Growth model with externalities for energetic transition via MFG with common external variable

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Abstract

This article introduces a novel mean-field game model for multi-sector economic growth in which a dynamically evolving externality, influenced by the collective actions of agents, plays a central role. Building on classical growth theories and integrating environmental considerations, the framework incorporates "common noise" to capture shared uncertainties among agents about the externality variable. We demonstrate the existence and uniqueness of a strong mean-field game equilibrium by reformulating the equilibrium conditions as a Forward-Backward Stochastic Differential Equation under the stochastic maximum principle and establishing a contraction argument to ensure a unique solution. We provide a numerical resolution for a specified model using a fixed-point approach combined with neural network approximations.

Key words: Mean field games, Common noise, Growth models, Externality.

MSC classification: Primary: 91A16, 91-03; Secondary: 60-08.

1 Introduction

Mean-field game (MFG) theory, first introduced by Lasry and Lions [22] and independently by Huang, Malhamé, and Caines [19], has emerged as a powerful framework for modelling interactions within large populations of agents. In these models, each agent optimises its strategy based on the aggregate behaviour of the population, often leading to Nash equilibria, which are easier to analyse in the infinite-agent limit. MFGs have found applications in economics, finance, and environmental modelling, providing essential tools for studying distributed decision-making in complex systems.

The incorporation of common noise, random external factors that affect all agents simultaneously, into MFGs has been an area of growing interest. This extension introduces additional complexity but also broadens the applicability of MFGs to real-world scenarios where agents are subject to shared uncertainties. Foundational work in this area includes studies by Ahuja [4], Cardaliaguet, Delarue, Lasry, and Lions [8], Carmona and Delarue [11, Vol. II] and Carmona, Delarue, and Lacker [12]. These contributions address the well-posedness of MFGs with common noise and provide insights into the convergence of finite-agent systems to their mean-field counterparts. Recent research by Djete [15] further explores the impact of common noise on interactions through controls.

In this paper, we propose a novel mean-field game model in which agents interact via a dynamically evolving common externality in a multi-sector economic growth framework. This externality, driven by aggregate agent actions, introduces a structure that differs from classical MFG formulations. While individual agent actions do not directly affect the externality, their collective behaviour influences its drift, leading to new challenges in both analysis and computation. Notably, unlike the master equation typically associated with games involving common

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noise, which generally includes a second-order derivative with respect to the measure, the master equation in our framework involves only first-order derivatives with respect to the measure as noted in Section 4.5. This structural distinction aligns with observations in Bertucci's work on monotone solutions for MFG master equations [5]. Our setting also resonates with work on trading by Cardaliaguet and Lehalle [10] and recent developments in mean-field game theory, such as the incorporation of noise through an additional variable in finite state spaces by Bertucci and Meynard [6]. Furthermore, the use of advanced techniques such as Malliavin calculus to handle MFGs with common noise, as explored by Tangpi and Wang [28], underscores the increasing complexity and versatility of these models. Our framework contributes to this growing body of work by offering a novel perspective on multi-sector economic growth under the influence of a common externality.

The proposed model builds upon classical growth theory, such as the two-sector models developed by Uzawa [29], and extends it to incorporate environmental and sustainability considerations. Research on green growth models by Smulders, Toman, and Withagen [26], Tahvonen and Kuuluvainen [27], and the Green Solow Model by Brock and Taylor [7] highlights the importance of integrating environmental externalities into growth dynamics. These approaches motivate our investigation of how agents' investment decisions in different types of capital affect common resources or external factors.

Recent advances in mean-field games have highlighted their potential to address growth and distributional dynamics in more complex settings. For example, Achdou et al. [2] use a mean-field framework to study interactions between firms in input markets, providing insights into competitive equilibria in complex economic networks. Similarly, Achdou et al. [3] study wealth and income distributions in macroeconomic contexts, providing continuous-time approaches to understanding inequality. Gomes, Lafleche, and Nurbekyan [17] extend mean-field game theory to model economic growth, addressing investment dynamics and sectoral interactions. In addition, Zhou and Huang [32] study stochastic growth games with common noise, illustrating the impact of common uncertainties on optimal strategies and equilibrium states. Taken together, these papers highlight the versatility of mean-field game models for capturing the interplay between individual decisions and aggregate economic outcomes.

This article is situated within a broader context of applications, including environmental economics and sustainable development. For instance, MFG models have been used to study the tragedy of the commons by Kobeissi, Mazari-Fouquer, and Ruiz-Balet [21] and to analyse the decarbonisation of financial markets by Lavigne and Tankov [23]. Our work extends these applications by providing a framework for multi-sector investment in a growth context with externalities driven by common noise.

Contributions. First, we contribute to the Integrated Assessment Models (IAMs) literature. IAMs are economic models of development, incorporating couplings with respect to the resources and environmental variables, see [30] for a survey. They aim at better understanding the nature of the coupling between climate and economics to recommand policies preventing and mitigating the effects of climate change on societies. For practical reasons, they often simplifies the overall economical system as one representative country.

Climate change is a perfect illustration of the tragedy of commons [18]. Climate can be seen as a common good shared by a finite number of interacting agents (the countries) deciding on their amount of effort to preserve its quality. Because the common good is shared, incentives to preserve its quality is reduced, and its effective conservation is determined as the output of a Nash equilibrium problem.

On the one hand, taking into account for the interaction among every country is impossible for computational reasons. On the other hand, strategic behaviours is a key aspect of the problematic, which might lead to fundamental misconception and irrelevant solutions if not taken into account. We fill this gap, proposing a limit model where the strategic behaviour of each country is taken into account, expected to approximate the finite player case, while being computationally tractable.

From a modelling perspective, we introduce the first economic growth model of mean field game type with interactions occurring through an external variable. While this modelling feature seems natural, it raises new mathematical challenges, see [5, 6, 24] for recent contributions. This modelling characteristic has also been considered in a mathematical finance context [10], but the associated mathematical difficulties vanishes due to structural assumptions on the cost functional of the agents. See [11, Vol. 1, Remark 1.20] for a discussion.

From a mathematical perspective, we address both the theoretical and computational challenges posed by our model. We demonstrate the existence and uniqueness of a strong mean-field game equilibrium. Using the stochastic maximum principle, we reformulate the equilibrium conditions as a Forward-Backward Stochastic Differential Equation (FBSDE) and establish a contraction argument to ensure the existence of a unique solution. If the contraction condition does not hold, existence and uniqueness can be recovered under regularity and monotonicity assumptions, see [24]. However, the monotonicity assumptions are not satisfied in our study, although we expect weak solutions to exist.

On the computational side, inspired by Carmona and Laurière [13], we develop a fixed-point algorithm combined with neural network approximations to solve the resulting optimisation problems. At each iteration, we fix the common contribution to the externality and approximate the optimal control using a neural network. The algorithm alternates between solving the optimisation problem faced by agents and updating the estimate of the aggregate contribution to externality. The main difference with the recently cited work is that a second neural network, and thus a second optimisation, is used to estimate the common contribution at each iteration. To stabilise convergence, particularly for large time horizons, we incorporate a fictitious time-stepping technique. This approach allows us to efficiently handle high-dimensional problems and capture the dynamic interaction between agents' controls and the evolving externality.

Structure of the paper. The rest of the paper is structured as follows. In Section 2, we describe the technological investment model in detail. Section 3 introduces the notations and mathematical tools used throughout the analysis. In Section 4, we present the theoretical analysis, including key assumptions, the stochastic maximum principle, and the proof of the existence and uniqueness of the equilibrium. Section 5 details our numerical approach and provides simulation results for a specified model.

2 Technological investment model

In this section we present and discuss the model under investigation in this article. The technical assumptions and mathematical proofs are left to the next section. In Section 2.1 we introduce the representative country in the economy. In Section 2.2 we describe the externality (or interaction) process p and the notion of Nash equilibrium.

2.1 Representative country

In this section we present a representative country in the economy. To do this, we fix an external variable p (which will be an externality variable), which we will discuss in the next section. We assume that there is a continuum of identical (which may be heterogeneous in terms of their parameters if they are identically and independently distributed) and atomless countries in the economy. The representative country optimises its utility of consumption through a vector of investments.

Capital dynamics. We consider $n > 0$ different types of capital k_t^i, valued in $\mathbb{R}^n$ for each time $t \in [0, T]$. In each period, the level of each capital $i \in \{1, \dots, n\}$ is increased by a given investment flow a_t^i minus capital depreciation δk_t^i, and randomly disturbed by a Gaussian noise whose variance depends on the level of the capital itself (i.e. the noise scales with the level of the capital). More precisely, the dynamics of the capital vector k is described by the following stochastic differential equation (or SDE)

$$\mathrm{d}k_t = (a_t - \delta k_t)\,\mathrm{d}t + \sigma(k_t)\mathrm{d}W_t, \quad k_0 = \kappa. \tag{1}$$

where δ is a $n \times n$ diagonal matrix of depreciation rates with positive diagonal elements, and $\sigma(k)$ is an $n \times n$ diagonal matrix of volatility rates scaled by the level of capital, i.e.

$$\sigma(k) = \left(\delta_{i,j}\sigma_i k^i\right)_{i,j=1,\dots,n}, \tag{2}$$

where $\delta_{i,j}$ is the Kronecker symbol and κ is a positive vector of the initial capital level of the representative country in the different technologies.

Example 2.1. *In a pollution model, we can consider two types of capital ($n = 2$): the brown capital k^1 which pollutes the environment, and the green capital k^2, which does not.*

Investment. The country can invest in all types of capital. We assume that each country faces a type of entropic penalty cost of investment

$$K(a) := \sum_{i=1}^{n} a^i \ln(a^i).$$

This cost prevents negative investments and bang-bang strategies.

Objective of each country. We assume that the production $F : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is a function of capital k and of the total level of externality p in $\mathbb{R}^d$. Given a stochastic trajectory of the externality p, each country solves the optimisation problem

$$\sup_{a \in L^2(\mathbb{F}, \mathbb{R}^n)} J[p](a) := \mathcal{U}[p](a) + \mathcal{V}[p](a) - \theta \mathcal{K}(a), \tag{P}$$

for some $\theta > 0$. The first term

$$\mathcal{U}[p](a) := \mathbb{E}\left[\int_0^T u(F(k_s^a, p_s) - \mathbb{1} \cdot a_s) e^{-\rho s} \mathrm{d}s \right],$$

denotes the expected and discounted utility derived from consumption (see remark below), where k^a is the controlled process solution of (1). The second term

$$\mathcal{V}[p](a) := \mathbb{E}\left[g(k_T^a, p_T) e^{-\rho T} \right],$$

is the expected terminal reward. The third term

$$\mathcal{K}(a) = \mathbb{E}\left[\int_0^T \sum_{i=1}^{n} a_s^i \ln(a_s^i) e^{-\rho s} \mathrm{d}s \right],$$

is the penalization cost of investment.

Remark 2.1. *Let c_t denote the level of consumption at time $t \in [0, T]$. Using the macroeconomic relation*

$$F(k_t, p_t) = \mathbb{1} \cdot a_t + c_t,$$

and substituted into the utility function, we can see that the objective of each country is simply to maximise its actualised utility of consumption.

2.2 Externality and equilibrium

In this section we describe the externality process and the notion of equilibrium. We start with the externality process, which is random and common to all players. We then introduce the notion of a mean-field game solution, i.e. the Nash equilibrium condition for this mean-field dynamic game.

Externality. The process p represents the externality faced by all the agents. It is valued in $\mathbb{R}^d$, the externality of production can be of different nature (CO_2, CH_4 concentration, mean surface temperature, level of resource stocks, biodiversity loss, etc). We assume that the dynamics of the externality is

$$\mathrm{d}p_t = \Phi(e_t, p_t)\mathrm{d}t + \gamma(p_t)\mathrm{d}W_t^0, \quad p_0 = \eta, \quad e_t = \mathbb{E}[\phi(k_t)|\mathcal{F}_t^0], \tag{3}$$

where $\gamma(p)$ is a $d \times d$ diagonal matrix of volatility rates scaled by the level of pollution, i.e.

$$\gamma(p) = \left(\delta_{i,j} \gamma_i p^i\right)_{i,j=1,\dots,d},$$

The mapping $\phi \colon \mathbb{R}^n \to \mathbb{R}^d$ transforms the capital level into the externality level.

Example 2.2. *In the context of greenhouse gas emissions, ϕ can be a simple linear map*

$$\phi(k) = \Gamma k, \tag{4}$$

where Γ is a matrix with positive entries encoding CO_2 equivalent emissions for each technology, for each pollutant gas.

For each capital, we can associate the variable e which is the total contribution of each country to the total level of the externality. The drift $\Phi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ represents the average dynamics of the externality. The dynamics of the externality is driven by a common noise W^0, creating a dependence of all agents in the game on this source of randomness. This implies that the coupling variable p is a common random environment for all players.

Nash equilibrium. The mean-field game problem is to find a tuple $(\bar{a}, \bar{p})$ such that

$$\bar{a} \in \operatorname*{argmax}_{a \in \mathcal{A}} J[\bar{p}](a), \quad \bar{p} = p^{\bar{a}}. \tag{N}$$

Let us describe this condition. Given an externality process, the objective of each country is to find an optimal investment strategy to maximise its criterion. Once each country has found its optimal strategy, and thus a trajectory of capital, this prescribes a new externality process via the externality dynamics. We say that we have found a mean-field solution (or a Nash equilibrium) if the new externality process is equal to the initial one.

3 Notations and Toolbox

In this section, we provide the main notations, Section 3.1, and a toolbox, Section 3.2, about well-known results on stochastic differential equations.

3.1 Notations

We denote by C^n the set of functions with n continuous derivatives, and by $C^{1,1}$ the set of functions belonging to C^1 with Lipschitz derivative. For a function f, the notation ∇f denotes its gradient. If f is a Lipschitz function of the form $(x, y) \mapsto f(x, y)$ we denote by $C_{f,x}$ its Lipschitz constant with respect to the variable x. Given a vector x in $\mathbb{R}^n$, we denote by $|x|$ the 2-norm of x.

Spaces of random variables and random processes. *$L^2(\mathcal{G}, \mathbb{R}^d)$ spaces.* For a given σ-field $\mathcal{G}$, we denote $L^0(\mathcal{G}, \mathbb{R}^d)$ the space of $\mathbb{R}^d$ valued and $\mathcal{G}$-measurable random variables. We denote by $L^2(\mathcal{G}, \mathbb{R}^d)$ the set of $X \in L^0(\mathcal{G}, \mathbb{R}^d)$ satisfying

$$\|X\|_{L^2(\mathcal{G},\mathbb{R}^d)} := \mathbb{E}\left[|X|^2\right]^{\frac{1}{2}} < +\infty.$$

We denote $L^\infty(\mathcal{G}, \mathbb{R}^d)$ the set of $X \in L^0(\mathcal{G}, \mathbb{R}^d)$ such that

$$\|X\|_{L^\infty(\mathcal{G})} := \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{i \in \{1,\dots,d\}} |X^i(\omega)| < +\infty.$$

$L^2(\mathbb{G}, \mathbb{R}^d)$ spaces. For a filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$ we denote by $L^0(\mathbb{G}, \mathbb{R}^d)$ the space of $\mathbb{R}^d$ valued $\mathbb{G}$-progressively measurable random processes. We denote $L^2(\mathbb{G}, \mathbb{R}^d)$ the set of $X \in L^0(\mathbb{G}, \mathbb{R}^d)$,

$$\|X\|_{L^2(\mathbb{G},\mathbb{R}^d)} := \mathbb{E}\left[\int_0^T |X_s|^2 \mathrm{d}s\right]^{\frac{1}{2}} < +\infty.$$

$S^2(\mathbb{G}, \mathbb{R}^d)$ spaces. We denote $S^2(\mathbb{G}, \mathbb{R}^d)$ the set of $X \in L^0(\mathbb{G}, \mathbb{R}^d)$ satisfying

$$\|X\|_{S^2(\mathbb{G},\mathbb{R}^d)} := \mathbb{E}\left[\sup_{t \in [0,T]} |X_t|^2\right]^{\frac{1}{2}} < +\infty.$$

We define $S^\infty(\mathbb{G},\mathbb{R}^d)$ the set of $X \in L^0(\mathbb{G},\mathbb{R}^d)$ such that

$$\|X\|_{S^\infty(\mathbb{G},\mathbb{R}^d)} \coloneqq \sup_{t\in[0,T]} \|X_t\|_{L^\infty(\mathcal{G}_t,\mathbb{R}^d)} < +\infty.$$

$M^2(\mathbb{G},\mathbb{R}^d)$ *spaces.* We denote $M^2(\mathbb{G},\mathbb{R}^d)$ the set of $X \in L^0(\mathbb{G},\mathbb{R}^d)$ satisfying

$$\|X\|_{M^2(\mathbb{G},\mathbb{R}^d)} \coloneqq \mathbb{E}\left[\int_0^T |X_s|^2 \mathrm{d}s\right]^{\frac{1}{2}} < +\infty.$$

For each space defined above, we omit the notation $\mathbb{R}^d$ when $d = 1$ in the following.

3.2 Toolbox

This section is a compilation of classical results on SDEs and BSDEs used in this article. We present results on the existence and uniqueness of stochastic differential equations borrowed from [31, Chapter 3-4]. This section may be skipped if the reader is reading this article for the first time.

SDE. Consider a time horizon $T > 0$ and a dimension $n > 0$. Consider the SDE

$$\mathrm{d}X_t = b_t(\omega, X_t)\mathrm{d}t + \sigma_t(\omega, X_t)\mathrm{d}W_t, \quad X_0 = \eta, \tag{5}$$

with unknown X, initial condition η, drift mapping for every $t \in \mathbb{R}_+$, $b_t \colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and diffusion mapping $\sigma_t \colon \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$. In the sequel, we drop the ω dependency. Assume that the data η, b and σ satisfy the following:

1. b and σ are $\mathbb{F}$ measurable.
2. Assume that b and σ are Lipschitz almost surely, almost everywhere that is to say there exists $C > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$|b_t(x_1) - b_t(x_2)| + |\sigma_t(x_1) - \sigma_t(x_2)| \leq C|x_1 - x_2|,$$

   for any $x_1, x_2 \in \mathbb{R}^d$.
3. $\eta \in L^2(\mathcal{F}_0,\mathbb{R}^d)$, $b_t(0) \in L^2(\mathbb{F},\mathbb{R}^d)$, $\sigma_t(0) \in L^2(\mathbb{F},\mathbb{R}^{d\times d})$.

Theorem 3.1. *There exists a unique solution $X \in S^2(\mathbb{F},\mathbb{R}^d)$ to equation (5).*

BSDE. This paragraph is dedicated to existence, uniqueness, and a priori estimates results for a general backward stochastic differential equation

$$-\mathrm{d}Y_t = f_s(Y_s, Z_s)\mathrm{d}s - Z_s\mathrm{d}W_s, \quad Y_T = \xi, \tag{6}$$

where the unknown is (Y, Z) and the data are the driver f and the terminal condition ξ. We start with the general n-dimensional case. We assume that

1. The driver $f \colon [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ is $\mathbb{F}$ measurable in all variables.
2. f is uniformly L-Lipschitz with respect to (y, z).
3. $\xi \in L^2(\mathcal{F}_T,\mathbb{R}^d)$ and $f_t(0, 0) \in L^2(\mathbb{F},\mathbb{R}^d)$.

Theorem 3.2. *There exists a unique solution $(Y, Z) \in S^2(\mathbb{F},\mathbb{R}^d) \times M^{2p}(\mathbb{F},\mathbb{R}^{d\times d})$ to equation (6), satisfying*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2 + \left(\int_0^T |Z_s|^2\mathrm{d}s\right)\right] \leq \mathbb{E}\left[|\xi_T|^2 + \left(\int_0^T f_s(0, 0)\mathrm{d}s\right)^2\right].$$

Remark 3.1. *In the essentially bounded case, that is to say when $\xi \in L^\infty(\mathcal{F}_T,\mathbb{R}^d)$ and $f(0, 0) \in L^\infty(\mathbb{F},\mathbb{R}^d)$, there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} |Y_t|^2 + \mathbb{E}\left[\int_0^T |Z_s|^2\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \leq C.$$

4 Mathematical analysis

In this section we present the mathematical analysis of the model. In Section 4.1 we state the main assumptions for our analysis. Then we start the proof of the main result of the article, which is the existence and uniqueness of a Nash equilibrium. This is done in several steps. Freezing the coupling variable, we characterise the optimal control of a representative country using a stochastic maximum principle approach in Section 4.2. Section 4.3 is dedicated to the study of the regularity of the optimal policy. Section 4.4 establishes the the existence and uniqueness of a Nash equilibrium via a contraction argument. Section 4.5 provides a discussion on the master equation associated to the mean field game model we consider in this article.

4.1 Assumptions

In this section we provide the main assumptions of this article.

Stochastic context. We fix a time horizon $T > 0$. We consider two complete filtered probability spaces $(\Omega^0, \mathcal{F}^0, \mathbb{F}^0, \mathbb{P}^0)$ and $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1)$, the first one carrying the Brownian motion W^0 and the second carrying the Brownian motion W. We then equip the product space $\Omega = \Omega^0 \times \Omega^1$ with the completion $\mathcal{F}$ of the product σ-field under the product probability measure $\mathbb{P} = \mathbb{P}^0 \otimes \mathbb{P}^1$, the extension of $\mathbb{P}$ to $\mathcal{F}$ being still denoted by $\mathbb{P}$. The right-continuous and complete augmentation of the product filtration is denoted by $\mathbb{F}$. In our setting, W is an idiosyncratic noise and W^0 is the common noise.

Assumptions on the utility function. The utility function $u : \mathbb{R}_+^* \to \mathbb{R}$ has the following properties:

i) u is increasing and strictly concave.

ii) u is of class C^2.

iii) $u'(c)$ tends to $+\infty$ when c goes to 0^+.

iv) for every $\varepsilon > 0$, $\max_{\varepsilon \leq c} |u''(c)| < +\infty$.

Assumptions on the production function. The production function $F : \mathbb{R}_+^n \times \mathbb{R}_+^d \to \mathbb{R}$ satisfies:

i) F is bounded by below by a positive constant $\underline{F}$.

ii) F is of class $C^{1,1}$.

iii) F is increasing and concave with respect to its first variable.

iv) $\sup_{(k,p)} |\nabla F(k,p)|$ is finite.

Assumptions on the terminal cost. We assume that the terminal cost $g : \mathbb{R}_+^n \times \mathbb{R}_+^d \to \mathbb{R}$ satisfies:

i) g is non negative.

ii) g is increasing and concave with respect to its first variable.

iii) g is of class $C^{1,1}$.

Assumptions on the function transforming the level of capital into externalities. We assume that $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^d$ is Lipschitz, i.e. there exists C_ϕ such that

$$\left|\phi(k^1) - \phi(k^2)\right| \leq C_\phi \left|k^1 - k^2\right|, \quad \forall k^1, k^2 \in \mathbb{R}_+^n.$$

Assumptions on the drift of the externality. We assume that $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^d_+$ is Lipschitz, i.e. there exist $C_{\Phi,e}$ and $C_{\Phi,p}$ such that

$$\left|\Phi(e^1,k^1) - \Phi(e^2,k^2)\right| \leq C_{\Phi,e}\left|e^1 - e^2\right| + C_{\Phi,p}\left|p^1 - p^2\right|, \quad \forall k^1,k^2 \in \mathbb{R}^n_+.$$

Examples of utility and production functions.

1. The classical Constant Relative Risk Aversion (CRRA) utility, which is often used in economic applications, satisfies these assumptions. More precisely, for any $c > 0$ the function

$$u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta}, & \text{if } \eta \in (0,1), \\ \ln(c) & \text{if } \eta = 1, \end{cases}$$

   where the parameter η measures the risk aversion, satisfies the hypothesis.

2. The assumptions on the production function are more restrictive: it follows from the required Lipschitz properties. The following example is a kind of Constant of Elasticity of Substitution (CES) production function where the share parameter is replaced by productivity coefficients depending on the externality $b_i(p)$:

$$F(k,p) = \left(\sum_{i=1}^n b_i(p)\min\left(k^i + \varepsilon, \overline{K}\right)^\gamma\right)^{\frac{\beta}{\gamma}},$$

   with $\varepsilon > 0$, $\gamma \in (0,1]$, and $\beta \in (0,1)$. The constant $\overline{K}$ corresponds to a space constraint: it ensures that if the functions b_i are bounded and Lipschitz, then $(k,p) \mapsto b_i(p)\min\left(k^i + \varepsilon, \overline{K}\right)^\gamma$ is Lipschitz.

   In others words, for F to satisfy the assumptions, the functions b_i must be $C^{1,1}$, satisfy

$$\frac{1}{C} \leq b_i(p) \leq C, \quad \frac{1}{C} \leq b_i'(p) \leq C,$$

   for a constant $C > 0$, and the min function needs to be replaced by a smoothen version.

   To see that the functions b_i affect the productivity of each sector, let us consider the simplest case: $\gamma = 1$, $n = 2$ and k^1, k^2 are less than $\overline{K} - \varepsilon$. Then the marginal productivity is given by

$$\begin{aligned} \frac{\partial F}{\partial k^1}(k,p) &= \beta\left(b_1(p)k^1 + b_2(p)k^2\right)^{\beta-1} b_1(p), \\ \frac{\partial F}{\partial k^2}(k,p) &= \beta\left(b_1(p)k^1 + b_2(p)k^2\right)^{\beta-1} b_2(p). \end{aligned}$$

   Therefore, if $b_1(p) < b_2(p)$, then increasing by 1 unit of k^2 is more profitable than increasing by 1 unit of k^1.

4.2 Stochastic maximum principle

Given a trajectory of externality p, we will use the stochastic maximum principle (see [25, Section 6.4.2]) to characterise the optimal control. For this introduce the generalised Hamiltonian of the optimal control problem:

$$H(a,k,p,y,z) = [a - (\delta+\rho)k]y + u(F(k,p) - a \cdot \mathbb{1}) - \theta\sum_{i=1}^n a^i \ln(a^i) + \operatorname{Tr}(\sigma(k)^T z).$$

Theorem 4.1. *Let a^* be an admissible control and $k^* = k^{a^*}$ be the associated controlled diffusion. Suppose that there exists a solution $(y^*, z^*, z^{0,*})$ to*

$$\begin{cases} -dy_t = \nabla_k H(a_t^*, k_t^*, y_t, z_t)\mathrm{d}t - z_t \mathrm{d}W_t - z_t^0 \mathrm{d}W_t^0, \\ \quad y_T = \nabla_k g(k_T, p_T), \end{cases}$$

such that

$$H(a_t^*, k_t^*, p_t, y_t^*, z_t^*) = \max_a H(a, k_t^*, p_t, y_t^*, z_t^*), \quad 0 \le t \le T, \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P} - a.s. \tag{7}$$

and

$$(k, a) \mapsto H(a, k, p, y_t^*, z_t^*) \quad \textit{is a concave function for almost all } (t, \omega). \tag{8}$$

Then a^* is an optimal control for (P). If moreover the map defined in (8) is strictly concave for almost all (t, ω) then the maximum is unique.

The scheme of the proof is standard and can be found in [25, Section 6.4.2]. The key of the proof relies on the concavity of the map $(a, k) \mapsto H(a, k, p, y, z)$.

Proof. Let us consider another admissible control a and its controlled process k. By definition of the criteria we have

$$J[p](a) - J[p](a^*) = \mathcal{U}[p](a) - \mathcal{U}[p](a^*) + \mathcal{V}[p](a) - \mathcal{V}[p](a^*) - \theta(\mathcal{K}(a) - \mathcal{K}(a^*)).$$

On the one hand, let us compute

$$\begin{aligned}
I_1 &= \mathcal{U}[p](a) - \mathcal{U}[p](a^*) - \theta(\mathcal{K}(a) - \mathcal{K}(a^*)) \\
&= \mathbb{E}\left[\int_0^T (u(F(k_t, p_t) - a_t \cdot \mathbb{1}) - u(F(k_t^*, p_t) - a_t^* \cdot \mathbb{1}))e^{-\rho t}\mathrm{d}t\right] \\
&\quad + \theta\mathbb{E}\left[\int_0^T (K(a_t) - K(a_t^*))e^{-\rho t}\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T (H(a_t, k_t, p_t, y_t^*, z_t^*) - H(a_t^*, k_t^*, p_t, y_t^*, z_t^*)e^{-\rho t}\mathrm{d}t\right] \\
&\quad - \mathbb{E}\left[\int_0^T [(a_t - a_t^* - (\delta + \rho)(k_t - k_t^*))y_t^* + \mathrm{Tr}((\sigma(k_t) - \sigma(k_t^*))^T z_t^*)]e^{-\rho t}\mathrm{d}t\right].
\end{aligned}$$

On the other hand, by concavity of the terminal condition with respect to its first variable, by the Itô formula we have

$$\begin{aligned}
I_2 &= \mathcal{V}[p](a) - \mathcal{V}[p](a^*) \le e^{-\rho T}\mathbb{E}\,[\,\nabla_k g(k_T^*, p_T) \cdot (k_T - k_T^*)] = \mathbb{E}\left[e^{-\rho T} y_T^* \cdot (k_T - k_T^*)\right] \\
&= \mathbb{E}\left[\int_0^T e^{-\rho t}[-\nabla_k H(a_t^*, k_t^*, p_t, y_t^*, z_t^*) - \rho y_t^*] \cdot (k_t - k_t^*)\mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\int_0^T e^{-\rho t}[y_t^*(a_t - a_t^* - \delta(k_t - k_t^*)) + \mathrm{Tr}((\sigma(k_t) - \sigma(k_t^*))^T z_t^*)]\mathrm{d}t\right].
\end{aligned}$$

Finally, we have

$$\begin{aligned}
J[p](a) - J[p](a^*) &= I_1 + I_2 \\
&\le \mathbb{E}\left[\int_0^T e^{-\rho t}[H(a_t, k_t, p_t, y_t^*, z_t^*) - H(a_t^*, k_t^*, p_t, y_t^*, z_t^*)]\mathrm{d}t\right] \\
&\quad - \mathbb{E}\left[\int_0^T e^{-\rho t}\nabla_k H(a_t^*, k_t^*, p_t, y_t^*, z_t^*) \cdot (k_t - k_t^*)\mathrm{d}s\right] \\
&\le 0,
\end{aligned}$$

using (7) and the concavity of the Hamiltonian with respect to its two first variables.

Moreover, it is easy to see that if a is different from a^* on a subset of $[0, T] \times \Omega$ with positive measure, and the Hamiltonian is strictly concave with respect to its first two variables, then the last inequality is strict, which provides the uniqueness of the optimal control. □

4.3 Regularity of the optimal control variable

By the stochastic maximum principle, the optimal control is a mapping $a\colon \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$,

$$a(k,p,y) = \operatorname*{argmax}_{\alpha \in \mathbb{R}^n} H(\alpha, k, p, y, z). \tag{9}$$

As we shall see, the right-hand side is independent of z, so we omit the dependence on this variable in the left-hand side. This section is devoted to the proof of the following proposition.

Proposition 4.1. *The mappings*

$$\mathbb{R}^n \times \mathbb{R}^d \times [0,\overline{y}]^n \ni (k,p,y) \mapsto a(k,p,y), \tag{10}$$

$$\mathbb{R}^n \times \mathbb{R}^d \times [0,\overline{y}]^n \times \mathbb{R}^{n\times n} \ni (k,p,y,z) \mapsto \nabla_k H(a(k,p,y),k,p,y,z), \tag{11}$$

are Lipschitz continuous.

The proof can be found at the end of this section. Let us first justify that the policy (9) is independent on z. By direct computation it can be seen that every $\bar{\alpha} \in \mathbb{R}^n$ which satisfies the first order condition $\nabla_a H(\bar{\alpha}, k, p, y, z) = 0$ is such that

$$\bar{\alpha}_i = \exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \mathbb{1}\cdot\bar{\alpha}) - 1\right)\right),$$

for all $i \in \{1,\dots,n\}$ which is independent of z. For all $(k,p,y) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$, we define the mapping $f[k,p,y]\colon \mathbb{R}^n \to \mathbb{R}$,

$$f[k,p,y](\xi) = \xi - \sum_{i=1}^n \exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi)\right) - 1\right),$$

and the equation

$$f[k,p,y](\xi) = 0. \tag{12}$$

Finally, provided that there exists a unique $\xi(k,p,y)$ solution to the last equation (which is the purpose of the next lemma), the optimal policy defined by (9) is given by

$$a_i(k,p,y) = \exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi(k,p,y))\right) - 1\right), \tag{13}$$

for each $i \in \{1,\dots,n\}$.

Lemma 4.1. *There exists a unique solution $\xi(k,p,y)$ to equation (12), satisfying*

$$0 < \xi(k,p,y) < F(k,p)$$

for each $(k,p,y) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$.

Proof. We can verify that for any $(k,p,y) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n$,

$$f[k,p,y](0) < 0, \qquad \lim_{\xi \to F(k,p)} f[k,p,y](\xi) = F(k,p) \geq \underline{F} > 0, \tag{14}$$

since we assume that F is lower bounded. The intermediate value theorem gives the existence of a solution to the equation (12) valued in $(0, F(k,p))$. The uniqueness comes from the strict monotonicity of $f[k,p,y]$.

□

From now on we define ξ as the solution of the equation (12) and the consumption strategy is given by $c(k,p,y) = F(k,p) - \xi(k,p,y)$ for all $(k,p,y) \in \mathbb{R}^n_+ \times \mathbb{R}^d_+ \times \mathbb{R}^n_+$.

Lemma 4.2. *For any positive real number $\overline{y}$. The mapping*

$$\mathbb{R}^n_+ \times \mathbb{R}^d_+ \times [0,\overline{y}]^n \ni (k,p,y) \mapsto c(k,p,y), \tag{15}$$

is bounded below by a positive constant $\eta_{\overline{y}}$.

Proof. Fix $(k,p,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^d \times [0,\overline{y}]^n$, by definition of ξ we have

$$\xi(k,p,y) \leq \exp\left(\frac{1}{\theta}\left(\overline{y} - u'(F(k,p) - \xi(k,p,y))\right) - 1\right).$$

Then using $c = F - \xi$, we have

$$c(k,p,y) \geq F(k,p) - \exp\left(\frac{1}{\theta}\left(\overline{y} - u'(c(k,p,y))\right) - 1\right).$$

We then obtain by assumption on the lower bound of F that

$$c(k,p,y) + \exp\left(\frac{1}{\theta}\left(\overline{y} - u'(c(k,p,y))\right) - 1\right) \geq \underline{F}.$$

Since $c + \exp\left(\frac{1}{\theta}\left(\overline{y} - u'(c)\right) - 1\right)$ goes to 0 when c tends to 0 and $\underline{F} > 0$, then c admits a positive bound from below, which concludes the proof. □

Lemma 4.3. *The function ξ is of class C^1.*

Proof. First observe that the mapping $(\xi,k,p,y) \mapsto f[k,p,y](\xi)$ is of class C^1. Then

$$\partial_\xi f[k,p,y](\xi) = 1 - \sum_{i=1}^n \frac{1}{\theta}\exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi)\right) - 1\right) u''(F(k,p) - \xi) \geq 1.$$

Therefore, we can apply the implicit mapping theorem to deduce that ξ is of class C^1. Moreover,

$$\nabla\xi(k,p,y) = -\partial_\xi f[k,p,y](\xi(k,p,y))^{-1}\nabla_{(k,p,y)} f[k,p,y](\xi(k,p,y)),$$

where

$$\nabla_{(k,p)} f[k,p,y](\xi) = \sum_{i=1}^n \frac{1}{\theta}\exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi)\right) - 1\right) u''(F(k,p) - \xi)\nabla_{(k,p)} F(k,p),$$

and

$$\nabla_y f[k,p,y](\xi) = -\left(\frac{1}{\theta}\exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi)\right) - 1\right)\right)_{i=1,\dots,n}.$$

□

Corollary 4.1. *For any $(k,p,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^d \times [0,\overline{y}]$,*

$$|\nabla_k \xi(k,p,y)| \leq |\nabla_k F(k,p)|, \quad |\nabla_p \xi(k,p,y)| \leq |\nabla_p F(k,p)|$$

and

$$|\nabla_y \xi(k,p,y) u''(F(k,p) - \xi(k,p,y))| \leq 1.$$

Finally, we prove the Proposition 4.1. We start with the mapping (10). We note that $c = F - \xi$. We check that $u'(c)$ is Lipschitz. To do this, we show that its gradient

$$\nabla_{(k,p,y)} u'(c) = u''(c)\nabla_{(k,p,y)} c,$$

is bounded. Indeed, Lemma 4.2 gives a bound by below of c. Then, we deduce that $u''(c)$ is bounded. This bound and Corollary 4.1 imply that $u''(c)\nabla_{(k,p,y)} c$ is bounded, therefore $u'(c)$ is Lipschitz.
By equation (13), the optimal control writes $a_i(k,p,y) = \exp\left(\frac{1}{\theta}(y_i - u'(c(k,p,y))) - 1\right)$, for each $i \in \{1,\dots,n\}$, so the mapping (10) is Lipschitz.

We now turn to (11). Using equation (2), defining $\hat{\sigma}$ such that $\hat{\sigma}_{i,j} = \sigma_i \delta_{i,j}$, $i = 1,\dots n$, $j = 1,\dots n$,

$$\nabla_k H(a(k,p,y),k,p,y,z) = -(\delta + \rho)y + \mathrm{diag}(\hat{\sigma}z) + u'(c(k,p,y))\nabla_k F(k,p).$$

The first two terms are linear, so we only need to show that the last term is Lipschitz. We have already established that $u'(c)$ is Lipschitz. Let us check that it is bounded. From Lemma 4.2 and the fact that u' is decreasing and positive, we get

$$0 < u'(c) \leq u'(\eta_{\overline{y}}) < +\infty.$$

On the other hand, by assumptions $\nabla_k F(k,p)$ is Lipschitz and bounded. Therefore, the product function

$$(k,p,y) \mapsto u'(c(k,p,y))\nabla_k F(k,p),$$

is Lipschitz, which ends the proof.

4.4 Existence and uniqueness of a mean-field game equilibrium

We are now in a position to tackle the mean field problem (N). By Theorem 4.1, a tuple (a,p) is solution to (N), if and only if there exists a solution to the following system

$$\begin{cases} \mathrm{d}p_t &= \Phi(\mathbb{E}[\phi(k_t)|\mathcal{F}^0_t], p_t)\mathrm{d}t + \gamma(p_t)\mathrm{d}W^0_t, & p_0 = \eta, \\ -\mathrm{d}y_t &= \nabla_k H(a(k_t,p_t,y_t),k_t,p_t,y_t,z_t)\mathrm{d}t - z_t\mathrm{d}W_t - z^0_t\mathrm{d}W^0_t, & y_T = \nabla_k g(k_T,p_T), \\ \mathrm{d}k_t &= (a(k_t,p_t,y_t) - \delta k_t)\mathrm{d}t + \sigma(k_t)\mathrm{d}W_t, & k_0 = \kappa, \end{cases} \tag{16}$$

where a is the mapping defined in (9) and examined in the previous section (which we recall here $a(k,p,y) = \operatorname{argmax}_{\alpha\in\mathbb{R}^n} H(\alpha,k,p,y,z)$). Let us rewrite the above system as a fixed-point problem:

- Given $k \in L^2(\mathbb{F},\mathbb{R}^n)$, we denote $\Theta_1(k)$ the solution to the first equation,
- Given a pair $(k,p) \in L^2(\mathbb{F},\mathbb{R}^n) \times L^2(\mathbb{F},\mathbb{R}^d)$, we denote $\Theta_2(k,p)$ the solution to the second equation,
- Given a couple $(y,p) \in L^2(\mathbb{F};\mathbb{R}^n) \times L^2(\mathbb{F},\mathbb{R}^d)$, we denote $\Theta_3(p,y)$ the solution to the third equation.

Find a solution (y,z,k,p) to the system of equation (16) is then equivalent to find a fixed-point to the mapping

$$\Theta\colon L^2(\mathbb{F},\mathbb{R}^n) \to L^2(\mathbb{F},\mathbb{R}^n)$$

defined as follow: for any $k \in L^2(\mathbb{F},\mathbb{R}^n)$

1. Associate $p = \Theta_1(k)$,
2. Then associate $(y,z,z^0) = \Theta_2(k,p)$,
3. Finally set $\Theta(k) = \Theta_3(p,y)$.

The following propositions establish the well-posedness and the Lipschitz properties of the maps Θ_i $(i=1,2,3)$. Under some appropriate assumptions the map Θ is a contraction. Therefore, existence and uniqueness follow from Picard's fixed point theorem.

Proposition 4.2. *The mapping $\Theta_1 : L^2(\mathbb{F},\mathbb{R}^n) \to L^2(\mathbb{F},\mathbb{R}^d)$ is well-defined and*

$$\|\Theta_1(k) - \Theta_1(k')\|_{L^2(\mathbb{F},\mathbb{R}^d)} \leq \sqrt{C_1}\,\|k-k'\|_{L^2(\mathbb{F},\mathbb{R}^n)}\,,$$

where $C_1 \coloneqq C_{\Phi,e}C_\phi T e^{(C_{\Phi,e}+2C_{\Phi,p}+C_\gamma^2)T}$, with $C_\gamma = \max_i \gamma_i$.

Proof. Step 1: Well-posedness of Θ_1. For any $k \in L^2(\mathbb{F},\mathbb{R}^n)$, the existence and uniqueness of the solution of the first equation of (16) follows from the Lipschitz properties of the coefficients. Since $\|\eta\|_{L^2(\Omega)} < +\infty$, the solution belongs to $L^2(\mathbb{F},\mathbb{R}^d)$. Therefore Θ_1 is well defined and completes the step.

Step 2: Lipschitz continuity. Let us check that the map Θ_1 is Lipschitz. It follows from the Lipschitz properties of Φ and Γ. Consider k^1 and k^2, two elements of $L^2(\mathbb{F},\mathbb{R}^n)$. For convenience we denote $p^i = \Theta_1(k^i)$, $\Delta p = p^1 - p^2$ and $\Delta k = k^1 - k^2$. We observe that $\Delta p_0 = 0$ and

$$\mathrm{d}\Delta p_t = \left(\Phi(\mathbb{E}[\phi(k_t^1)|\mathcal{F}_t^0], p_t^1) - \Phi(\mathbb{E}[\phi(k_t^2)|\mathcal{F}_t^0], p_t^2)\right)\mathrm{d}t + \gamma(\Delta p_t)\mathrm{d}W_t^0.$$

Therefore, for any $t \in [0,T]$, Ito's formula yields that

$$\begin{aligned}\|\Delta p_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^d)} =& \mathbb{E}\left[\int_0^t 2\Delta p_s \left(\Phi(\mathbb{E}[\phi(k_s^1)|\mathcal{F}_s^0], p_s^1) - \Phi(\mathbb{E}[\phi(k_s^2)|\mathcal{F}_s^0], p_s^2)\right)\mathrm{d}s\right] \\ &+ \mathbb{E}\left[\int_0^t \mathrm{Tr}(\gamma(\Delta p_s)^T\gamma(\Delta p_s))\mathrm{d}s\right].\end{aligned}$$

Using Fubini's theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\Delta p_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^d)} \le & 2\int_0^t \|\Delta p_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)} \left\|\Phi(\mathbb{E}[\phi(k_s^1)|\mathcal{F}_s^0], p_s^1) - \Phi(\mathbb{E}[\phi(k_s^2)|\mathcal{F}_s^0], p_s^2)\right\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s \\ &+ C_\gamma^2\int_0^t \|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s.\end{aligned}$$

Using the Lipschitz continuity of Φ and the fact that

$$\left\|\mathbb{E}[\phi(k_s^1)|\mathcal{F}_s^0] - \mathbb{E}[\phi(k_s^2)|\mathcal{F}_s^0]\right\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)} \le \left\|\phi(k_s^1) - \phi(k_s^2)\right\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)}$$

we have

$$\begin{aligned}\|\Delta p_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^d)} \le & 2C_{\Phi,e}\int_0^t \|\Delta p_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\left\|\phi(k_s^1) - \phi(k_s^2)\right\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s \\ &+ 2C_{\Phi,p}\int_0^t \|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s + C_\gamma^2\int_0^t \|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s.\end{aligned}$$

Using the Lipschitz continuity of ϕ, we finally end up with

$$\|\Delta p_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^d)} \le \left(C_{\Phi,e} + 2C_{\Phi,p} + C_\gamma^2\right)\int_0^t \|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s + C_{\Phi,e}C_\phi\int_0^t \|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s.$$

Thus, Grönwall's Lemma leads to

$$\|\Delta p_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^d)} \le C_{\Phi,e}C_\phi e^{\left(C_{\Phi,e}+2C_{\Phi,p}+C_\gamma^2\right)t}\int_0^t \|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)}\mathrm{d}s,$$

for any $t \in [0,T]$. Taking the integral over time, Fubini's theorem finally gives us

$$\|\Delta p\|^2_{L^2(\mathbb{F},\mathbb{R}^d)} \le C_1\|\Delta k\|^2_{L^2(\mathbb{F},\mathbb{R}^n)},$$

concluding the step and the proof. □

We now turn to Θ_2. For ease of notation, we introduce the function $\upsilon\colon \mathbb{R}_+^n \times \mathbb{R}_+^d \times \mathbb{R}^n \to \mathbb{R}$,

$$\upsilon(k,p,y) = u'(c(k,p,y))\nabla_k F(k,p).$$

We recall that when y is bounded, υ is Lipschitz (see the end of the proof of Proposition 4.1) and we denote $C_{\upsilon,k}$, $C_{\upsilon,p}$ and $C_{\upsilon,y}$ its Lipschitz constants with respect to k, p and y.

Proposition 4.3. *The mapping $\Theta_2 : L^2(\mathbb{F},\mathbb{R}^n) \times L^2(\mathbb{F},\mathbb{R}^d) \to S^2(\mathbb{F},\mathbb{R}^n) \times M^2(\mathbb{F},\mathbb{R}^{n\times n}) \times M^2(\mathbb{F},\mathbb{R}^{n\times d})$ is well-defined. There exists $C_y > 0$ such that for any (k,p)*

$$\left\|\Theta_2^1(k,p)\right\|_{S^\infty(\mathbb{F},\mathbb{R}^n)} \le C_y,$$

where $\Theta_2^1(k,p)$ *denotes the first component of* $\Theta_2(k,p)$. *In addition,*

$$\left\|\Theta_2^1(k^1,p^1)-\Theta_2^1(k^2,p^2)\right\|^2_{L^2(\mathbb{F},\mathbb{R}^n)} \leq C_2\left\|k^1-k^2\right\|^2_{L^2(\mathbb{F},\mathbb{R}^n)} + C_3\left\|p^1-p^2\right\|^2_{L^2(\mathbb{F},\mathbb{R}^d)},$$

where

$$C_2 = (C^2_{\nabla_k g,k} + TC^2_{\upsilon,k})\frac{1}{\nu}\left(e^{\nu T}-1\right),$$
$$C_3 = (C^2_{\nabla_k g,p} + TC^2_{\upsilon,p})\frac{1}{\nu}\left(e^{\nu T}-1\right),$$

with $\nu = -2(\delta+\rho) + C^2_{\upsilon,y} + C^2_\sigma + C_{\upsilon,k} + C_{\upsilon,p}$, *with* $C_\sigma = \max_i \sigma_i$.

Proof. Step 1: Well-posedness of Θ_2. The Lipschitz properties of the coefficients ensure well-posedness as stated in Theorem 3.2.

Step 2: S^∞ *estimate.* Using standard arguments, we start by linearising the BSDE. The fundamental theorem of calculus gives us

$$\begin{aligned} u'(F(k_t,p_t)-\xi^*(k_t,p_t,y_t)) &= u'(F(k_t,p_t)-\xi^*(k_t,p_t,0)) \\ &\quad - \int_0^1 u''(F(k_t,p_t)-\xi^*(k_t,p_t,sy_t))\nabla_y\xi^*(k_t,p_t,sy_t)dsy_t. \end{aligned}$$

Therefore, by setting

$$\alpha_t = -(\delta+\rho) - \int_0^1 u''(F(k_t,p_t)-\xi^*(k_t,p_t,sy_t))\nabla_y\xi^*(k_t,p_t,sy_t)ds\nabla_k F(k_t,p_t),$$

we observe that y satisfies

$$-dy_t = (u'(F(k_t,p_t)-\xi^*(k_t,p_t,0))\nabla_k F(k_t,p_t) + \alpha_t y_t + \mathrm{diag}(\hat\sigma z_t))\mathrm{d}t - z_t\mathrm{d}W_t - z^0_t\mathrm{d}W^0_t.$$

Note that using Corollary 4.1, $\|\alpha_t\|_{L^\infty} \leq \delta+\rho+\|\nabla_k F\|_\infty$.
Let $\mathbb{Q}$ be the equivalent probability measure to $\mathbb{P}$ given by $\mathrm{d}\mathbb{Q} = \mathcal{E}(\int \mathrm{diag}(\hat\sigma)\mathrm{d}W)_T\mathrm{d}\mathbb{P}$ where $\mathcal{E}(\int \mathrm{diag}(\hat\sigma)\mathrm{d}W)_T$ denotes the stochastic exponential associated with $\mathrm{diag}(\hat\sigma)$. Note that the Novikov's condition is trivially satisfied since σ is assumed to be constant.

We deduce from Ito's formula and by taking the conditional expectation under $\mathbb{Q}$, that for any $t\in[0,T]$ we have

$$y_t e^{-\int_t^T \alpha_u du} = \mathbb{E}_{\mathbb{Q}}\left[\nabla_k g(k_T,p_T) + \int_t^T u'(F(k_\tau,p_\tau)-\xi^*(k_\tau,p_\tau,0))\nabla_k F(k_t,p_t)e^{-\int_\tau^T \alpha_u du}d\tau\middle|\mathcal{F}_t\right].$$

We observe that all components of y_t are non-negative and bounded. Indeed, α is bounded in $S^\infty(\mathbb{F})$, $\nabla_k g$ and $\nabla_k F$ are uniformly bounded and non-negative, so that

$$(k,p)\mapsto u'(F(k,p)-\xi^*(k,p,0)),$$

from the bound given in lemma 4.2 and the concavity of u. Finally, for any $t\in[0,T]$

$$\operatorname{ess\,sup}|y_t| \leq (\|\nabla_k g\|_\infty + Tu'(\eta_0)\|\nabla_k F\|_\infty)\,e^{\left(\delta+\rho+\|\nabla_k F\|_\infty\right)T}$$

where η_0 is defined in Lemma 4.2.

Step 3: Lispchitz continuity of Θ_2. Let (k^1,p^1) and (k^2,p^2) be in $L^2(\mathbb{F},\mathbb{R}^n)\times L^2(\mathbb{F},\mathbb{R}^d)$. Let $(y^i,z^i,z^{0,i}) = \Theta_2(k^i,p^i)$ for $i\in\{1,2\}$ and

$$\Delta y = y^1-y^2,\quad \Delta z = z^1-z^2,\quad \Delta z^0 = z^{0,1}-z^{0,2},\quad \Delta p = p^1-p^2,\quad \Delta k = k^1-k^2.$$

We have

$$-\mathrm{d}\Delta y_t = (-(\delta+\rho)\Delta y_t + \mathrm{diag}(\hat\sigma\Delta z_t) + \Delta\upsilon_t)\,\mathrm{d}t - \Delta z_t\mathrm{d}W_t - \Delta z^0_t\mathrm{d}W^0_t,$$

with terminal condition $\Delta y_T = \nabla_k g(k_T^1, p_T^1) - \nabla_k g(k_T^2, p_T^2)$ and where $\Delta \upsilon_t = \upsilon(k_t^1, p_t^1, y_t^1) - \upsilon(k_t^2, p_t^2, y_t^2)$. Let $\Lambda_t = e^{t\nu}$ for some real parameter ν (which might be negative) to be determined. Now by the Itô formula we have that

$$\Lambda_t|\Delta y_t|^2 + \int_t^T \Lambda_s \left(|\Delta z_s|^2 + |\Delta z_s^0|^2\right) \mathrm{d}s = \Lambda_T|\Delta y_T|^2 - \int_t^T \Lambda_s \nu |\Delta y_s|^2 \mathrm{d}s - 2\int_t^T \Lambda_s \Delta y_s^T \mathrm{d}\Delta y_s$$

$$= |\Delta y_T|^2 - \int_t^T \Lambda_s \nu |\Delta y_s|^2 \mathrm{d}s + 2\int_t^T \Lambda_s \left(-(\delta+\rho)|\Delta y_s|^2 + \Delta y_s^T \operatorname{diag}(\hat\sigma \Delta z_s) + \Delta y_s^T \Delta \upsilon_s\right) \mathrm{d}s - M_t,$$

where $M_t = -\int_t^T \Lambda_s \Delta y_s^T \Delta z_s \mathrm{d}W_s - \int_t^T \Lambda_s \Delta y_s^T \Delta z_s^0 \mathrm{d}W_s^0$ is a martingale term and we recall that $|z|^2 = \mathrm{Tr}(z^T z)$ for any $z \in \mathbb{R}^{n\times d}$. By Fenchel's and Cauchy-Schwarz inequalities we have that

$$\begin{aligned}\Delta y_s^T \operatorname{diag}(\hat\sigma \Delta z_s) + \Delta y_s^T \Delta \upsilon_s \leq& \frac{1}{2} C_\sigma^2 |\Delta y_s|^2 + \frac{1}{2}|\Delta z_s|^2 + |\Delta y_s||\Delta \upsilon_s| \\ \leq& \frac{1}{2}(2C_{\upsilon,y} + C_\sigma^2 + C_{\upsilon,k} + C_{\upsilon,p})|\Delta y_s|^2 \\ &+ \frac{1}{2}|\Delta z_s|^2 + \frac{1}{2}(C_{\upsilon,k}|\Delta k_s|^2 + C_{\upsilon,p}|\Delta p_s|^2).\end{aligned}$$

Combining with the previous equality yields,

$$\begin{aligned}\Lambda_t|\Delta y_t|^2 \leq& \Lambda_T|\Delta y_T|^2 + (-\nu - 2(\delta+\rho) + 2C_{\upsilon,y} + C_\sigma^2 + C_{\upsilon,k} + C_{\upsilon,p}) \int_t^T \Lambda_s |\Delta y_s|^2 \mathrm{d}s \\ &+ \int_t^T \Lambda_s (C_{\upsilon,k}|\Delta k_s|^2 + C_{\upsilon,p}|\Delta p_s|^2) \mathrm{d}s - M_t.\end{aligned}$$

Choosing $\nu = -2(\delta+\rho) + 2C_{\upsilon,y} + C_\sigma^2 + C_{\upsilon,k} + C_{\upsilon,p}$ the expression simplifies,

$$\Lambda_t|\Delta y_t|^2 \leq \Lambda_T|\Delta y_T|^2 + \int_t^T \Lambda_s (C_{\upsilon,k}|\Delta k_s|^2 + C_{\upsilon,p}|\Delta p_s|^2) \mathrm{d}s - M_t.$$

Recalling that the terminal condition is Lipschitz

$$|\Delta y_T|^2 \leq C_{\nabla_k g,k}^2 |\Delta k_T|^2 + C_{\nabla_k g,p}^2 |\Delta p_T|^2,$$

we further estimate

$$\Lambda_t|\Delta y_t|^2 \leq \Lambda_T \left(C_{\nabla_k g,k}^2 |\Delta k_T|^2 + C_{\nabla_k g,p}^2 |\Delta p_T|^2\right) + \int_t^T \Lambda_s (C_{\upsilon,k}|\Delta k_s|^2 + C_{\upsilon,p}|\Delta p_s|^2) \mathrm{d}s - M_t.$$

Dividing by Λ_t both sides and taking the expectation yields

$$\begin{aligned}\|\Delta y_t\|_{L^2(\mathcal{F}_t,\mathbb{R}^n)}^2 \leq& \Lambda_{t,T} \left(C_{\nabla_k g,k}^2 \|\Delta k_T\|_{L^2(\mathcal{F}_T,\mathbb{R}^n)}^2 + C_{\nabla_k g,p}^2 \|\Delta p_T\|_{L^2(\mathcal{F}_T,\mathbb{R}^d)}^2\right) \\ &+ \int_t^T \Lambda_{t,s} (C_{\upsilon,k}^2 \|\Delta k_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^n)}^2 + C_{\upsilon,p}^2 \|\Delta p_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)}^2) \mathrm{d}s,\end{aligned}$$

where $\Lambda_{t,s} = \Lambda_s \Lambda_t^{-1}$. Taking the integral with respect to time both sides,

$$\|\Delta y_t\|_{L^2(\mathbb{F},\mathbb{R}^n)}^2 \leq \left((C_{\nabla_k g,k}^2 + TC_{\upsilon,k})\|\Delta k\|_{L^2(\mathbb{F},\mathbb{R}^n)}^2 + (C_{\nabla_k g,p}^2 + TC_{\upsilon,p})\|\Delta p\|_{L^2(\mathbb{F},\mathbb{R}^n)}^2\right) \int_0^T \Lambda_{t,T} \mathrm{d}t.$$

Using that $\int_0^T \Lambda_{t,T} \mathrm{d}t = \Lambda_T \int_0^T e^{\nu(T-t)} \mathrm{d}t = \frac{1}{\nu}\left(e^{\nu T} - 1\right)$, concludes the step and the proof. □

Proposition 4.4. *The mapping* $\Theta_3 : L^2(\mathbb{F},\mathbb{R}^d) \times L^2(\mathbb{F},\mathbb{R}^n) \to L^2(\mathbb{F},\mathbb{R}^n)$, *is well-defined and*

$$\left\|\Theta_3(p^1,y^1) - \Theta_3(p^2,y^2)\right\|_{L^2(\mathbb{F})}^2 \leq C_4 \left\|p^1 - p^2\right\|_{L^2(\mathbb{F})}^2 + C_5 \left\|y^1 - y^2\right\|_{L^2(\mathbb{F})}^2,$$

where

$$\begin{aligned}C_4 &= C_{a,p} T e^{\left(C_{a,p} + C_{a,y} + 2C_{a,k} + C_\sigma^2 - 2\delta\right)T}, \\ C_5 &= C_{a,y} T e^{\left(C_{a,p} + C_{a,y} + 2C_{a,k} + C_\sigma^2 - 2\delta\right)T}.\end{aligned}$$

Proof. Step 1: Well-posedness of Θ_3. For any $(p,y) \in L^2(\mathbb{F},\mathbb{R}^d) \times L^2(\mathbb{F},\mathbb{R}^n)$, the existence and uniqueness of the solution of the third equation of (16) comes from the Lipschitz properties of the coefficients. In addition, since $\|\kappa\|_{L^2(\Omega)} < +\infty$, the solution belongs in $L^2(\mathbb{F},\mathbb{R}^n)$. Therefore, Θ_3 is well-defined.
Step 2: Lipschitz continuity. Let us verify that the map Θ_3 is Lipschitz. It is a consequence of the Lipschitz property of a. We denote $k^i = \Theta_3(p^i, y^i)$ for $i \in \{1,2\}$ and

$$\Delta a_t = a(k_t^1, p_t^1, y_t^1) - a(k_t^2, p_t^2, y_t^2).$$

Observe that $\Delta k_0 = 0$ and

$$d\Delta k_t = (\Delta a_t - \delta \Delta k_t)\,\mathrm{d}t + \sigma(\Delta k_t)\mathrm{d}W_t.$$

Therefore, by Itô's formula we have

$$\begin{aligned}\|\Delta k_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^n)} &= \mathbb{E}\left[\int_0^t 2\Delta k_s^T \mathrm{d}\Delta k_s + \int_0^t \mathrm{Tr}(\sigma(\Delta k_s)^T\sigma(\Delta k_s))ds\right] \\ &= \mathbb{E}\left[\int_0^t 2\Delta k_s^T\left(\Delta a_t - \delta\Delta k_s\right)\mathrm{d}s + \int_0^t \mathrm{Tr}(\sigma(\Delta k_s)^T\sigma(\Delta k_s))\mathrm{d}s\right].\end{aligned}$$

By Fubini's theorem, Cauchy-Schwarz inequality and linearity of σ, we have

$$\begin{aligned}\|\Delta k_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^n)} \le& 2\int_0^t \left(\|\Delta k_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\|\Delta a_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^n)} - \delta\|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\right)\mathrm{d}s \\ &+ C_\sigma^2\int_0^t \|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\mathrm{d}s.\end{aligned}$$

Now by the Lipschitz property of function a, we have

$$\begin{aligned}\|\Delta k_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^n)} \le& 2\int_0^t \|\Delta k_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\left(C_{a,p}\|\Delta p_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^d)} + C_{a,y}\|\Delta y_s\|_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\right)\mathrm{d}s \\ &+ (2C_{a,k} + C_\sigma^2 - 2\delta)\int_0^t \|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\mathrm{d}s.\end{aligned}$$

Now the Fenchel inequality yields that

$$\begin{aligned}\|\Delta k_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^n)} \le& \int_0^t \left(C_{a,p}\|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)} + C_{a,y}\|\Delta y_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\right)\mathrm{d}s \\ &+ (C_{a,p} + C_{a,y} + 2C_{a,k} + C_\sigma^2 - 2\delta)\int_0^t \|\Delta k_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\mathrm{d}s.\end{aligned}$$

By Grönwall's Lemma we have

$$\|\Delta k_t\|^2_{L^2(\mathcal{F}_t,\mathbb{R}^n)} \le e^{\left(C_{a,p}+C_{a,y}+2C_{a,k}+C_\sigma^2-2\delta\right)t}\int_0^t \left(C_{a,p}\|\Delta p_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^d)} + C_{a,y}\|\Delta y_s\|^2_{L^2(\mathcal{F}_s,\mathbb{R}^n)}\right)\mathrm{d}s.$$

Taking the integral in time, we deduce that

$$\|\Delta k\|^2_{L^2(\mathbb{F},\mathbb{R}^n)} \le Te^{\left(C_{a,p}+C_{a,y}+2C_{a,k}+C_\sigma^2-2\delta\right)T}\left(C_{a,p}\|\Delta p\|^2_{L^2(\mathbb{F},\mathbb{R}^d)} + C_{a,y}\|\Delta y\|^2_{L^2(\mathbb{F},\mathbb{R}^n)}\right).$$

□

We can now establish the main result of this paper under the following condition:

$$C_4C_1 + C_5(C_2 + C_3C_1) < 1. \tag{17}$$

Theorem 4.2. *There exists a unique equilibrium if* (17) *holds.*

Proof. From Proposition 4.2, 4.3 and 4.4, and from direct computations

$$\left\|\Theta(k^1) - \Theta(k^2)\right\|_{L^2(\mathbb{F},\mathbb{R}^n)} \leq (C_4C_1 + C_5(C_2 + C_3C_1))^{\frac{1}{2}} \left\|k^1 - k^2\right\|_{L^2(\mathbb{F},\mathbb{R}^n)}.$$

The conclusion follows by a direct application of Picard's fixed-point theorem. □

We end this section with an interpretation on the constants involved in the contraction condition (17) and then present several situations in which the contraction property occurs.

To improve the discussions readability, we recall the definition of each constant and our notational convention: if f is a Lipschitz function of the form $(x, y) \mapsto f(x, y)$, we denote by $C_{f,x}$ its Lipschitz constant with respect to the variable x. We also recall the constants C_1 to C_5,

- $C_1 = C_{\Phi,e}C_\phi Te^{\left(C_{\Phi,e}+2C_{\Phi,p}+C_\gamma^2\right)T}$,
- $C_2 = (C^2_{\nabla_k g,k} + TC^2_{\upsilon,k})\frac{1}{\nu}\left(e^{\nu T} - 1\right)$,
- $C_3 = (C^2_{\nabla_k g,p} + TC^2_{\upsilon,p})\frac{1}{\nu}\left(e^{\nu T} - 1\right)$,
- $C_4 = C_{a,p}Te^{\left(C_{a,p}+C_{a,y}+2C_{a,k}+C_\sigma^2-2\delta\right)T}$,
- $C_5 = C_{a,y}Te^{\left(C_{a,p}+C_{a,y}+2C_{a,k}+C_\sigma^2-2\delta\right)T}$,

with $\nu = -2(\delta + \rho) + C^2_{\upsilon,y} + C^2_\sigma + C_{\upsilon,k} + C_{\upsilon,p}$, $C_\gamma = \max_i \gamma_i$ and $C_\sigma = \max_i \sigma_i$.

Interpretation of the constants. *Constant* C_1*.* The constant C_1 measures how much the drift of the external variable Φ depends on the aggregate contribution of agents e and the level of the external variable p. In economic terms, it captures the sensitivity of the external variable to collective behaviour. A high C_1, driven by large Lipschitz constants $C_{\Phi,e}, C_{\Phi,p}$ or significant stochastic intensity C_γ, indicates that the external variable is highly reactive to agents' decisions, which can amplify feedback effects in the system.

Constant C_2 *and* C_3*.* The constants C_2 and C_3 describe how changes in agents' states k and the external variable p affect their individual goals. C_2 reflects the sensitivity of agents' goals to changes in their own states, while C_3 measures how agents' goals respond to the external variable. These constants grow with the time horizon T and the strength of interactions between agents and their environment, highlighting the amplified responses over longer time periods.

Constant C_4 *and* C_5*.* The constants C_4 and C_5 describe how agents' optimal control strategies a are influenced by the external variable p and the adjoint variable y. Both share a common multiplicative factor:

$$Te^{\left(C_{a,p}+C_{a,y}+2C_{a,k}+C_\sigma^2-2\delta\right)T}.$$

This factor represents the amplification of feedback effects over time, driven by the time horizon T, stochastic fluctuations C_σ, and sensitivity parameters. A higher depreciation rate δ offsets these effects, stabilising the system. The term $C_{a,p}$ in C_4 measures how strongly agents' controls respond to changes in the external variable p, while $C_{a,y}$ in C_5 captures the influence of adjoint variable y on agents' controls. Higher values of $C_{a,p}$ or $C_{a,y}$ indicate stronger coupling between agents' decisions and the system, increasing the potential for amplified feedback loops.

Together, these constants reflect the interplay between direct and indirect effects in the system. Direct interactions are captured by C_4C_1, where agents' decisions directly influence the external variable's dynamics. Indirect feedback effects, represented by $C_5(C_2 + C_3C_1)$, account for how changes in the external variable propagate through agents' objectives and decisions. The contraction condition holds when the amplification of interactions due to time horizon, sensitivity parameters, and stochastic fluctuations is sufficiently small, or when stabilising factors such as depreciation effectively limit the propagation of feedback in the system.

Discussion on the contraction. We can identify three main regimes that helps the contraction to hold. We mean by "help" that a combination of the following regime might lead to contraction.

Small time horizon. As expected, if the time horizon T is small enough, then the contraction condition holds. This is a standard requirements to establish the well-posedness and uniqueness of FBSDEs.

Small interaction and large production. Small interactions helps to get contraction. If the sensitivity with respect to the aggregate contribution of agents of the drift of the external variable is small, meaning that $C_{\phi,e}$ is small, then C_1 is small. In addition, if the production is large, we expect that the sensitivity of the marginal production with respect to k to be small. Using the Corollary 4.1 and its proof, we deduce that $C_{\upsilon,k}$ is small when $\|\nabla_k F\|_\infty$ and $C_{\nabla_k F,k}$ are small. If moreover $C_{\nabla_k g,k}$ is small, then the constant C_2 is small, ensuring that the contraction condition holds.

Small sensibility of the control by large regularization. The contraction condition can be satisfied for small enough values of the constants C_4 and C_5, induced by small enough values of the constants $C_{a,y}$ and $C_{a,p}$. The latter corresponds to low sensibility of the feedback control with respect to the adjoint y and the externality p. Recalling that the feedback control is given by

$$a_i(k,p,y) = \exp\left(\frac{1}{\theta}\left(y_i - u'(F(k,p) - \xi(k,p,y))\right) - 1\right),$$

and y_i for each $i \in \{1,\dots,d\}$ is bounded by Proposition 4.3, the Lipschitz constant $C_{a,y}$ can be as small as desired for large values of the regularisation parameter θ. The same reasoning applies for the constant $C_{a,p}$. The function F is Lipschitz with respect to p and by the second inequality of Corollary 4.1, ξ is Lipschitz with respect to p. Since the consumption $c = F(k,p) - \xi(k,p,y)$ is bounded from below, the derivative u' is Lipschitz. Then the larger the constant θ, the smaller the constant $C_{a,y}$.

4.5 Link with the master equation

Mean field games can be studied using the master equation. It is a partial differential equation defined on an infinite dimensional space where its solution should be the value of the game. The well-posedness of this equation motivates the need to develop sharper existence and uniqueness results for the FBSDE system. Indeed, if existence and uniqueness can be ensured for the solutions of the system, then one can define the master field as follows:

$$\begin{aligned}&\mathcal{U}(t,k,p,m)\\ &= \mathbb{E}\left[\int_t^T \left(u(c_\tau) + K(a(k_\tau,p_\tau,y_\tau)\right)e^{-\rho\tau}\mathrm{d}\tau + g(k_T,p_T)e^{-\rho(T-t)}\middle| k_t = k, p_t = p, \mathbb{P}_\kappa = m\right],\end{aligned}$$

with $c_\tau = F(k_\tau,p_\tau) - a(k_\tau,p_\tau,y_\tau)\cdot \mathbb{1}$ and $(k_\tau,p_\tau,y_\tau)_{\tau\in[t,T]}$ being the unique (strong) solution to

$$\begin{cases} \mathrm{d}p_\tau &= \Phi(\mathbb{E}[\phi(k_\tau)|\mathcal{F}^0_\tau],p_\tau)\mathrm{d}\tau + \gamma(p_\tau)\mathrm{d}W^0_\tau, & p_t = p,\\ -\mathrm{d}y_\tau &= \nabla_k H(a(k_\tau,p_\tau,y_\tau),k_\tau,p_\tau,y_\tau,z_\tau)\mathrm{d}\tau - z_\tau\mathrm{d}W_\tau - z^0_\tau\mathrm{d}W^0_\tau, & y_T = \nabla_k g(k_T,p_T),\\ \mathrm{d}k_\tau &= (a(k_\tau,p_\tau,y_\tau) - \delta k_\tau)\mathrm{d}\tau + k_\tau\sigma\mathrm{d}W_\tau, & k_t = \kappa,\end{cases}$$

with $\mathbb{P}_\kappa$ the law of κ and where a is the optimal investment policy defined in (9). If $\mathcal{U}$ is regular enough, it is expected that it is solution of the following master equation:

$$\begin{aligned}&\partial_t U - \rho U + \mathcal{H}(k,p,\nabla_k U) + \Delta_k\left(\frac{\sigma(k)}{2}U\right) + \Delta_p\left(\frac{\gamma(p)}{2}U\right)\\ &+ \int_{\mathbb{R}^n_+} D_k\frac{\delta U}{\delta m}D_q\mathcal{H}dm + \int_{\mathbb{R}^n_+}\mathrm{div}_k\left(D_k\frac{\delta U}{\delta m}\right)\frac{\sigma}{2}dm = 0,\end{aligned}$$

where

$$\mathcal{H}(k,p,\nabla U) = \sup_{a\in\mathbb{R}^n} \{u(F(k,p) - \mathbb{1}\cdot a) + \nabla_k U(a-\delta k) - K(a)\}\,.$$

The sens of the derivative with respect to the probability measure m needs to be specified. We refer to [5] for such definitions. The author introduced a notion of monotone solutions, which he used in a similar framework in paragraph 5.2 of the same paper.

Comment on the framework. We end this section with a comparison of our framework with the study [24]. In the latter work, the authors obtain existence and uniqueness for a mean-field game problem with one external variable by studying the master equation. In order to prove existence and uniqueness of the master equation in the long run (i.e. for any finite time horizon $T > 0$), they consider a monotonous regime, which we do not encounter in our study.

5 Numerical simulations

This section is dedicated to numerical resolution. We describe the numerical method in Section 5.1. We specify the model in Section 5.2. This example is concerned with an economy with two types of capital: a brown capital with high productivity but high exposure to climate change and a green capital with low productivity but insensible to climate change. We discuss the numerical results at the end of the section.

5.1 Algorithm

To solve Problem (P), we rely on a fixed-point method that fixes the pollution p: Once p is fixed, the problem is still in high dimension and we use a neural network to approximate the optimal control. Then it is possible to solve the Pontryagin optimality equations given by (16) by discretizing the problem in time and adapting to the mean field case one of the algorithms developed in [16].
We do not follow this approach, but use a direct one as proposed in [13]: Let N_T be a positive integer, let $\Delta_t = \frac{T}{N_T}$ and $t_n = n\Delta t, n = 0,\ldots,N_T$. At each fixed-point iteration j of the algorithm, assuming we have an approximation R_t^{j-1} of $\mathbb{E}[\phi(k_t)|\mathcal{F}_t^0]$, the computation consists of two parts:

1. First, we approximate the control at each time t_n by a single feedforward network

$$a^\xi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^n \longrightarrow \mathbb{R}^n_+$$

with parameters ξ, taking time as input as in [14]. Then we solve:

$$\begin{aligned}\xi_j^* = \underset{\xi}{\mathrm{argmax}}\Delta t \sum_{i=0}^{N_T-1} \mathbb{E}\left[\left(u(F(k^j_{t_i},p^j_{t_i}) - \mathbb{1}\cdot a^\xi(t_i,p^j_{t_i},k^j_{t_i})) - \theta K(a^\xi(t_i,p^j_{t_i},k^j_{t_i}))\right)e^{-\rho t_i}\right] \\ + \mathbb{E}\left[g(k^j_T,p^j_T)e^{-\rho T}\right], \qquad (18)\end{aligned}$$

where for $i = 1,\ldots,N_T-1$,

$$\begin{aligned}p^j_{t_{i+1}} =& p^j_{t_i} + \Phi(R^{j-1}_{t_i},p^j_{t_i})\Delta t + \gamma(p^j_{t_i})(W^0_{t_{i+1}} - W^0_{t_i}),\\ k^j_{t_{i+1}} =& k^j_{t_i} + (a^\xi(t_i,p^j_{t_i},k^j_{t_i}) - \delta k^j_{t_i})\Delta t + \sigma(k^j_{t_i})(W_{t_{i+1}} - W_{t_i}),\end{aligned}$$

and $p^j_0 = \eta$, $k^j_0 = \kappa$,

2. Then we estimate R^j_t. Introducing a second feedforward network

$$b^{\hat\xi} \colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d,$$

with parameters $\hat{\xi}$, we solve

$$\hat{\xi}_j^* = \underset{\hat{\xi}}{\operatorname{argmax}} \sum_{i=1}^{N_T} \mathbb{E}[\big(b^{\hat{\xi}}(t_i, p_{t_i}^j) - \phi(k_{t_i}^j)\big)^2], \tag{19}$$

and we set

$$R_t^j = b^{\hat{\xi}_j^*}(t, p_t^j). \tag{20}$$

The algorithm is stopped when $\sum_{i=1}^{N_T} \mathbb{E}[(p_{t_i}^{j+1} - p_{t_i}^j)^2] < \epsilon$.
In practice, in order to avoid oscillations during iterations when T is high, we use a fictitious version of the algorithm (see for example [9]) and we replace equation (20) by

$$R_t^j = \frac{1}{j+1} \sum_{k=0}^{j} b^{\hat{\xi}_k^*}(t, p_t^k), \tag{21}$$

with $b^{\hat{\xi}_0^*} = 0$.
To solve the equations (18), (19), we use the ADAM gradient descent algorithm [20] in Tensorflow [1] with the tanh activation function, a learning rate equal to 10^{-3}, 3 hidden layers of 20 neurons each.

5.2 Numerical results

We have used the parameters in Table 1 below for the simulation. We consider two sectors of activity ($n = 2$): a brown one source of pollution and a green one. The utility function is a power function and the production function is the limit of a Constant Elasticity of Substitution (CES) production function when the substitution parameter goes to 1, i.e. we consider the limit of

$$\big(b_1(p)(k^1)^\gamma + b_2(k^2)^\gamma\big)^{\frac{\beta}{\gamma}},$$

when γ tends to 1, where β belongs to $(0, 1)$, k^1 represents the capital level in the brown sector and k^2 in the green sector. The coefficients $b_1(p)$ and b_2 are the productivity coefficients for each sector: we assume that the green sector is not affected by the environmental variable, while the brown sector is negatively affected by it. The function b_1 is a logistic function. We consider a Dirac mass as the initial distribution of capital.

Parameters	Value
δ	0.1
σ	0.04
σ_0	0.1
$u(c)$	$c^{0.8}$
ρ	0.1
F(k,p)	$(b_1(p)k^1 + b_2k^2)^{0.3}$
$b_1(p)$	$1/\left(1 + \exp\left(0.5p - 0.1\right)\right)$
b_2	0.4
$g(k, p)$	$u(F(k, p))e^{-\rho T}/\rho$
$\varepsilon(a)$	$-0.1 \sum_{i=1}^{2} a^i \ln(a^i)$
m_0	$\delta_{(0.2,0.2)}$
γ_0	0.15
$\Phi(e, p)$	$0.3e - 0.1p$
$\xi(k)$	$0.5k^1$

Table 1: Model parameters

Before providing a detailed analysis, we will briefly introduce Figures 1, 2, 3, and 4. Figure 1 shows the evolution over time of the pollution level, the productivity coefficient $b_1(p)$ of the brown sector as well as the productivity coefficient b_2 of the green sector, the average production and consumption for two pollution scenarios.

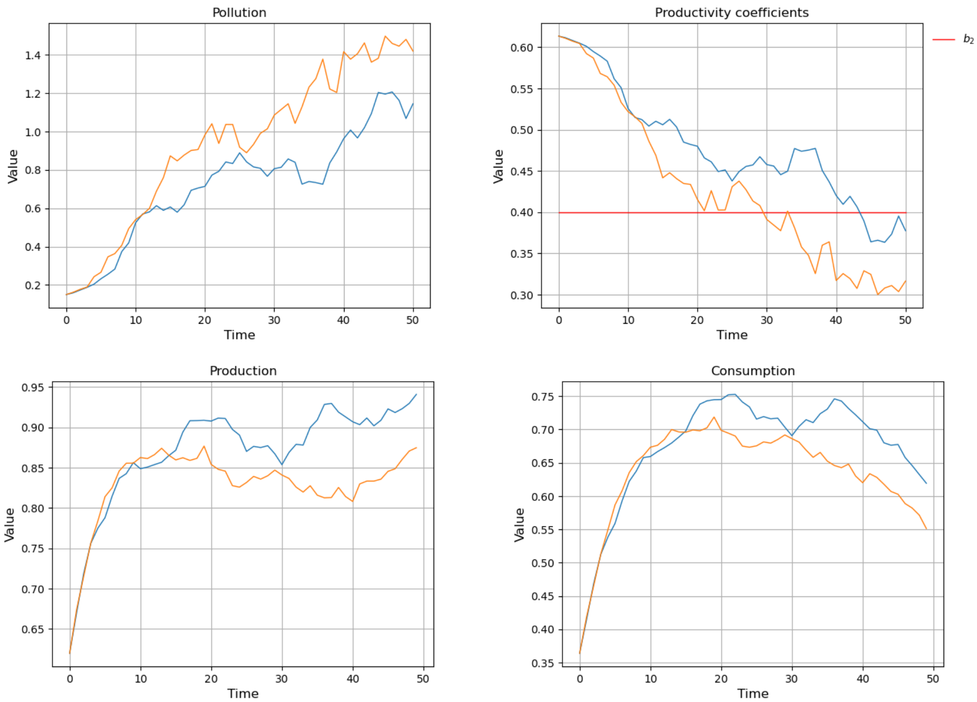

Figure 1: Two realisations of the pollution process.

We observe an increase in average production, which leads to a corresponding rise in average consumption. This growth results in higher pollution levels, causing a decline in the brown productivity coefficient $b_1(p)$.

Figures 2 and 3 below, show the investment distributions (on the left) and the capital distributions (on the right) as a function of time.

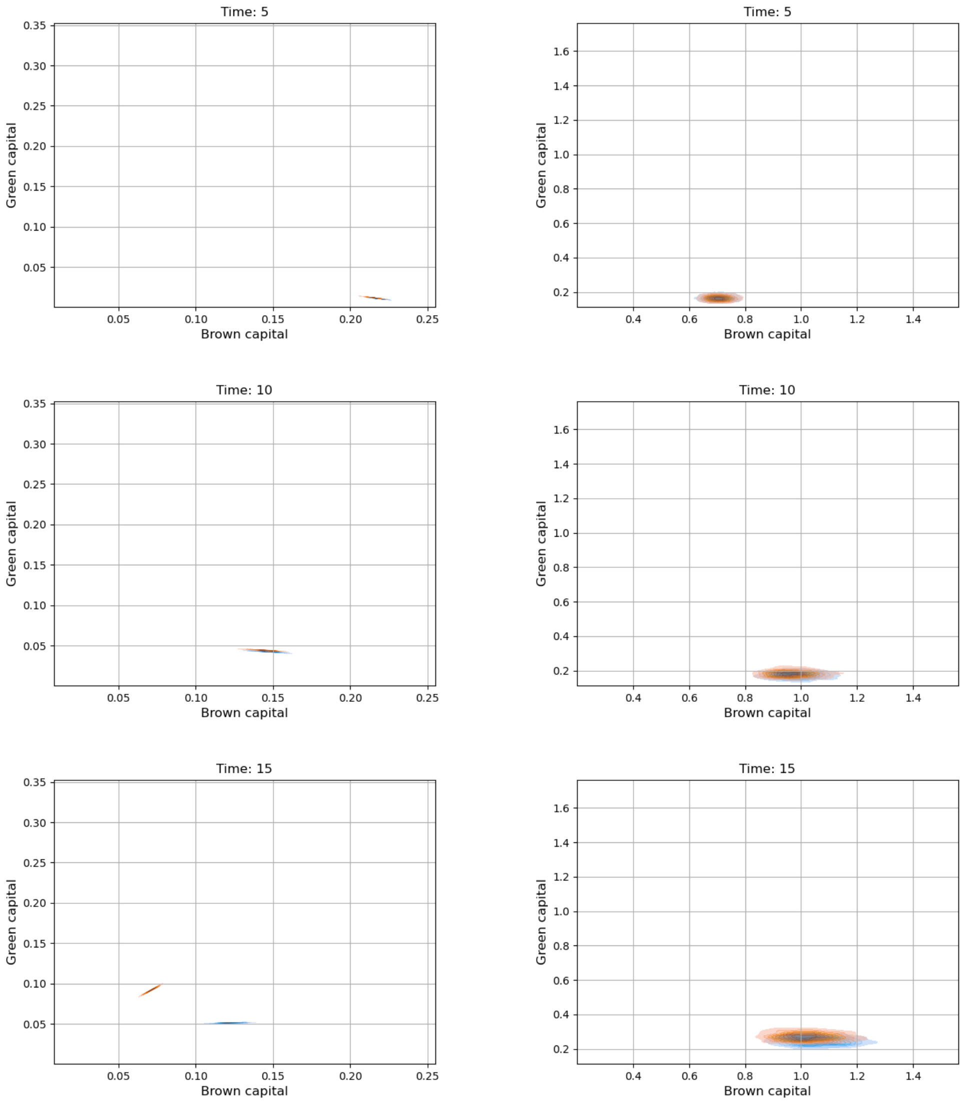

Figure 2: Distributions of the investment a_t (on the left) and the capital k_t (on the right) for time 5, 10, and 20.

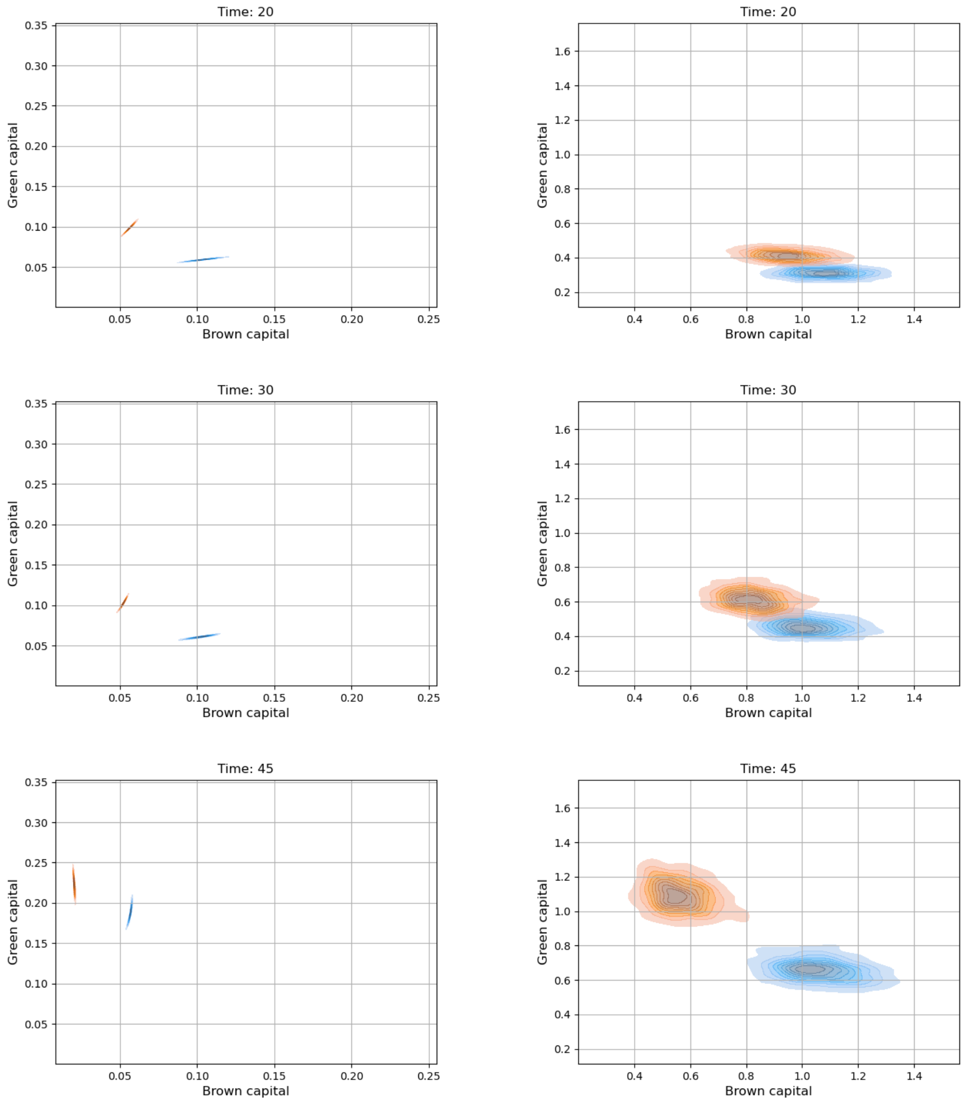

Figure 3: Distributions of the investment a_t (on the left) and of the capital k_t (on the right) for time 20, 30, and 45.

Figure 4 illustrates the 5%, 95% quantiles, and the mean of the pollution process. The widening spread between the quantiles highlights the significant impact of common noise, which initially increases before stabilizing. Similarly, pollution levels rise and eventually stabilize.

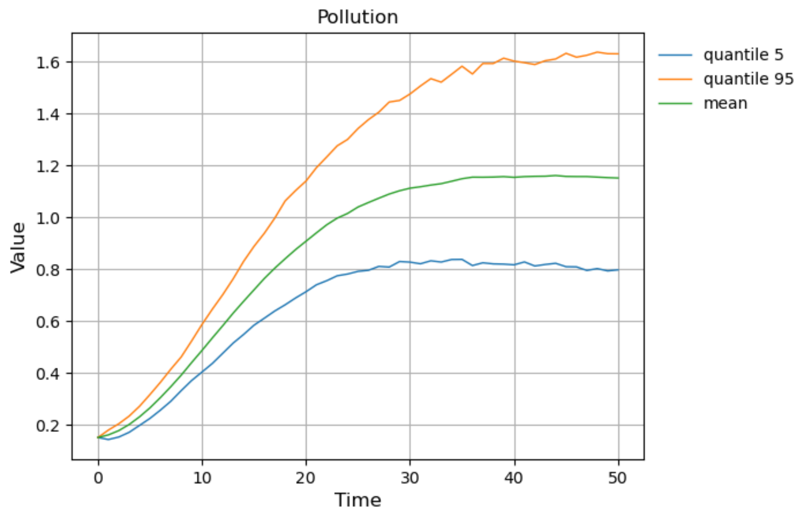

Figure 4: Quantiles 5%, 95% and mean of the pollution process

Discussions on the outputs. Examining the pollution dynamics illustrated in Figure 1, we observe a rapid increase in pollution in both scenarios, driven by the swift expansion of the brown sector. This trend can be attributed to the productivity coefficients: Figure 1 highlights the fact that, at first, the brown sector's productivity coefficient, $b_1(p)$, is bigger than the green sector's productivity coefficient, b_2 (represented by the red line in the figure). Consequently, agents are initially incentivized to invest in the brown sector, as depicted in Figure 2.

At time $t = 10$, there is a shift as countries begin investing in the green sector, despite the brown sector still appearing more attractive. This shift reflects agents' expectations of increasing pollution, which would eventually render the brown sector less competitive. As a result, it becomes advantageous to develop the green sector, marking the beginning of a transition.

Returning to the pollution process, its volatility grows over time. This is linked to the rising pollution levels and the scaling of the volatility rate with these levels. By time $t = 10$, random shocks result in the pollution trajectories of the two scenarios diverging. Consequently, the optimal investment strategies also differ. Figures 2 and 3 reveal that the scenario with higher pollution levels (the orange scenario) allocates more investment to the green sector. These differing policies lead to diverging capital distributions: by time $t = 45$, the brown sector in the blue scenario is approximately twice as developed as in the orange scenario, whereas the green sector shows the opposite pattern. In summary, higher pollution levels encourage greater development of the green sector, leading to slower pollution growth in the orange scenario compared to the blue one. This dynamic is reflected in the pollution drift, which is higher in the blue scenario. Figure 4 further supports that there is a stabilisation effect due to the drift in equation (3) since the pollution process appears to converge toward a steady state.

Finally, we note that the random shocks influencing the pollution process in the orange scenario have had a negative impact on production and consumption levels compared to the blue scenario.

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