# A class of nonstationary GARCH models with application to gas prices

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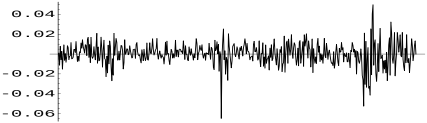
# Outline

# Volatility models

Introduced for financial series which, after differentiation, look like this:

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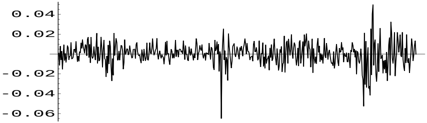


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But the empirical autocorrelations of the squares are generally statistically significant.

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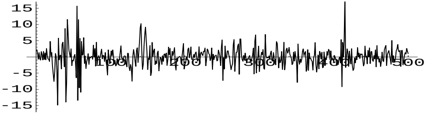
# Standard GARCH(1,1) Model

$$\begin{cases} \epsilon_t &= \sigma_t \eta_t, \qquad (\eta_t)\ iid,\ E\eta_t = 0,\ \ Var(\eta_t) = 1 \\ \sigma_t^2 &= \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \qquad \omega > 0, \alpha, \beta \geq 0 \end{cases}$$

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$$E \log(\alpha\eta_0^2 + \beta) < 0$$

or second-order stationary solution:

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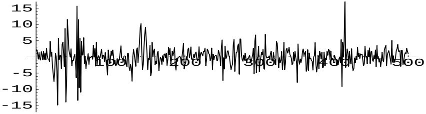


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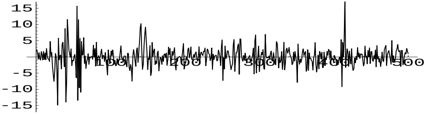


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# Advantages and limits

These models are able to capture

- the leptokurticity of the distributions
- volatility clustering
- dependence without correlation

but not

- seasonal behaviors
- dependence with respect to exogenous variables (ex: temperature for the energy prices)

# A GARCH(1,1) driven by an exogenous process

$$\begin{cases} \epsilon_t &= \sigma_t \eta_t, \qquad (\eta_t) \; iid \; (0,1) \\ \sigma_t^2 &= \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2, \end{cases}$$

where
- $\omega(\cdot) > 0, \alpha(\cdot), \beta(\cdot) \geq 0$
- $(s_t)$ is a sequence of numbers $s_t \in E = \{1, \dots, d\}$ (realizations of a process $(S_t)$).

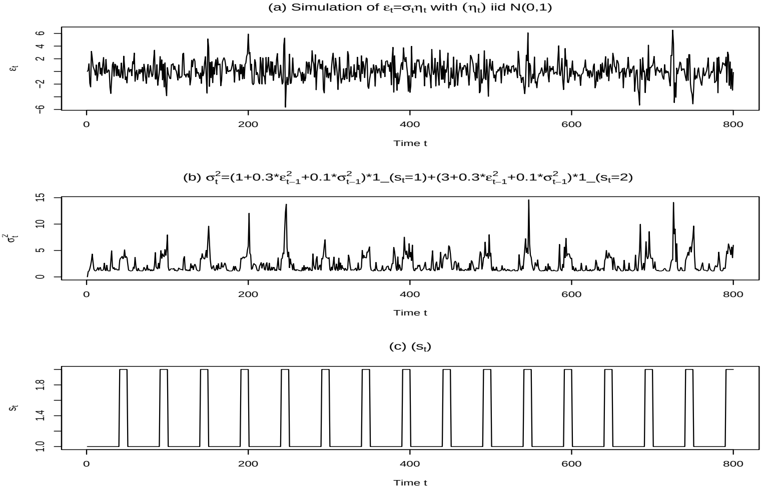
For energy prices, $s_t$ could be an integer giving information about : the day in the week (e.g. week-end or not), the level of temperature...

# Example: 2 regimes

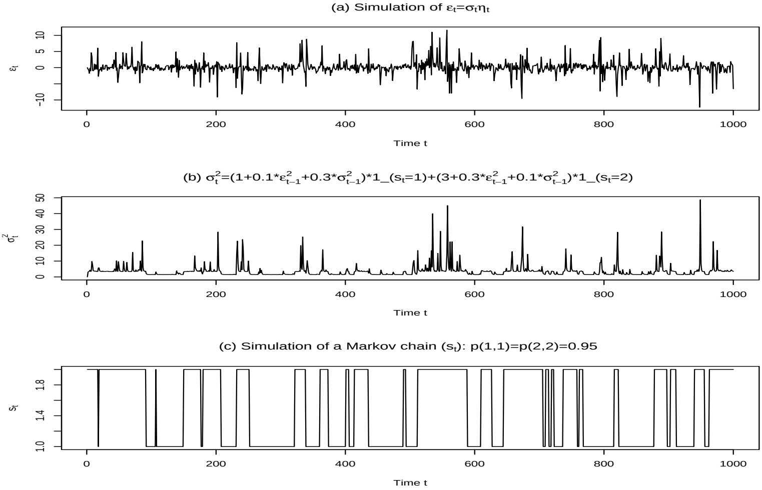
$$\begin{cases} \epsilon_t & = \sigma_t \eta_t, \qquad (\eta_t) \ iid \ (0,1) \\ \sigma_t^2 & = \begin{cases} \omega(1) + \alpha(1)\epsilon_{t-1}^2 + \beta(1)\sigma_{t-1}^2 & \text{si } s_t = 1 \\ \omega(2) + \alpha(2)\epsilon_{t-1}^2 + \beta(2)\sigma_{t-1}^2 & \text{si } s_t = 2 \end{cases} \end{cases}$$

$$\omega(2), \omega(2) > 0, \alpha(1) \geq 0, \beta(1) \geq 0, \alpha(2) \geq 0, \beta(2) \geq 0.$$

# Example : $(s_t)$ periodic



# Example : $(s_t)$ realization of a Markov chain



# TS models with time-dependent coefficients

- **Non stationary processes:** Priestley (1965), Whittle (1965), Hallin (1986)
- **Locally stationary processes:** Dalhaus (1997)
- **Periodic models:** Periodic ARMA (Anderson and Vecchia (1983), Lund and Basawa (2000)); Periodic GARCH (Bollerslev and Ghysels (1996))
- **ARMA with time-varying coefficients:** Kwoun and Yajima (1986), Bibi and Francq (2003), Francq and Gautier (2004), Azrak and Mélard (2006)
- **Non stationary volatility models:** Engle and Rangel (2005), Dalhaus and Subba Rao (2006), Amado and Teräsvirta (2008)

# Existence of non explosive solutions

$$\sigma_t^2 \quad = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2$$

## Proposition

*For $j = 1, \ldots d$ assume that for all $t$,*

$$\lim_{n\to\infty} \textit{Frequency of } j \textit{ among } \{s_t, s_{t-1}, \ldots, s_{t-n}\} := \pi_j.$$

*Then, the stability condition is*

$$\gamma_0 := \sum_{j=1}^{d} \pi_j E\{\log \alpha(j)\eta_0^2 + \beta(j)\} < 0.$$

# Remarks

- A sufficient condition for stability: stationarity of each regime.

$$E\{\log \alpha(j)\eta_0^2 + \beta(j)\} < 0, \qquad j = 1, \dots, d.$$

- A necessary condition: $\prod_{j=1}^{d} \beta^{\pi_j}(j) < 1.$
- In the ARCH(1) case (no coefficients $\beta$), the condition is more explicit:

$$\prod_{j=1}^{d} \alpha^{\pi_j}(j) < e^{-E \log \eta_0^2}.$$

- If $\gamma_0 > 0$, for any initial value $\sigma_0^2$

$$\sigma_t^2 \to +\infty, \ a.s. \quad t \to \infty.$$

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- If for some regime $\alpha(j) = \beta(j) = 0$ and $\pi_j > 0$, the model is stable.
- Conditional and unconditional variances are time-dependent: under existence conditions

$$\text{var}(\epsilon_t) = \omega(s_t) + \sum_{n=1}^{\infty} \left( \prod_{i=0}^{n-1} (\alpha + \beta)(s_{t-i}) \right) \omega(s_{t-n}).$$

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# Existence of moments

## Proposition

*If, for some positive integer $m$,*

$$\gamma_m = \prod_{j=1}^{d} \left[ E\{\alpha(j)\eta_0^2 + \beta(j)\}^m \right]^{\pi_j} < 1,$$

*the model is stable and the solution $(\epsilon_t)$ is such that $E\epsilon_t^{2m} < \infty$.*

# Comparison with the Markov-Switching models

$$\begin{cases} \epsilon_t & = \sigma_t \eta_t, \qquad (\eta_t) \ iid \ (0,1) \\ \sigma_t^2 & = \omega(S_t) + \alpha(S_t)\epsilon_{t-1}^2 + \beta(S_t)\sigma_{t-1}^2 \end{cases}$$

where $(S_t)$ is a stationary, irreducible and aperiodic Markov chain on $\{1, \dots, d\}$.

- Existence of a strictly stationary solution under the same condition $\gamma_0 < 0$ (where the $\pi_j$ are the stationary probabilities)

- But the moment conditions are different (depend on the transition probabilities)

From a statistical point of view, $(S_t)$ is not observed which makes the likelihood generally intractable.

# Estimation

Model:

$$\epsilon_t = \sqrt{h_t}\eta_t, \quad h_t = \omega_0(s_t) + \alpha_0(s_t)\epsilon_{t-1}^2 + \beta_0(s_t)h_{t-1}.$$

Parameters:

$$\theta = (\omega(1), \dots, \omega(d), \alpha(1), \dots, \alpha(d), \beta(1), \dots, \beta(d))'$$

Parameter space: $\Theta \subset ]0, +\infty[^d \times [0, \infty[^{2d}$.

**The sequence $(s_t)$ is known.**

# Gaussian Quasi-likelihood

Observations: $(\epsilon_1, \dots, \epsilon_n)$ [and $(s_1, \dots, s_n)$].

$$L_n(\theta; \epsilon_1, \dots, \epsilon_n) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\sigma_t^2}\right),$$

where for $t \geq 2$, with initial values,

$$\sigma_t^2 = \sigma_t^2(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2.$$

$\hat{\theta}_n$: QML estimator of $\theta_0$

# Use of the process $(S_t)$

**A0:** $(s_t)$ is the realization of a process $(S_t)$ which is stationary, ergodic, and independent of $(\eta_t)$.

If

$$\gamma_0 = \sum_{j=1}^{d} \pi_j E\{\log\{\alpha_0(j)\eta_0^2 + \beta_0(j)\}\} = E\{\log\{\alpha_0(S_t)\eta_0^2 + \beta_0(S_t)\}\} < 0,$$

there exists a strictly stationary solution $(\epsilon_{S,t})$ to

$$\epsilon_{S,t} = \sigma_{S,t}\eta_t, \quad \sigma_{S,t}^2 = \omega_0(S_t) + \alpha_0(S_t)\epsilon_{S,t-1}^2 + \beta_0(S_t)\sigma_{S,t-1}^2.$$

# Assumptions

**A1:** $\theta_0 \in \Theta$ and $\Theta$ is compact

**A2:** $\sum_{j=1}^{d} \pi_j E\{\log a_0(j,\eta_0)\} < 0 \qquad (a_0(j,\eta_0) = \alpha_0(j)\eta_0^2 + \beta_0(j))$
$\forall \theta \in \Theta,\ \prod_{j=1}^{d} \beta^{\pi_j}(j) < 1.$

**A3:** There exist $r, \rho \in (0,1)$ and $C > 0$ such that

$$\forall i > 0, \quad E\left\{a_0^r(S_t,\eta_{t-1}) \dots a_0^r(S_{t-i},\eta_{t-i-1})\right\} < C\rho^{i+1}.$$

**A4:** $\eta_t^2$ has a non degenerate distribution and $E\eta_t^2 = 1$.

**A5:** For all $i$, $\alpha_0(e_i) + \beta_0(e_i) \neq 0$ and there exist $\ell \in \{1,\dots,d\}$ and $k > 0$ such that $\alpha_0(e_\ell)\mathbb{P}(S_{t-k} = e_\ell, S_t = e_i) > 0$.

# Remarks on Assumption A3:

- Vanishes for an independent process $(S_t)$: under **A2**,

$$Ea_0^r(S_t,\eta_t) < 1, \quad \text{for some } r > 0$$

(Berkes, Horváth and Kokoszka (2003)).

- If $(S_t)$ is a stationary, irreducible, and aperiodic Markov chain **A3** is satisfied if $\rho(\mathbb{P}_r) < 1$, where

$$\mathbb{P}_r = \begin{pmatrix} p(1,1)E\{a_0^r(1,\eta_t)\} & \cdots & p(d,1)E\{a_0^r(1,\eta_t)\} \\ \vdots & & \vdots \\ p(1,d)E\{a_0^r(d,\eta_t)\} & \cdots & p(d,d)E\{a_0^r(d,\eta_t)\} \end{pmatrix}.$$

# Asymptotic distribution

## Proposition

*Under* **A0-A5***, for almost all sequence* $(s_t)$*,*

$$\hat{\theta}_n \to \theta_0, \quad a.s. \quad \text{as } n \to \infty.$$

*If, in addition,* $\theta_0$ *is in the interior of* $\Theta$ *and* $\kappa_\eta = E\eta_t^4 < \infty$*,*

$$\sqrt{n}(\hat{\theta}_n - \theta) \stackrel{d}{\rightsquigarrow} \mathcal{N}(0, (\kappa_\eta - 1)J^{-1})$$

*where*

$$J = E_{S,\eta}\left(\frac{1}{\sigma^4_{S,t}(\theta_0)} \frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta} \frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta'}\right)$$

# Estimation of the asymptotic covariance matrix

A consistent estimator of $J$ is

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma_t^4(\hat{\theta}_n)}\frac{\partial\sigma_t^2(\hat{\theta}_n)}{\partial\theta}\frac{\partial\sigma_t^2(\hat{\theta}_n)}{\partial\theta'},$$

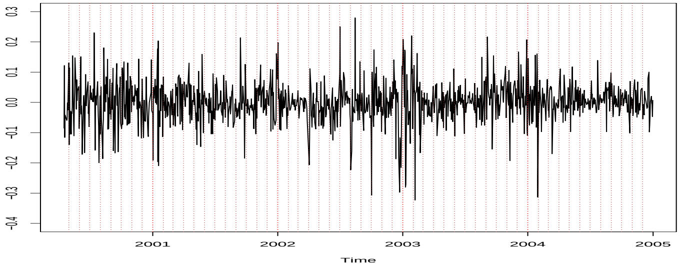
where

$$\sigma_t^2(\hat{\theta}_n) = \hat{\omega}_n(s_t) + \hat{\alpha}_n(s_t)\epsilon_{t-1}^2 + \hat{\beta}_n(s_t)\sigma_{t-1}^2(\hat{\theta}_n)$$

# Application to the modeling of gas volatility

Series of the gas spot price (Zeebrugge market) filtered from level effects (trends, cointegration with the Brent)

$\rightarrow$ series $\epsilon_t$ of log returns




$s_t$: classes of temperature levels

# Estimated models

- 1 regime (standard GARCH)

$$h_t = \underset{(0.0000)}{0.0003} + \underset{(0.0006)}{0.13} \epsilon_{t-1}^2 + \underset{(0.0011)}{0.79} h_{t-1}$$

- 3 regimes

$$h_t = \begin{cases} \underset{(0.0002)}{0.0003} + \underset{(0.05)}{0.13} \epsilon_{t-1}^2 + \underset{(0.06)}{0.80} h_{t-1} & \text{when } T_t < 9, \\ \underset{(0.0004)}{0.0011} + \underset{(0.10)}{0.37} \epsilon_{t-1}^2 + \underset{(0.16)}{0.36} h_{t-1} & \text{when } 9 \le T_t \le 14, \\ \underset{(0.0001)}{0.0004} + \underset{(0.06)}{0.14} \epsilon_{t-1}^2 + \underset{(0.10)}{0.76} h_{t-1} & \text{when } T_t > 14. \end{cases}$$

$$\pi_1 = 0.35, \quad \pi_2 = 0.32, \quad \pi_3 = 0.33.$$

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# Estimated models

- 5 regimes

$$h_t = \begin{cases} \underset{(0.0004)}{0.0008} + \underset{(0.08)}{0.15}\ \epsilon_{t-1}^2 + \underset{(0.11)}{0.80}\ h_{t-1} & \text{when } T_t < 6, \\ \underset{(0.0003)}{0.0010} + \underset{(0.04)}{0.00}\ \epsilon_{t-1}^2 + \underset{(0.09)}{0.80}\ h_{t-1} & \text{when } 6 \leq T_t \leq 9, \\ \underset{(0.0004)}{0.0015} + \underset{(0.12)}{0.46}\ \epsilon_{t-1}^2 + \underset{(0.17)}{0.21}\ h_{t-1} & \text{when } 9 < T_t \leq 14, \\ \underset{(0.0005)}{0.0007} + \underset{(0.12)}{0.32}\ \epsilon_{t-1}^2 + \underset{(0.17)}{0.62}\ h_{t-1} & \text{when } 14 < T_t \leq 16, \\ \underset{(0.0003)}{0.0003} + \underset{(0.05)}{0.04}\ \epsilon_{t-1}^2 + \underset{(0.13)}{0.81}\ h_{t-1} & \text{when } T_t > 16. \end{cases}$$

$$\pi_1 = 0.16, \quad \pi_2 = 0.19, \quad \pi_3 = 0.32, \quad \pi_4 = 0.15, \quad \pi_5 = 0.18.$$

# Estimated models

- Periodic model (no temperature)

$$h_t = \begin{cases} \underset{(0.0001)}{0.0001} + \underset{(0.1)}{0.20} \; \epsilon^2_{t-1} + \underset{(0.07)}{0.77} \; \sigma^2_{t-1} & \text{when } t \in Q_1, \\ \underset{(0.0018)}{0.0023} + \underset{(0.09)}{0.09} \; \epsilon^2_{t-1} + \underset{(0.74)}{0.000} \; \sigma^2_{t-1} & \text{when } t \in Q_2, \\ \underset{(0.004)}{0.000} + \underset{(0.03)}{0.01} \; \epsilon^2_{t-1} + \underset{(0.14)}{0.99} \; \sigma^2_{t-1} & \text{when } t \in Q_3, \\ \underset{(0.0003)}{0.0004} + \underset{(0.10)}{0.25} \; \epsilon^2_{t-1} + \underset{(0.12)}{0.69} \; \sigma^2_{t-1} & \text{when } t \in Q_4. \end{cases}$$

$$\pi_1 = 0.25, \quad \pi_2 = 0.25, \quad \pi_3 = 0.25, \quad \pi_4 = 0.25.$$

# Comparison of estimated models

Table: Likelihoods of the estimated models and Kurtosis of the standardized returns

| | GARCH $(d=1)$ | 3 regimes $(d=3)$ | 5 regimes $(d=5)$ | 7 regimes $(d=7)$ | Periodic $(d=4)$ |
|---|---|---|---|---|---|
| $\log L_n$ | 5173 | 5179 | 5210 | 5223 | 5217 |

Wald and LR tests ($5\%$ level):

- GARCH(1,1) not rejected against the 3 regimes model
- GARCH(1,1) rejected against the models with $d > 3$
- Rejection of the model with 3 regimes against the 5 and 7 regimes models
- Rejection of the model with 5 regimes against the 7 regimes model

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# Comparison of estimated models

MSE : mean square error of prediction of $\epsilon^2$

Table: MSE ($\times 10^{-5}$) of predictions (last 500 observations)

| GARCH ($d=1$) | 5 regimes ($d=5$) | 7 regimes ($d=7$) | Periodic ($d=4$) |
|---|---|---|---|
| 9.319 | 9.014 | 9.051 | 9.259 |

# Summary and conclusions

- Standard GARCH models are not appropriate for series displaying non stationarities.
- The proposed model is conditional to an exogenous process. More flexible than purely periodic models.
- Solutions, when existing, are non stationary. The existence conditions depend on the GARCH coefficients and the frequencies of occurence of the different regimes.
- QML estimation requires additional assumptions on the exogenous process. Numerical implementation is not more difficult than with standard GARCH models.
- Taking into account the temperature allows to better model the volatility of gas prices. A 7 regimes model seems to be the most satisfactory.