{2}} St_4 \\ \sigma_t^2 = 1 + 0.04\epsilon_{t-1}^2 + 0.9\sigma_{t-1}^2 \end{cases}$$

The volatility parameter of the Gaussian model is larger than that of the Student-innovation model.

Now consider the VaR's at level 1%.

The risk parameter of the first model is $\theta_0^* = (5.41, 0.27, 0.9)$, whereas that of the second model is $\theta_0^* = (7.01, 0.28, 0.9)$.

The first model is more volatile but less risky than the second one for the VaR at 1%.

1. Conditional risk in volatility models
2. Risk parameter in volatility models
3. Estimating the risk parameter
   - QML estimators of general risk parameters
   - One-step VaR estimation
   - Comparison with two-step VaR estimators

Conditional risk

Vast literature on the estimation of the volatility parameter (under $E\eta_1^2 = 1$).

The most widely used method is the QML (Quasi-maximum likelihood):

- asymptotic theory valid under mild assumptions (strict stationarity but no moments of the observed process);
- does not require to know the distribution of η_t.

Some references on QML estimation for GARCH:

- **ARCH(q) or GARCH(1,1):** Weiss (Econ. Theory, 1986), Lee and Hansen (Econ. Theory, 1994), Lumsdaine (Econometrica, 1996),
- **GARCH(p,q):** Berkes, Horváth and Kokoszka (Bernoulli, 2003), Francq and Zakoïan (Bernoulli, 2004), Hall and Yao (Econometrica, 2003), Mikosch and Straumann (Ann. Statist., 2006).
- **More general stationary GARCH models:** Straumann and Mikosch (Ann. Statist., 2006), Robinson and Zaffaroni (Ann. Statist., 2006), Bardet and Wintenberger (Ann. Statist., 2009), Meitz and Saikkonen (Econ. Theory, 2011).
- **Explosive ARCH(1) and GARCH(1,1):** Jensen and Rahbek (Econometrica, 2004 and Econ. Theory, 2004), Francq and Zakoïan (Econometrica, 2012).

Two strategies for QML estimation of the conditional risk

Based on two formulations of the conditional risk:

$$r_{t-1}(\epsilon_t) = \begin{cases} \sigma_t(\theta_0) r(\eta_t), & \text{with } E\eta_1^2 = 1, \\ \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \theta_0^*), & \text{with } r(\eta_1^*) = 1. \end{cases}$$

1. Standard Gaussian QML estimation + nonparametric estimation of $r(\eta_t)$.
2. Non Gaussian QML estimation under the identifiability assumption $r(\eta_1^*) = 1$.

Non Gaussian QML estimator under $r(\eta_1^*) = 1$.

Given observations $\epsilon_1, \dots, \epsilon_n$, and arbitrary initial values $\tilde{\epsilon}_i$ for $i \leq 0$, let

$$\tilde{\sigma}_t(\theta) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, \dots; \theta).$$

This random variable will be used to approximate

$$\sigma_t(\theta) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1, \epsilon_0, \epsilon_{-1}, \dots; \theta).$$

We choose an arbitrary, *instrumental*, positive density h, and we define the QML criterion

$$\tilde{Q}_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} g(\epsilon_t, \tilde{\sigma}_t(\theta)), \qquad g(x, \sigma) = \log \frac{1}{\sigma} h\left(\frac{x}{\sigma}\right).$$

Let the QMLE, for some compact space $\Theta \subset \mathbb{R}^m$,

$$\hat{\theta}_n^* = \arg\max_{\theta \in \Theta} \tilde{Q}_n(\theta).$$

Technical assumptions for the consistency:

A1: (ϵ_t) is a strictly stationary and ergodic solution of the model.

A2: Almost surely, $\sigma_t(\theta) \in (\underline{\omega}, \infty]$ for any $\theta \in \Theta$ and for some $\underline{\omega} > 0$. Moreover, $\sigma_t(\theta_0^*)/\sigma_t(\theta) = 1\ a.s.$ iff $\theta = \theta_0^*$.

In addition, we assume that the function $\sigma \to Eg(\eta_1^*, \sigma)$ is valued in $[-\infty, +\infty)$ and has a unique maximum at 1:

A3: $Eg(\eta_1^*, \sigma) < Eg(\eta_1^*, 1), \qquad \forall \sigma > 0, \quad \sigma \neq 1.$

A4: h is continuous on $\mathbb{R}$, differentiable except on a finite set A, and there exist constants $\delta \geq 0$ and $C_0 > 0$ such that for all $u \in A^c$, $|uh'(u)/h(u)| \leq C_0(1 + |u|^\delta)$ with $E|\epsilon_0|^\delta < \infty$.

A5: There exist a random variable C_1 measurable with respect to $\{\epsilon_u, u < 0\}$ and a constant $\rho \in (0, 1)$ such that $\sup_{\theta \in \Theta} |\sigma_t(\theta) - \tilde{\sigma}_t(\theta)| \leq C_1 \rho^t$.

Interpretation of **A3**

Consistency of the risk parameter estimator

If **A0-A5** hold, the non-Gaussian QML estimator satisfies

$$\hat{\theta}_n^* \to \theta_0^*, \quad a.s.$$

Remark: the innovation distribution is subject to two conditions

$$r(\eta_1^*) = 1 \quad \text{and} \quad Eg(\eta_1^*, \sigma) < Eg(\eta_1^*, 1).$$

Can we find a density h making them compatible?

Choice of the QML density h

Assume that, for some measurable function $\psi : \mathbb{R} \to \mathbb{R}$,

$$r(X) = 1 \quad \text{iff} \quad E\{\psi(X)\} = 0.$$

Assume **A4** holds with $A = \emptyset$. Then **A3** holds for any distribution of η_0^* satisfying $r(\eta_0^*) = 1$ iff the density h is such that

$$x\{\log h(x)\}' = \lambda\psi(x) - 1, \quad \text{for all } x,$$

for some constant $\lambda \neq 0$.

Provides a practical way to choose the QML density h.

Examples

- $r(X) = \|X\|_s = (E|X|^s)^{1/s}, \quad s > 0$.

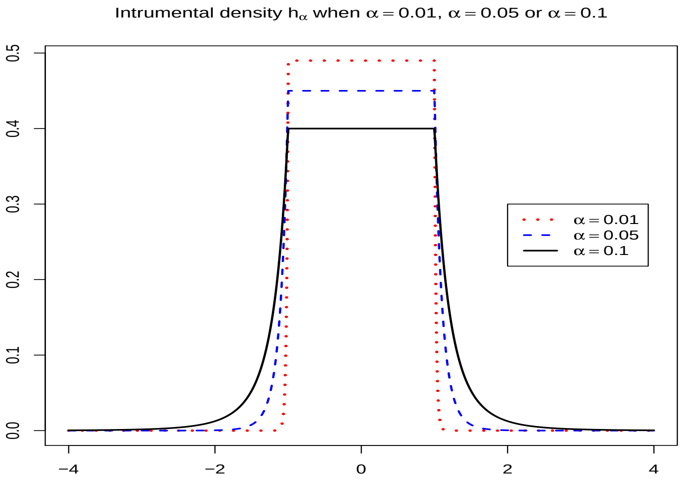
  We have $\psi(X) = |X|^s - 1$ and we find

$$h(x) \ \propto \ x^{-(1-\lambda)} \exp(-\lambda|x|^s/s), \quad \forall\lambda > 0.$$

- VaR at level α: $\quad r(X) = -F_X^{-1}(\alpha)$.

  If $\mathbb{P}^X = \mathbb{P}^{-X}$ and $\alpha \in (0, 0.5)$, $\psi(X) = \mathbf{1}_{\{|X|>1\}} - 2\alpha$, we find

$$h_\alpha(x) \ \propto \ |x|^{2\lambda\alpha-1}\{|x|^{-\lambda}\mathbf{1}_{\{|x|>1\}} + \mathbf{1}_{\{|x|\le 1\}}\}, \quad \forall\lambda > 0.$$


Intrumental density h_α when $\alpha = 0.01$, $\alpha = 0.05$ or $\alpha = 0.1$
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
$\alpha = 0.01$
$\alpha = 0.05$
$\alpha = 0.1$

Additional assumptions for the asymptotic normality

A6: $\theta_0^* \in \overset{\circ}{\Theta}$.

A7: $x' \frac{\partial \sigma_t(\theta_0^*)}{\partial \theta} = 0, \quad a.s. \quad \Rightarrow \quad x = 0.$

A8: The function $\theta \mapsto \sigma(x_1, x_2, \dots; \theta)$ has continuous second-order derivatives, and for C_1, ρ as in **A5**,

$$\sup_{\theta \in \Theta} \left\| \frac{\partial \sigma_t(\theta)}{\partial \theta} - \frac{\partial \tilde{\sigma}_t(\theta)}{\partial \theta} \right\| + \left\| \frac{\partial^2 \sigma_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \tilde{\sigma}_t(\theta)}{\partial \theta \partial \theta'} \right\| \leq C_1 \rho^t.$$

A9: h is twice differentiable with $\left| u^2 \left(h'(u)/h(u) \right)' \right| \leq C_0(1 + |u|^\delta)$ for all $u \in \mathbb{R}$ and $E|\epsilon_0|^{2\delta} < \infty$.

A10: There exists a neighborhood $V(\theta_0^*)$ of θ_0^* such that

$$E \sup_{\theta \in V(\theta_0^*)} \left\| \frac{1}{\sigma_t(\theta)} \frac{\partial^2 \sigma_t(\theta)}{\partial \theta \partial \theta'} \right\|^2 < \infty.$$

Asymptotic normality of the risk parameter estimator

Let $g_1(x,\sigma) = \partial g(x,\sigma)/\partial\sigma$ and $g_2(x,\sigma) = \partial g_1(x,\sigma)/\partial\sigma$.

Under **A0-A10** and if $Eg_2(\eta_0^*, 1) \neq 0$,

$$\sqrt{n}\left(\hat{\theta}_n^* - \theta_0^*\right) \xrightarrow{d} \mathcal{N}(0, 4\tau_{h,f}^2 I^{-1})$$

where

$$I = I(\theta_0^*) = E\left(\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}(\theta_0^*)\right) \quad \text{and} \quad \tau_{h,f}^2 = \frac{Eg_1^2(\eta_0^*, 1)}{\left\{Eg_2(\eta_0^*, 1)\right\}^2}.$$

But this does not apply to the VaR (**A9** not satisfied).

Definition of the VaR parameter

Model reparameterization:

$$\begin{cases} \epsilon_t = \sigma_t^* \eta_t^*, \quad P[\eta_t^* < -1] = \alpha, \\ \sigma_t^* = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \theta_{0,\alpha}). \end{cases}$$

where

- $\eta_t^* = -\eta_t / F^{-1}(\alpha)$ (provided $F^{-1}(\alpha) < 0$)
- $\theta_{0,\alpha} = \theta_0^* = H(\theta_0, -F^{-1}(\alpha))$: the *VaR parameter* at level α.

The theoretical VaR is now given by

$$\text{VaR}_t(\alpha) = \sigma_t^*.$$

Definition of the VaR parameter estimator

QML estimator of $\theta_{0,\alpha}$:

$$\hat{\theta}_{n,\alpha} = \arg\max_{\theta\in\Theta} \sum_{t=1}^{n} \log \frac{1}{\tilde{\sigma}_t(\theta)} h_\alpha\left(\frac{\epsilon_t}{\tilde{\sigma}_t(\theta)}\right)$$

where

$$h_\alpha(x) = \lambda\alpha(1-2\alpha)|x|^{2\lambda\alpha-1}\{|x|^{-\lambda}\mathbf{1}_{\{|x|>1\}} + \mathbf{1}_{\{|x|\leq 1\}}\}$$

The estimator $\hat{\theta}_{n,\alpha}$ does not depend on the constant $\lambda > 0$.

Interpretation as a quantile regression estimator

If the distribution of η_0^* is symmetric, we have

$$\log|\epsilon_t| = \log\sigma_t^* + \log|\eta_t^*|, \qquad P[\log|\eta_0^*| < 0] = 1 - 2\alpha,$$

Let $\rho_\alpha(u) = u(\alpha - \mathbf{1}_{\{u\leq 0\}})$.

Then

$$\begin{aligned}\hat{\theta}_{n,\alpha} &= \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{t=1}^{n} \rho_{1-2\alpha}\left\{\log\left(\frac{|\epsilon_t|}{\tilde{\sigma}_t(\theta)}\right)\right\} \\ &= \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{t=1}^{n} \left|\log\left(\frac{|\epsilon_t|}{\tilde{\sigma}_t(\theta)}\right)\right| \\ &\qquad \times \left((1-2\alpha)\mathbf{1}_{\{|\epsilon_t|>\tilde{\sigma}_t(\theta)\}} + 2\alpha\mathbf{1}_{\{|\epsilon_t|<\tilde{\sigma}_t(\theta)\}}\right).\end{aligned}$$

Consistency and A.N. of the VaR parameter estimator

B1: The law of η_1^* is symmetric, admits a density in a neighborhood of 1 and satisfies $E|\log|\eta_1^*|| < \infty$.

If **A0-A2**, **A5** and **B1** hold, for all $\alpha \in (0, 1/2)$,

$$\hat{\theta}_{n,\alpha} \to \theta_{0,\alpha}, \quad a.s.$$

Under additional technical assumptions, there exists a sequence of local minimizers $\hat{\theta}_{n,\alpha}$ of the criterion satisfying

$$\sqrt{n}(\hat{\theta}_{n,\alpha} - \theta_{0,\alpha}) \xrightarrow{d} \mathcal{N}\left(0, \Xi_\alpha := \frac{2\alpha(1-2\alpha)}{4f^{*2}(1)} J_\alpha^{-1}\right)$$

where $J_\alpha = ED_t(\theta_{0,\alpha})D_t'(\theta_{0,\alpha})$ and $D_t(\theta) = \sigma_t^{-1}(\theta)\partial\sigma_t(\theta)/\partial\theta$.

Confidence intervals for the VaR

One-step consistent estimator of the VaR parameter:

$$\widehat{\mathrm{VaR}}_t(\alpha) = \tilde{\sigma}_t(\hat{\theta}_{n,\alpha}).$$

Asymptotic confidence interval for $\mathrm{VaR}_t(\alpha)$
(at the level $(1-\alpha_0)\%$):

$$\tilde{\sigma}_t(\hat{\theta}_{n,\alpha}) \pm \frac{\Phi^{-1}_{1-\alpha_0}}{\sqrt{n}} \left\{ \frac{\partial \tilde{\sigma}_t(\hat{\theta}_{n,\alpha})}{\partial \theta'} \widehat{\Xi}_\alpha \frac{\partial \tilde{\sigma}_t(\hat{\theta}_{n,\alpha})}{\partial \theta} \right\}^{1/2},$$

The VaR evaluation is subject to estimation risk.

Even when the model is correctly specified, the market risk, as measured by the theoretical VaR, is not perfectly known, but is likely to belong to the confidence interval.

Two-step estimator

$$\text{VaR}_t(\alpha) = -\sigma_t(\theta_0)F_\eta^{-1}(\alpha).$$

Under the usual condition $E\eta_t^2 = 1$, and $E\eta_t^4 < \infty$,

- θ_0 is estimated by standard QML (estimator $\hat{\theta}_n$);
- the theoretical quantile $\xi_\alpha := F_\eta^{-1}(\alpha)$ is estimated using the estimated rescaled innovations:

$$\hat{\eta}_t = \frac{\epsilon_t}{\tilde{\sigma}_t(\hat{\theta}_n)}.$$

Let $\xi_{n,\alpha}$ denote the empirical α-quantile of $\hat{\eta}_1, \dots, \hat{\eta}_n$.

An estimator of the VaR at level α is then given by

$$\widetilde{\text{VaR}}_t(\alpha) = -\tilde{\sigma}_t(\hat{\theta}_n)\xi_{n,\alpha}.$$

Comparing the one-step and two-step estimators

Estimators of the VaR: the one-step estimator

$$\widehat{\mathrm{VaR}}_t(\alpha) = \tilde{\sigma}_t(\hat{\theta}_{n,\alpha})$$

and the two-step estimators

$$\widetilde{\mathrm{VaR}}_t(\alpha) = -\tilde{\sigma}_t(\hat{\theta}_n)\xi_{n,\alpha} = \tilde{\sigma}_t\{H(\hat{\theta}_n, -\xi_{n,\alpha})\}.$$

and, under the assumption of symmetric errors distribution,

$$\widetilde{\widetilde{\mathrm{VaR}}}_t(\alpha) = \tilde{\sigma}_t(\hat{\theta}_n)\tilde{\xi}_{n,1-2\alpha} = \tilde{\sigma}_t\{H(\hat{\theta}_n, \tilde{\xi}_{n,1-2\alpha})\},$$

where $\tilde{\xi}_{n,1-2\alpha}$ is the empirical $(1-2\alpha)$-quantile of $|\hat{\eta}_1|, \dots, |\hat{\eta}_n|$.

A comparison of the VaR estimators can then be based on the asymptotic accuracies of the estimators of $\theta_{0,\alpha}$:

$$\hat{\theta}_{n,\alpha}, \quad \hat{\theta}_{n,\alpha}^{2step} := H(\hat{\theta}_n, -\xi_{n,\alpha}), \quad \hat{\theta}_{n,\alpha}^{S2step} := H(\hat{\theta}_n, \tilde{\xi}_{n,1-2\alpha}).$$

Asymptotic distribution of the two-step estimators

Requires deriving the joint asymptotic distributions of $(\hat{\theta}_n', -\xi_{n,\alpha})$ and $(\hat{\theta}_n', \tilde{\xi}_{n,1-2\alpha})$ under $E\eta_t^2 = 1$.

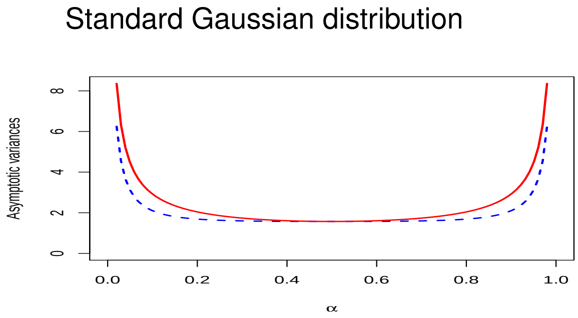
An additional assumption is needed: $\kappa_4 = E\eta_t^4 < \infty$.

We find that the asymptotic variance of $-\xi_{n,\alpha}$, the empirical quantile of $\hat{\eta}_t$'s, is

$$\zeta_\alpha = \underbrace{\frac{\alpha(1-\alpha)}{f^2(\xi_\alpha)}}_{\text{If no estimation}} + \underbrace{\frac{\xi_\alpha p_\alpha}{f(\xi_\alpha)} + \xi_\alpha^2 \frac{\kappa_4 - 1}{4}}_{\text{Effect of estimation}} .$$

where $\xi_\alpha = F_\eta^{-1}(\alpha)$ and $p_\alpha = E\left(\eta_1^2 \mathbf{1}_{\{\eta_1 < \xi_\alpha\}}\right) - \alpha$.

Asymptotic variances of empirical quantiles, with or without estimation (dotted and full lines)

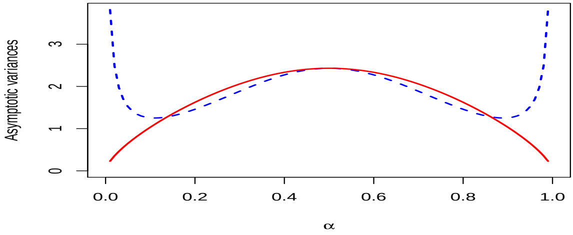


Asymptotic variances of empirical quantiles, with or without estimation (dotted and full lines)

GED(ν) distribution

density $f(x) \propto \exp\{-0.5|x|^{1/\nu}\}$

$\nu = 0.25$



Comparison of VaR estimators for standard GARCH with symmetric innovations

Under the previous assumptions (in particular $E\eta_t^4 < \infty$ for the two-step estimator),

$$\text{Var}_{as}\{\sqrt{n}(\hat{\theta}_{n,\alpha} - \theta_{0,\alpha})\} \preceq \text{Var}_{as}\left\{\sqrt{n}\left(\hat{\theta}_{n,\alpha}^{S2tep} - \theta_{0,\alpha}\right)\right\}$$
$$\text{iff} \quad \Delta_\alpha \leq 0,$$

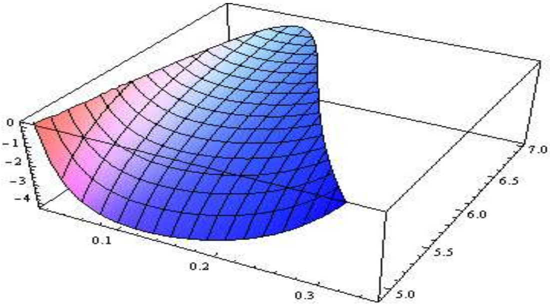
where $\Delta_\alpha = \frac{2\alpha(1-2\alpha)}{\xi_\alpha^2 f^2(\xi_\alpha)} - (\kappa_4 - 1)$.

Remark: the coefficient Δ_α only depends on the distribution of η_t, not on the true parameter value.

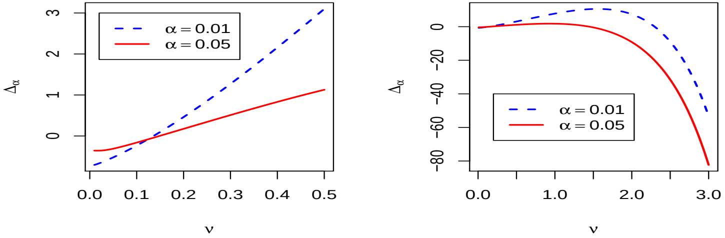
For fat-tailed distributions the one-step estimator will be better.

Surface $\Delta_\alpha \leq 0$ (1-step estimator better than 2-step)

Student(ν) distribution (standardized)
$\nu \in [4.9, 7]$ and $\alpha \in [0.01, 0.35]$



Δ_α for GED(ν)



Simulation experiments

Table: Empirical relative efficiency of the 1-step method with respect to the 2-step method for estimating the VaR parameter. ARCH(1) model with Student innovations. Number of replication: $N = 1,000$.

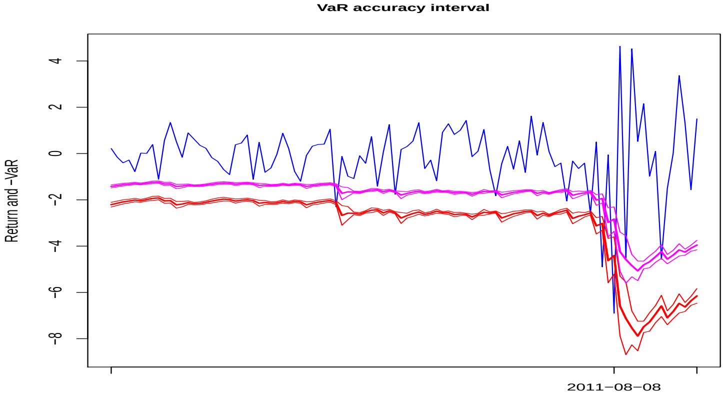
	$n = 500$							$n = 5,000$						
	ν							ν						
	1	2	3	4	5	6	∞	1	2	3	4	5	6	∞
$\alpha = 5\%$														
$\omega_{0,\alpha}$	7.5	2.8	1.7	1.3	1	0.9	0.9	13.9	6.6	2.7	1.3	1.1	1	0.8
$\alpha_{0,\alpha}$	7.3	3.6	1.7	1.3	1	1.0	0.8	22.2	8.7	3.2	1.3	1.1	1	0.9
$\alpha = 1\%$														
$\omega_{0,\alpha}$	6.1	1.6	1.0	0.8	0.7	0.7	0.7	41.1	3.6	1.6	0.9	0.8	0.8	0.7
$\alpha_{0,\alpha}$	3.8	1.8	2.6	0.8	0.7	0.7	0.7	13.7	6.0	2.1	0.9	0.8	0.7	0.7

Real data: January, 2, 1991 to August, 26, 2011

Table: Estimators of the VaR parameter $\theta_{0,\alpha}$ at level $\alpha = 5\%$ of GARCH(1,1) models. Estimations of Δ_α based on residuals of the 2-step and 1-step methods: $\Delta_\alpha < 0$ indicates superiority of the 1-step method.

		$\omega_{0,\alpha}$	$\alpha_{0,\alpha}$	$\beta_{0,\alpha}$	$\hat{\Delta}^{S2step}_{5\%}$	$\hat{\Delta}_{5\%}$
Nikkei	$\hat{\theta}^{S2step}_{n,5\%}$	0.08 (0.02)	0.33 (0.05)	0.87 (0.02)	-3.86	-4.54
	$\hat{\theta}_{n,5\%}$	0.04 (0.01)	0.30 (0.03)	0.88 (0.01)		
NSE	$\hat{\theta}^{S2step}_{n,5\%}$	0.16 (0.06)	0.26 (0.06)	0.87 (0.03)	-3.11	-3.30
	$\hat{\theta}_{n,5\%}$	0.18 (0.05)	0.31 (0.05)	0.85 (0.02)		
SP500	$\hat{\theta}^{S2step}_{n,5\%}$	0.02 (0.00)	0.20 (0.02)	0.92 (0.01)	-2.10	-2.31
	$\hat{\theta}_{n,5\%}$	0.02 (0.00)	0.19 (0.01)	0.92 (0.01)		
SPTSX	$\hat{\theta}^{S2step}_{n,5\%}$	0.02 (0.01)	0.17 (0.03)	0.93 (0.01)	-0.06	-0.52
	$\hat{\theta}_{n,5\%}$	0.04 (0.01)	0.23 (0.03)	0.90 (0.01)		

Estimated VaR's and VaR accuracy intervals



Log returns of the SP500 and estimated VaR's at the 5% and 1% levels, from April 6, 2011 to August 26, 2011. Estimation of the VaR parameter is based on the 500 previous values.

Conclusions

- Notion of conditional risk/VaR parameter.
- Facilitates the asymptotic comparison of risk evaluation procedures.
- Reparameterization allows for one-step estimation and easier evaluation of confidence intervals for the VaR.
- For standard GARCH models the ranking of the two methods only depends on the sign of the scalar Δ_α.
- This coefficient involves the risk level α and simple characteristics of the innovations distribution.
- The one-step method is typically more efficient in presence of fat tailed innovations.

Examples

- Standard GARCH(p,q) (Engle (82), Bollerslev (86)):

$$\sigma_t^2 = \omega_0 + \sum_{i=1}^{q} \alpha_{0i}\epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{0j}\sigma_{t-j}^2$$

- Asymmetric Power GARCH model: for $\delta > 0$,

$$\sigma_t^\delta = \omega_0 + \sum_{i=1}^{q} \alpha_{0i+}(\epsilon_{t-i}^+)^\delta + \alpha_{0i-}(-\epsilon_{t-i}^-)^\delta + \sum_{j=1}^{p} \beta_{0j}\sigma_{t-j}^\delta$$

- ARCH(∞) (Robinson (91)), introduced to capture long memory:

$$\sigma_t^2 = \psi_{00} + \sum_{i=1}^{\infty} \psi_{0i}\epsilon_{t-i}^2$$

Return

Interpretation of the identifiability assumption

A3: $Eg(\eta_1^*, \sigma) < Eg(\eta_1^*, 1) \qquad \forall \sigma > 0, \quad \sigma \neq 1.$

If η_1^* has a density f, let $h_\sigma(x) = \sigma^{-1}h(\sigma^{-1}x)$ and the Kullback-Leibler "distance" $K(f,f^*) = E\log(f/f^*)(\eta_1^*)$. Then

A3: $K(f,h) < K(f,h_\sigma) \qquad \forall \sigma > 0, \quad \sigma \neq 1$

Remark: When $h = f$ (MLE), **A3** vanishes.

Return