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Mean Reflected SDEs and Propagation of Chaos

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Overview

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Overview

Introduction-Motivation

Reflected SDEs in mean

Propagation of chaos and simulation

Numerical illustrations

Generalizations and problems

Refleted SDEs

- B Brownian motion in $\mathbf{R}^d$, $\mathcal{F}$ its augmented filtration
- Skorokhod problem
  - ⋆ Given the barrier $\{L_t\}_{0\le t\le T}$ and the initial condition $X_0 \ge L_0$

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s + \boldsymbol{K_t}, \quad t \ge 0$$

$$\boldsymbol{X_t} \ge \boldsymbol{L_t}, \qquad \int_0^t (X_s - L_s)dK_s = 0, \qquad t \ge 0.$$

  - ⋆ X, K continuous real processes,
  - ⋆ K is nondecreasing with $K_0 = 0$
  - ⋆ Tanaka 79', Lions-Sznitman 84', ...

Reflected SDEs in mean

- We consider a reflected SDE

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s + \boldsymbol{K_t}, \quad t \geq 0$$

- The reflection is not on X_t itself but involves its law
- Given an increasing function h, the constraint is

$$\forall t \geq 0, \quad \mathbb{E}[\boldsymbol{h(X_t)}] \geq \mathbf{0}$$

- The Skorokhod condition becomes

$$\int_0^t \mathbb{E}[h(X_s)]\,dK_s = 0, \quad t \geq 0.$$

Motivation : Risk Measures

- A risk measure is an application $\rho : L^2(\mathcal{F}_T) \longrightarrow \mathbf{R}$ such that
  1. $X \leq Y \Longrightarrow \rho(X) \geq \rho(Y)$;
  2. $\rho(X + m) = \rho(X) - m$.
     - ⋆ Convex risk measures: H. Föllmer, A. Schied
     - ⋆ Coherent (convex + positively homogenous): P. Artzner, F. Delbaen, J.-M. Eber, D. Heath
- The acceptance set is

$$\mathcal{A}_\rho = \{X : \rho(X) \leq 0\}$$

- Given a set $\mathcal{A}$, one can define a risk measure by setting

$$\rho(X) = \inf\{m \in \mathbf{R} : m + X \in \mathcal{A}\}$$

Motivation : Risk Measures

- If u is a nondecreasing function, one can choose as acceptance set

$$\mathcal{A} = \{X : \mathbb{E}[\,u(X)] \geq \alpha\} = \{X : \mathbb{E}[\,h(X)] \geq 0\}, \quad h(x) = u(x) - \alpha$$

- If one invests in the stock S following the strategy π, the value of the portofolio is given by

$$X_t = X_0 + \int_0^t \pi_t\, dS_t = X_0 + \int_0^t \mu\pi_t S_t + \int_0^t \sigma\pi_t S_t\, dB_t, \quad t \geq 0$$

- The investor can follow the strategy he wants as soon as
  - ⋆ $\boldsymbol{X_t}$ **remains an acceptable position** for a given risk measure.

- Examples
  - ⋆ $VaR_\alpha(X) = \inf\{m : \mathbb{P}(m + X < 0) \leq \alpha\}, \quad h(x) = \mathbf{1}_{x \geq 0} - (1 - \alpha)$
  - ⋆ $AVaR_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha VaR_s(X)ds = \mathbb{E}\,[-X \,|\, -X \geq VaR_\alpha(X)]$ (if X is continuous)

Overview

A simple example

- Let us solve the following reflected SDE

$$X_t = X_0 - \gamma t + \sigma B_t + K_t, \quad t \geq 0,$$

$$\mathbb{E}[X_t] \geq u, \qquad \int_0^t (\mathbb{E}[X_s] - u)\, dK_s = 0, \qquad t \geq 0$$

  - ⋆ with $\gamma > 0$, $\mathbb{E}[X_0] > u$.

- The solution is

$$X_t = X_0 - \gamma t + \sigma B_t + (\mathbb{E}[\boldsymbol{X_0}] - \gamma \boldsymbol{t} - \boldsymbol{u})_-$$

$$\boldsymbol{K_t} = \gamma(\boldsymbol{t} - \boldsymbol{t}^*)_+, \qquad \mathbb{E}[X_0] - \gamma t^* = u$$

A simple example

- Starting from the previous solution, for $\alpha \in \mathbf{R}$, set

$$\mathcal{E}_t^\alpha = \exp\left(\alpha B_t - \alpha^2 t/2\right), \quad K_t^\alpha = \int_0^t \mathcal{E}_s^\alpha \, dK_s$$

- Let X^α be the "solution" to the SDE

$$X_t^\alpha = X_0 - \gamma t + \sigma B_t + K_t^\alpha, \quad t \geq 0$$

- Then (X^α, K^α) is still a solution to the reflected SDE:
  - ⋆ $\mathbb{E}\left[X_t^\alpha\right] = \mathbb{E}\left[X_t\right]$ since $\mathbb{E}\left[\mathcal{E}_t^\alpha\right] = 1$
  - ⋆ we have the Skorokhod condition since $dK^\alpha << dK$.

No Uniqueness if K is allowed to be random

A simple example

- **There is no minimal solution**
- Assume $(\overline{X}, \overline{K})$ is a minimal solution then

$$\begin{aligned}\overline{X}_t \leq X_t^\alpha &= X_0 - \gamma t + \sigma B_t + K_t^\alpha, \\ &= X_0 - \gamma t + \sigma B_t + \int_0^t \mathcal{E}_s^\alpha dK_s\end{aligned}$$

- As a byproduct, $\alpha \to +\infty$,

$$\forall t > 0, \quad \overline{X}_t \leq X_0 - \gamma t + \sigma B_t, \qquad \mathbb{E}\left[\overline{X}_t\right] = \mathbb{E}\left[X_0\right] - \gamma t < u,$$

- The constraint is not satisfied for t large enough

Deterministic solution

Definition

By a **deterministic** solution we mean a couple (X, K) of progressively measurable processes s.t.

1. (X, K) is continuous ;
2. K is nondecreasing with $K_0 = 0$ and **deterministic** ;
3. (X, K) is square integrable ;
4. the equation is satisfied:

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) \cdot dB_s + K_t, \quad t \geq 0,$$

$$\mathbb{E}\left[h(X_t)\right] \geq 0, \qquad \int_0^t \mathbb{E}\left[h(X_s)\right] dK_s = 0, \qquad t \geq 0.$$

Assumptions

- We assume that $b : \mathbf{R} \longrightarrow \mathbf{R}$ and $\sigma : \mathbf{R} \longrightarrow \mathbf{R}^d$ are Lipschitz continuous
- $X_0 \in \mathrm{L}^p$ for $p > 4$
- The function $h : \mathbf{R} \longrightarrow \mathbf{R}$ is nondecreasing and for some $0 < m \leq M$

$$m|x - y| \leq |h(x) - h(y)| \leq M|x - y|, \quad (0 < m \leq h'(x) \leq M).$$

- This assumption is rather strong! But so far ...

Existence and Uniqueness result

- We want to solve the SDE

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s + K_t, \quad t \geq 0,$$

$$\mathbb{E}\left[h(X_t)\right] \geq 0, \qquad \int_0^t \mathbb{E}\left[h(X_s)\right] dK_s = 0, \qquad t \geq 0.$$

- We assume that X_0 is square integrable and $\mathbb{E}\left[h(X_0)\right] \geq 0$

Theorem (R. Elie, Y. Hu, PhB)

The previous reflected SDE has a unique deterministic solution.

Proof: fixed point argument

- Let Y be a given process and let us solve

$$X_t = X_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s) \cdot dB_s + K_t, \qquad \mathbb{E}\left[h(X_t)\right] \geq 0$$

- We set

$$U_t = X_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s) \cdot dB_s$$

- Since $\mathbb{E}\left[h(X_t)\right] = \mathbb{E}\left[h(U_t + K_t)\right]$, we have

$$K_t \geq G_0(U_t) = G_0\left(\mu_{U_t}\right)$$

where $G_0 : \mathrm{L}^2 \longrightarrow \mathbf{R}$ is defined by

$$G_0(X) = \inf\{x \geq 0 : \mathbb{E}\left[h(x + X)\right] \geq 0\}$$

- K_t is nonincreasing and we have

$$K_t \geq \sup_{s \leq t} G_0(U_s)$$

Proof

- We define (X, K) by setting

$$K_t = \sup_{s\leq t} G_0(U_s), \qquad X_t = U_t + K_t.$$

- By definition of K, we have $\mathbb{E}\left[h(X_t)\right] \geq 0$
- Since $G_0(U_t) = \sup_{s\leq t} G_0(X_s) > 0$ dK-a.e.

$$\int_0^t \mathbb{E}\left[h(X_s)\right] dK_s = \int_0^t \mathbb{E}\left[h(U_s + G_0(U_s))\right] \mathbf{1}_{G_0(X_s)>0}\, dK_s = 0.$$

- It remains to prove that $Y \longrightarrow X$ is a contraction.
- The key point is the following observation

$$|G_0(X) - G_0(X')| \leq \frac{M}{m} W_1\left(\mu_X, \mu_{X'}\right) \leq \frac{M}{m} \mathbb{E}\left[|X - X'|\right].$$

Properties

- $t \longmapsto K_t$ is $1/2$-Hölder continuous. This comes directly from

$$K_{t+h} - K_t = \sup_{0 \leq s \leq h} G_0 \left(X_t + \int_t^{t+s} b(X_u) du + \int_t^{t+s} \sigma(X_u)\, dB_u \right)$$

- If (X, K) is a solution, Itô's formula gives when h is smooth

$$\mathbb{E}\left[h(X_{t+h})\right] = \mathbb{E}\left[h(X_t)\right] + \int_t^{t+h} \mathbb{E}\left[\mathcal{L}h(X_s)\right] ds + \int_t^{t+h} \mathbb{E}\left[h'(X_s)\right] dK_s$$

- Thus, $dK_t << dt$ and

$$K'_t = \mathbf{1}_{\mathbb{E}[h(X_t)=0]} \frac{\mathbb{E}\left[\mathcal{L}h(X_t)\right]_-}{\mathbb{E}\left[h'(X_t)\right]}$$

Risk Measures

- In the same way, if ρ is a risk measure defined on L^2, we can solve

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) \cdot dB_s + K_t, \quad t \geq 0$$
$$\rho(X_t) \leq 0, \qquad \int_0^t \rho(X_s) dK_s = 0, \qquad t \geq 0.$$

- In this case, $G_0(X) = \rho^+(X)$.

Theorem

If $\rho : \mathrm{L}^2 \longrightarrow \mathbf{R}$ is a Lipschitz risk measure, then the reflected SDE has a unique deterministic solution.

- $|\rho(X) - \rho(Y)| \leq C\, \mathbb{E}\left[|X - Y|^2|\right]^{1/2}$

Examples

- Typical examples are coherent risk measures

$$\rho(X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{Q}\}$$

  - ⋆ $\mathcal{Q}$ is a set of probabilities absolutely continuous w.r.t. $\mathbb{P}$

- As soon as the set of densities is bounded in L^2, ρ is Lipschitz
- In particular,

$$AVaR_\alpha(X) = \sup\left\{\mathbb{E}^{\mathbb{Q}}[-X] : \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\right\}$$

Overview

Simulations ?

- Let us consider the reflected SDE

$$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)dB_s + K_t, \quad t \geq 0,$$
$$\mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)]\,dK_s = 0, \quad t \geq 0.$$

- The idea is to take advantage of the definition of K :

$$K_{t+h} - K_t = \sup_{t \leq r \leq t+h} G_0\left(X_t + \int_t^r b(X_s)\,ds + \int_t^r \sigma(X_s)dB_s\right),$$
$$G_0(X) = \inf\{x \geq 0 : \mathbb{E}\left[h(x+X)\right] \geq 0\} = G_0(\mu_X).$$

- The natural discretization is

$$X_{t+h} = X_t + h\,b(X_t) + \sigma(X_t)\left(B_{t+h} - B_t\right) + K_{t+h} - K_t,$$
$$K_{t+h} - K_t = G_0\left(X_t + h\,b(X_t) + \sigma(X_t)\left(B_{t+h} - B_t\right)\right),$$

Simulations

- But we have to compute G_0
- One can consider the following system

$$X^i_{t+h} = X^i_t + h\,b(X^i_t) + \sigma(X^i_t)\,(B^i_{t+h} - B^i_t) + K_{t+h} - K_t, \quad 1 \le i \le N$$

$$K_{t+h} - K_t = G_0\left(\left\{X^i_t + h\,b(X^i_t) + \sigma(X^i_t)\,(B^i_{t+h} - B^i_t)\right\}_{1\le i\le N}\right),$$

$$G_0\left(\left\{X^i\right\}_{1\le i\le N}\right) = \inf\left\{x \ge 0 : \frac{1}{N}\sum_{i=1}^{N} h\left(x + X^i\right) \ge 0\right\} = G_0\left(\mu^N_X\right).$$

- We split the analysis into two parts:
  - ⋆ The system of particles
  - ⋆ The discretization

Propagation of chaos

- We introduce the following system of particles: for $1 \leq i \leq N$,

$$X_t^i = x_0 + \int_0^t b(X_s^i)\, ds + \int_0^t \sigma(X_s^i) dB_s^i + K_t^N, \quad t \geq 0,$$

$$\frac{1}{N}\sum_{i=1}^N h(X_t^i) \geq 0, \quad \int_0^t \frac{1}{N}\sum_{i=1}^N h(X_s^i)\, dK_s^N = 0, \quad t \geq 0.$$

  - ⋆ B^i independent BM.

- This system is a reflected diffusion with an oblique reflection: the direction of the reflexion is $(1, \ldots, 1)^t$

- $\overline{X}^i$ are independent copies of X

Result

Theorem (Chaudru de Raynal, Guillin, Labart, PhB)

If h is bi-Lipschitz, then

$$\mathbb{E}\left[\left|X_t^i - \overline{X}_t^i\right|^2\right] \leq C\, N^{-1/2}.$$

In the function h is smooth, $\mathcal{C}^2$ with bounded derivatives,

$$\mathbb{E}\left[\left|X_t^i - \overline{X}_t^i\right|^2\right] \leq C\, N^{-1}.$$

Proof

- For $x \in \mathbf{R}$ and $\nu \in \mathrm{M}^1$,

$$H(x,\nu) = \int h(x+y)\nu(dy), \quad G_0(\nu) = \inf\{x \geq 0 : H(x,\nu) \geq 0\},$$
$$G(\nu) = \inf\{x : H(x,\nu) \geq 0\}$$

Proposition

We have the following properties:

1. *H is a bi-Lipschitz function*
2. *G_0 is Lipschitz continuous:*

$$|G_0(\nu) - G_0(\nu')| \leq \frac{M}{m} W_1(\nu,\nu').$$

3. *More precisely,*

$$|G_0(\nu) - G_0(\nu')| \leq \frac{1}{m}\left|\int h(G(\nu)+y)(\nu(dy) - \nu'(dy))\right|$$

Proof

- Let us recall that

$$K_t = \sup_{s \le t} G_0(U_s), \quad U_s = x_0 + \int_0^s b(X_r)dr + \int_0^s \sigma(X_r)dB_r$$

- In other words, if we call μ_s the distribution of U_s

$$K_t = \sup_{s \le t} G_0(\mu_s).$$

- Let μ_s^N be the empirical distribution of the

$$U_s^i = x_0 + \int_0^s b(X_r^i)dr + \int_0^s \sigma(X_r^i)dB_r^i$$

- We have, in the same way, $K_t^N = \sup_{s \le t} G_0(\mu_s^N)$

Proof

- We compute $|X_t^i - \overline{X}_t^i|$ ($b \equiv 0$):

$$
\begin{aligned}
|X_t^j - \overline{X}_t^j| &\leq \left| \int_0^t \left(\sigma(X_s^j) - \sigma(\overline{X}_s^j) \right) dB_s^j \right| + \left| \sup_{s \leq t} G_0(\mu_s^N) - \sup_{s \leq t} G_0(\mu_s) \right| \\
&\leq \left| \int_0^t \left(\sigma(X_s^j) - \sigma(\overline{X}_s^j) \right) dB_s^j \right| + \sup_{s \leq t} \left| G_0(\mu_s^N) - G_0(\overline{\mu}_s^N) \right| \\
&\quad + \sup_{s \leq t} \left| G_0(\overline{\mu}_s^N) - G_0(\mu_s) \right| .
\end{aligned}
$$

- Gronwall's lemma for the first two terms: the speed of convergence is given by

$$\sup_{s \leq t} \left| G_0(\overline{\mu}_s^N) - G_0(\mu_s) \right|$$

  - ⋆ Not so easy with the sup

Proof

- We have

$$\mathbb{E}\left[\sup_{s\le t}\left|G_0(\overline{\mu}_s^N)-G_0(\mu_s)\right|^2\right]\le\frac{1}{m^2}\mathbb{E}\left[\sup_{s\le t}\left|\int h(G(\mu_s)+y)(\overline{\mu}_s^N(dy)-\mu_s(dy))\right|^2\right]$$

- When h is not smooth, we improve a result from Rachev and Ruschendorf

$$\begin{aligned}\mathbb{E}\left[\sup_{s\le t}\left|G_0(\overline{\mu}_s^N)-G_0(\mu_s)\right|^2\right]&\le C\,\mathbb{E}\left[\sup_{s\le t}W_1(\overline{\mu}_s^N,\mu_s)\right]\\&\le C\,N^{-1/2}\end{aligned}$$

- When h is smooth, we can use Itô's formula to compute

$$\mathbb{E}\left[\sup_{s\le t}\left|\int h(G(\mu_s)+y)(\overline{\mu}_s^N(dy)-\mu_s(dy))\right|^2\right]\le C\,N^{-1}$$

Discretization

Theorem (Chaudru de Raynal, Guillin, Labart, PhB)

If h is bi-Lipschitz, then

$$\mathbb{E}\left[\left|X_t^{h,N,i} - \overline{X}_t^i\right|^2\right] \leq C\,\left(N^{-1/2} + h|\log h|\right).$$

If h is smooth

$$\mathbb{E}\left[\left|X_t^{h,N,i} - \overline{X}_t^i\right|^2\right] \leq C\,\left(N^{-1} + h|\log h|\right).$$

Theorem (Ghannoum, Labart, PhB)

There is no need of $|\log h|$*.*

Overview

Linear constraint

- $X_t = X_0 - \beta t + a \int_0^t X_s + \sigma B_t + K_t,$
- $\mathbb{E}[X_t] \geq p$, $K_t = (ap - \beta)(t - t^*)_+$, $t^* = \frac{1}{a}\left(\log\left(x_0 + \beta/a\right) - \log\left(p + \beta/a\right)\right)$

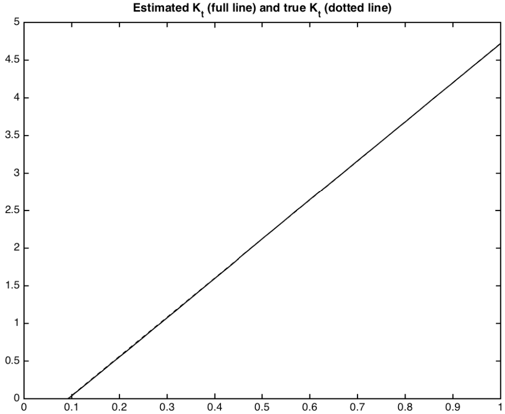

Figure: $n = 500,\ N = 10000,\ T = 1,\ \beta = 2.1,\ a = 1,\ \sigma = 1,\ x_0 = 1,\ p = 3.6$

Nonlinear constraint

- $X_t = X_0 - \beta t + a \int_0^t X_s + \sigma B_t + K_t$
- $h(x) = x + \alpha \sin(x) - p$

$$K_t = e^{-at} d \sup_{s \leq t} \left(F_s^{-1}(0)\right)^+,$$

$$F_t(x) = \Bigg\{ e^{-at}\left(x_0 - \beta\left(\frac{e^{at}-1}{a}\right) + x\right) + \alpha \exp\left(-e^{-at}\frac{\sigma^2}{a}\sinh(at)\right) \\ \times \sin\left(e^{-at}\left(x_0 - \beta\left(\frac{e^{at}-1}{a}\right) + x\right)\right) - p \Bigg\}.$$

Nonlinear constraint

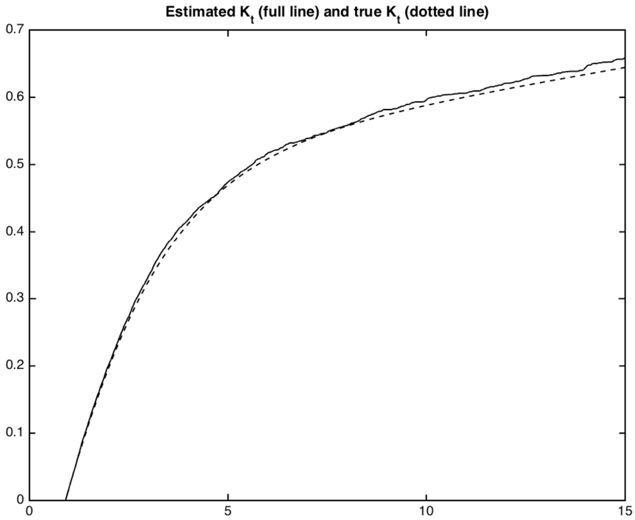

Figure: $n = 100,\ N = 10000,\ T = 15,\ \beta = 10^{-2},\ \sigma = 1,\ p = \pi/2,\ \alpha = .9,\ x_0 + \alpha \sin(x_0) - p = 0$

A different approach

- h smooth, K has a density w.r.t. Lebesgue measure

$$K_t = \int_0^t \mathbf{1}_{\mathbb{E}[h(X_s)]=0}\, \mathbb{E}\left[h'(X_s)\right]^{-1} \mathbb{E}\left[\mathcal{L}h(X_s)\right]^- \, ds$$

- The solution to the mean reflected SDE is the solution to the classical McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s)\, dB_s + \int_0^t f\left(\mathbb{P}_{X_s}\right)\, ds,$$

$$f(\mu) = \mathbf{1}_{\mu(h)=0} \frac{\mu\left(\mathcal{L}h\right)^-}{\mu\left((h')\right)}$$

- The numerical scheme resulting from the McKean-Vlasov SDE seems to converge
  - ⋆ Analysis in progress

Overview

Generalizations

- Generalizations
  - ⋆ SDEs with jumps (Abir Ghannoum)
  - ⋆ BSDEs when f does not depend on z (Hélène Hibon)
- Between generalizations and problems
  - ⋆ Multidimensional case
  - ⋆ Link with PDEs

Multidimensional case

- $h : \mathbf{R}^n \longrightarrow \mathbf{R}$
  - ⋆ h concave
  - ⋆ $0 < m^2 \leq |\nabla h(x)|^2 \leq M^2$
- We consider the normal reflected SDE

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) \cdot dB_s + \int_0^t \nabla h(X_s) dK_s, \quad t \geq 0,$$
$$\mathbb{E}\left[h(X_t)\right] \geq 0, \qquad \int_0^t \mathbb{E}\left[h(X_s)\right] dK_s = 0, \qquad t \geq 0.$$

First result

There exists a unique solution with K deterministic

Problems

- Propagation of chaos for BSDEs when f depends on z
- Regularity of h: $h'(x) \geq m > 0$
- Mixed reflexion depending on both the law and the path
  - ⋆ So far, $X_t \geq \mathbb{E}[X_t] - \alpha$

Thank you for your attention