Buy-low and sell-high investment strategies

Mihail Zervos
Department of Mathematics
London School of Economics

Joint work with Timothy C. Johnson and Fares Alazemi

The investment problem

- We consider an asset with price process X that is modelled by the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{1}$$

where W is a standard one-dimensional Brownian motion.

- For example, X can be a geometric Brownian motion, which is given by

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \tag{2}$$

for some constants b and $\sigma \neq 0$.

Alternatively, X can be a mean-reverting constant elasticity of variance (CEV) processes, which is given by

$$dX_t = \kappa(\vartheta - X_t)\,dt + \sigma X_t^{\ell}\,dW_t. \tag{3}$$

for some constants $\kappa, \vartheta, \sigma > 0$ and $\ell \in [\frac{1}{2}, 1[$ such that $2\kappa\vartheta > \sigma^2$ if $\ell = \frac{1}{2}$. Such CEV processes have been proposed in the empirical finance literature as better models for a range of asset prices, particularly, in the commodity markets.

- An investor follows a strategy that consists of sequentially buying and selling one share of the asset. We use a controlled finite variation càglàd process Y that takes values in $\{0, 1\}$ to model the investor's position in the market.

  In particular,

$$Y_t = 1 \text{ if the investor holds the asset at time } t,$$
$$Y_t = 0 \text{ if the investor does not hold the asset at time } t,$$

  while, the jumps of Y occur at the sequence of times $(\tau_n,\ n \geq 1)$ at which the investor buys or sells.

- We assume that selling (resp., buying) the asset involves a fixed transaction cost $c_{\mathrm{s}} > 0$ (resp., $c_{\mathrm{b}} > 0$). Therefore,

  - if the investor sells one share of the asset at time τ_j, then she/he receives a payoff equal to $X_{\tau_j} - c_{\mathrm{s}}$, and
  - if the investor buys one share of the asset at time τ_j, then she/he faces a cost that is equal to $-(X_{\tau_j} + c_{\mathrm{b}})$.

- Given an initial condition $(Y_0, X_0) = (y, x) \in \{0,1\} \times \,]0, \infty[$, the investor's objective is to select an investment strategy Y that maximises the expected present value of all transactions, namely, the performance criterion

$$J_{y,x}(Y) = \lim_{n \to \infty} \sum_{j=1}^{n} \mathbb{E}_x \Bigg[e^{-\Lambda_{\tau_j}} \big[\big(X_{\tau_j} - c_{\mathrm{s}} \big) \mathbf{1}_{\{\Delta Y_{\tau_j} = -1\}} - \big(X_{\tau_j} + c_{\mathrm{b}} \big) \mathbf{1}_{\{\Delta Y_{\tau_j} = 1\}} \big] \mathbf{1}_{\{\tau_j < \infty\}} \Bigg]. \tag{4}$$

where

$$\Lambda_t = \int_0^t r(X_s) \, ds, \tag{5}$$

for some function $r > 0$.

- The problem's value function v is defined by

$$v(y, x) = \sup_{Y \in \mathcal{A}_{y,x}} J_{y,x}(Y), \quad \text{for } y \in \{0, 1\} \text{ and } x > 0, \tag{6}$$

where $\mathcal{A}_{y,x}$ is the set of admissible investment strategies

The case of a geometric Brownian motion

- Suppose that

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t \tag{7}$$

and that $r(x) = r_0 > 0$, for some constants b, $\sigma \neq 0$ and $r_0 > 0$.

  – If $r_0 < b$, then the strategy "buy at time 0 if not long in the market and sell at time T" has payoff such that

$$\begin{aligned}
&\left\{-(x + c_{\text{b}})\right\} + \mathbb{E}\left[e^{-r_0 T}\left(X_T - c_{\text{s}}\right)\right] \\
&\qquad = \left\{-(x + c_{\text{b}})\right\} + xe^{(b-r_0)T}\mathbb{E}\left[e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}\right] - e^{-r_0 T}c_{\text{s}} \\
&\qquad = \left\{-(x + c_{\text{b}})\right\} + xe^{(b-r_0)T} - e^{-r_0 T}c_{\text{s}}, \qquad (8)
\end{aligned}$$

which converges to ∞ as $T \to \infty$.

– If $r_0 > b$, then
  * if the investor does not hold the asset at time 0, then it is optimal to never enter the market;
  * if the investor holds the asset at time 0, then it is optimal to sell it as soon as its price exceeds the level $rc_{\mathrm{s}}(m-1)/[(r-b)m]$.

- The geometric Brownian motion is the most common model in mathematical finance: this might explain why buy-low and sell-high investment models have not attracted much interest in the literature.

- If we consider more general diffusion models, then the picture is rather different! Of special interest are the mean-reverting (CEV) processes, which are given by

$$dX_t = \kappa(\vartheta - X_t)\,dt + \sigma X_t^{\ell}\,dW_t, \tag{9}$$

for some constants $\kappa, \vartheta, \sigma > 0$ and $\ell \in [\frac{1}{2}, 1[$ such that $2\kappa\vartheta > \sigma^2$ if $\ell = \frac{1}{2}$.

Heuristic derivation of the Hamilton-Jacobi-Bellman (HJB) equation

- Suppose that, at time 0, the investor is long in the market. The investor is then faced with the choice of one of the following two possible actions:
  - Sell the asset and continue optimally, which implies that
  $$v(1,x) \geq x - c_{\mathrm{s}} + v(0,x). \tag{10}$$
  - Wait for Δt and then continue optimally, which implies that
  $$v(1,x) \geq \mathbb{E}_x\left[e^{-\Lambda_{\Delta t}} v(1, X_{\Delta t})\right].$$

  Using Itô's formula, we calculate
  $$0 \geq \mathbb{E}_x\left[\int_0^{\Delta t} e^{-\Lambda_s}\left(\frac{1}{2}\sigma^2 v_{xx}(1,\cdot) + b v_x(1,\cdot) - r v(1,\cdot)\right)(X_s)\, ds\right].$$
  Dividing by Δt and passing to the limit $\Delta t \downarrow 0$, we obtain
  $$\frac{1}{2}\sigma^2(x) v_{xx}(1,x) + b(x) v_x(1,x) - r(x) v(1,x) \leq 0. \tag{11}$$

In view of the Markovian nature of the problem, one of the above actions should be optimal at any point $x \in \mathcal{I}$. Therefore,

$$\left[v(0,x) + x - c_{\mathrm{s}} - v(1,x)\right] \left[\frac{1}{2}\sigma^2(x)v_{xx}(1,x) + b(x)v_x(1,x) - r(x)v(1,x)\right] = 0. \tag{12}$$

In view of (10)–(12), we expect that the value function v should satisfy

$$\max\left\{\frac{1}{2}\sigma^2(x)v_{xx}(1,x) + b(x)v_x(1,x) - r(x)v(1,x),\; v(0,x) + (x - c_{\mathrm{s}}) - v(1,x)\right\} = 0. \tag{13}$$

- If the investor does not hold the asset at time 0, then we can use similar arguments to conclude that the value function v should satisfy

$$\max\left\{\frac{1}{2}\sigma^2(x)v_{xx}(0,x) + b(x)v_x(0,x) - r(x)v(0,x),\right.$$
$$\left. v(1,x) - (x + c_{\mathrm{b}}) - v(0,x)\right\} = 0. \tag{14}$$

- The preceding considerations suggest that the problem's value function v should identify with a solution w of the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w_{xx}(y,x) + b(x)w_x(y,x) - r(x)w(y,x),\right.$$
$$\left. w(1-y,x) - w(y,x) + yH_{\mathrm{s}}(x) - (1-y)H_{\mathrm{b}}(x)\right\} = 0, \quad y = 0, 1, \tag{15}$$

where H_{b} and H_{s} are given by

$$H_{\mathrm{b}}(x) = x + c_{\mathrm{b}} \quad \text{and} \quad H_{\mathrm{s}}(x) = x - c_{\mathrm{s}}, \quad \text{for } x > 0. \tag{16}$$

Assumptions

- The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are Borel-measurable,

$$\sigma^2(x) > 0, \quad \text{for all } x > 0, \tag{17}$$

and

$$\int_{\underline{\alpha}}^{\overline{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)}\, ds < \infty \quad \text{for all } 0 < \underline{\alpha} < \overline{\beta} < \infty. \tag{18}$$

- The associated solution of (1) is non-explosive.
- The function $r :]0, \infty[\to]0, \infty[$ is Borel-measurable and uniformly bounded away from 0, i.e., there exists a constant $r_0 > 0$ such that $r(x) \geq r_0$ for all $x > 0$. Also,

$$\int_{\bar{\alpha}}^{\bar{\beta}} \frac{r(s)}{\sigma^2(s)}\, ds < \infty \quad \text{for all } 0 < \bar{\alpha} < \bar{\beta} < \infty. \tag{19}$$

- In the presence of these assumptions, the general solution of the homogeneous ODE

$$\mathcal{L}g(x) := \frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0 \tag{20}$$

exists in the classical sense and is given by

$$g(x) = A\varphi(x) + B\psi(x), \tag{21}$$

for some constants $A, B \in \mathbb{R}$ and some C^1 functions $\varphi, \psi :]0, \infty[\to]0, \infty[$ with absolutely continuous first derivatives.

- The functions $\varphi, \psi :]0, \infty[\to]0, \infty[$ can be chosen so that

$$\psi(x) = \psi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right] \text{ for all } x < y, \tag{22}$$

$$\varphi(x) = \varphi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right] \text{ for all } y < x, \tag{23}$$

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{24}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0, \tag{25}$$

and

$$\lim_{x \downarrow 0} \varphi(x) = \lim_{x \to \infty} \psi(x) = \infty, \tag{26}$$

where T_y is the first hitting time of $\{y\}$, which is defined by $T_y = \inf\{t \geq 0 \mid X_t = y\}$. Also, the processes $\left(e^{-\Lambda_t}\varphi(X_t)\right)$ and $\left(e^{-\Lambda_t}\psi(X_t)\right)$ are local martingales.

- *Example* (*geometric Brownian motion*).
  Suppose that
  $$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \tag{27}$$
  for some constants b and $\sigma \neq 0$, and that $r(x) = r_0$ for some constant $r_0 > 0$. Then
  $$\varphi(x) = x^m \quad \text{and} \quad \psi(x) = x^n, \tag{28}$$
  where $m < 0 < n$ are the solutions of
  $$\frac{1}{2}\sigma^2 k^2 + \left(b - \frac{1}{2}\sigma^2\right) k - r_0 = 0. \tag{29}$$

- *Example* (*square-root mean-reverting process*).
  Suppose that
  $$dX_t = \zeta(\vartheta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t \tag{30}$$
  for some constants $\zeta, \vartheta, \sigma > 0$ such that $\zeta\vartheta - \frac{1}{2}\sigma^2 > 0$, and that $r(x) = r_0$ for some constant $r_0 > 0$. Then
  $$\varphi(x) = \mathrm{U}\left(\frac{r_0}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}x\right) \text{ and } \psi(x) = {}_1\mathrm{F}_1\left(\frac{r_0}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}x\right). \tag{31}$$

- The inaccessible boundary point 0 is called

$$\text{natural if } \lim_{x \downarrow 0} \psi(x) = 0$$

$$\text{and entrance if } \lim_{x \downarrow 0} \psi(x) > 0.$$

If X is the geometric Brownian motion

$$dX_t = bX_t\, dt + \sigma X_t\, dW_t, \tag{32}$$

then 0 is a natural boundary point, while, if X is the mean-reverting (CEV) process

$$dX_t = \kappa(\vartheta - X_t)\, dt + \sigma X_t^\ell\, dW_t, \tag{33}$$

then 0 is an entrace boundary point.

- The problem data is such that

$$\lim_{x\to\infty}\frac{H(x)}{\psi(x)}=0 \quad \text{and} \quad \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\mathcal{L}H(X_t)\big|\,dt\right]<\infty. \tag{34}$$

  where H is the identity function, i.e., $H(x)=x$ for all $x>0$. Also, $c_{\mathrm{b}}, c_{\mathrm{s}}>0$, and there exist constants $0\le x_{\mathrm{b}}<x_{\mathrm{s}}$ such that

$$\mathcal{L}H_{\mathrm{b}}(x)\begin{cases}>0, & \text{if } x_{\mathrm{b}}>0 \text{ and } x<x_{\mathrm{b}},\\ <0, & \text{if } x>x_{\mathrm{b}},\end{cases} \quad \text{and } \mathcal{L}H_{\mathrm{s}}(x)\begin{cases}>0, & \text{if } x<x_{\mathrm{s}},\\ <0, & \text{if } x>x_{\mathrm{s}}.\end{cases} \tag{35}$$

- If 0 is a natural boundary point, then the above assumptions can all hold only if $x_{\mathrm{b}}=0$.

The solution of the problem

- A first possibility arises when it is optimal for the investor to sell as soon as the asset price exceeds a given level $\alpha > 0$, if $y = 1$, and never enter the market otherwise. In view of the heuristics explaining the structure of the HJB equation, we look for a solution w such that

$$\frac{1}{2}\sigma^2(x)w_{xx}(1,x) + b(x)w_x(1,x) - r(x)w(1,x) = 0, \quad \text{for } x < \alpha, \tag{36}$$

$$w(1,x) = H_{\mathrm{s}}(x) \equiv x - c_{\mathrm{s}}, \quad \text{for } x \geq \alpha, \tag{37}$$

and

$$w(0,x) = 0, \quad \text{for } x > 0. \tag{38}$$

We recall that the solution of (36) is given by

$$w(1,x) = A\varphi(x) + B\psi(x), \tag{39}$$

for some constants $A, B \in \mathbb{R}$, that φ (resp., ψ) is decreasing (resp., increasing), and

$$\lim_{x\downarrow 0} \varphi(x) = \lim_{x\to\infty} \psi(x) = \infty. \tag{40}$$

We therefore look for a solution of the HJB equation (15) of the form given by

$$w(0,x) = 0 \quad \text{and} \quad w(1,x) = \begin{cases} A\psi(x), & \text{if } x < \alpha, \\ H_{\text{s}}(x), & \text{if } x \geq \alpha, \end{cases} \tag{41}$$

for some constant A.

To determine the parameter A and the free-boundary point α, we appeal to the so-called "principle of smooth fit" of sequential switching and we require that $w(1,\cdot)$ is C^1 along α, which yields the system of equations

$$A\psi(\alpha) = H_{\text{s}}(\alpha) \quad \text{and} \quad A\psi'(\alpha) = H'_{\text{s}}(\alpha). \tag{42}$$

This system is equivalent to

$$A = \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)} = \frac{H_{\mathrm{s}}'(\alpha)}{\psi'(\alpha)} \tag{43}$$

and

$$q(\alpha) = 0, \tag{44}$$

where

$$q(x) = \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds. \tag{45}$$

There exists a unique $\alpha > 0$ satisfying equation (44). Furthermore, the function w defined by (41) for this value of α, where $A > 0$ is given by (43), satisfies the HJB equation (15) if and only if the problem data is such that either $x_{\mathrm{b}} = 0$ or

$$x_{\mathrm{b}} > 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{H_{\mathrm{b}}(x)}{\psi(x)} \equiv \lim_{x \downarrow 0} \frac{x + c_{\mathrm{b}}}{\psi(x)} \geq \frac{\alpha - c_{\mathrm{s}}}{\psi(\alpha)} \equiv \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)}. \tag{46}$$

- The second possibility arises when it is optimal for the investor to sequentially enter and exit the market. In this case, we postulate that the value function of our control problem should identify with a solution w of the HJB equation (15) that has the form given by the expressions

$$w(1,x) = \begin{cases} A\psi(x), & \text{if } x < \gamma, \\ B\varphi(x) + H_{\mathrm{s}}(x), & \text{if } x \geq \gamma, \end{cases} \tag{47}$$

$$w(0,x) = \begin{cases} A\psi(x) - H_{\mathrm{b}}(x), & \text{if } x \leq \beta, \\ B\varphi(x), & \text{if } x > \beta, \end{cases} \tag{48}$$

for some constants A, B and free-boundary points β, γ such that $0 < \beta < \gamma$. To determine these variables, we conjecture that the functions $w(1,\cdot)$ and $w(0,\cdot)$ should be C^1 at the free-boundary points γ and β, respectively, which yields the system of equations

$$A\psi(\gamma) = B\varphi(\gamma) + H_{\mathrm{s}}(\gamma), \quad B\varphi(\beta) = A\psi(\beta) - H_{\mathrm{b}}(\beta), \tag{49}$$

$$A\psi'(\gamma) = B\varphi'(\gamma) + H'_{\mathrm{s}}(\gamma) \quad \text{and} \quad B\varphi'(\beta) = A\psi'(\beta) - H'_{\mathrm{b}}(\beta). \tag{50}$$

These equations are equivalent to the expressions

$$A = -\int_{\beta}^{\infty} \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds = -\int_{\gamma}^{\infty} \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds, \tag{51}$$

$$B = \int_{0}^{\beta} \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds = \int_{0}^{\gamma} \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds, \tag{52}$$

and the requirement that the free-boundary points $0 < \beta < \gamma$ should satisfy the system of equations

$$q_{\varphi}(\beta, \gamma) = 0 \quad \text{and} \quad q_{\psi}(\beta, \gamma) = 0, \tag{53}$$

where

$$q_{\varphi}(x, z) = \int_{x}^{\infty} \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - \int_{z}^{\infty} \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds \tag{54}$$

and

$$q_{\psi}(x, z) = \int_{0}^{x} \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - \int_{0}^{z} \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds. \tag{55}$$

The system of equations (53) has a unique solution $0 < \beta < \gamma$ if and only if the problem data is such that

$$x_{\mathrm{b}} > 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{H_{\mathrm{b}}(x)}{\psi(x)} \equiv \lim_{x \downarrow 0} \frac{x + c_{\mathrm{b}}}{\psi(x)} < \frac{\alpha - c_{\mathrm{s}}}{\psi(\alpha)} \equiv \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)}, \tag{56}$$

where $\alpha > 0$ is the unique solution of (44).

In this case, the function w defined by (47)–(48), for $A > 0$ and $B > 0$ given by (51) and (52), satisfies the HJB equation (15).