Common Agency dilemma with information asymmetry.

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Motivations and general situation

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Goal: Design a contract that maximises the utility of the Principal under constraints.

Examples

- Optimal remuneration of an employee,
- How regulators with imperfect information and limited policy instruments can motivate firms to reduce pollution,
- How a company can optimally compensate its executives,
- How banks achieve optimal securitization of mortgage loans
- How investors should pay their portfolio managers
- An insurer who proposes a car insurance to customers...

However:

The actions of the Agent are observable/contractible or not.

There are characteristics of the Agent which are unknown to the Principal.

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A Stackelberg-like equilibrium between the Principal and the Agent:

- compute the best-reaction function of the Agent given a contract
- determine his corresponding optimal effort
- use this in the utility function of the Principal to maximise over all contracts.

Moral hazard: 1-Principal/1-Agent

Some reminders on a model with a Principal and one Agent.

The Holmström-Milgrom problem

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$$U_0^A(\xi) := \sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \underbrace{\left(\xi - \int_0^T k(a_s) ds \right)}_{\text{salary - cost of his effort}} \right]. \tag{1}$$

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"(1) $\Longleftrightarrow$ solving a **Backward SDE** with a unique solution (Y, Z) ",

$$Y_t = \xi + \int_t^T \left(-\frac{R_A}{2} |Z_s|^2 + \sup_a \{ b(s, X_s, a_s) Z_s - k(a_s) \} \right) ds - \int_t^T Z_s dW_s$$

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$$U_0^A(\xi) = -e^{-R_A Y_0}, \quad \text{optimal effort: } a^\star(Z).$$

The Holmström-Milgrom problem and some extensions

We get the following representation for admissible contract ξ

$$\xi = Y_0 - \int_0^T \left(-\frac{R_A}{2} |Z_s|^2 + \sup_a \left(b(s, X, a) Z_s - k(a) \right) \right) ds + \int_0^T Z_s dW_s.$$

The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi) \geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(X_T - \xi) \right],$$

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A stochastic control problem with

- State variables: the output X and the value function of the Agent,
- controlled variable: Z and Y_0.

$\hookrightarrow$ HJB equation associated with it

see Sannikov (07'), Cvitanić, Possamaï, Touzi (14', 17').

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In a model with N-Agents hired by a Principal.

- The Agents are assumed to be rational and aim at finding Nash equilibria.

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  Model in continuous time with exponential utilities: Koo, Shim and Sung (08'), Elie and Possamaï (16').

"Nash equilibrium $\Longleftrightarrow$ multi-dimensional (qg-)BSDE."

$\hookrightarrow$ Problem of the Principal: an HJB equation with $2N$ state variables.

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In a model with N-Agents hired by a Principal.

- A Planner manages the effort of the N-Agents for their well-being.

M. (17') *Moral hazard in welfare economics: on the advantage of Planner's advices to manage employees' actions.*

"Pareto optima $\overset{\text{scalarization}}{\Longleftrightarrow}$ solving a MOOP $\Longleftrightarrow$ one dim. BSDE."

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- Conditions ensuring that Nash equilibria are Pareto efficient.

Common Agency: N-Principals/1-Agent

We now consider a problem with N-Principals hiring a common Agent.

Toy example: A beekeeper and crop-owners, via pollination processes, which leads to externalities,
Intergovernmental Science-Policy Platform on Biodiversity and Ecosystem Services, *The Assessment Report on Pollinators*.

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Other examples:

- Baron, 85'. Regulators (EPA and a public utility commission) and a firm to prevent pollution.
- Braverman and Stiglitz, 82'. Farmers and landlords (sharecropping).
- Tirole, 03'. Foreign borrowing issues.
- Government management: different levels of the same Minister compensates a firm.

Common Agency

General approaches to analyse common agency problems with discrete times models:

- Bernheim and Whinston, *Common Agency*, Econometrica, 1986.

$\hookrightarrow$ Sufficient and necessary condition to describe the equilibrium+ A non-cooperative equilibrium between Principals is efficient only for the first best level of effort.

- Dixit, Grossman and Helpman, *Common Agency and coordination*, Journal of political economy, 1997.

$\hookrightarrow$ Non-quasilinear utility functions, efficiency of equilibria in the Pareto sense.

The model

To make it simple we consider $N = 2$ Principals.

- The output $dX_t := \Sigma dW_t$ is a 2-dim. process with Σ invertible.
- Action of the Agent $\nu := (\nu^1, \nu^2)^\top$ such that

$$dX_t = b(\nu_t)dt + \Sigma dW_t^\nu, \text{ under } \mathbb{P}^\nu.$$

- Cost function: $c(\nu_t)$.

The problem of the Agent

Let ξ^1 and ξ^2 be the contracts proposed by the Principals. The problem of the Agent is thus for $\xi := (\xi^1, \xi^2)^\top$

$$U_0^A(\xi) := \sup_\nu \mathbb{E}^{\mathbb{P}^\nu}\left[-e^{-R_A\left(\xi^1+\xi^2-\int_0^T c(\nu_s)ds\right)}\right].$$

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As usual, we consider $\nu^\star(z) \in \text{ argmax } (z \cdot b(\nu) - c(\nu))$

$$Y_t^\xi = \xi \cdot \mathbf{1}_2 + \int_t^T \left(-\frac{R_A}{2}\|\Sigma^\top Z_s^\xi\|^2 + Z_s^\xi \cdot b(\nu^\star(Z_s^\xi)) - c(\nu^\star(Z_s^\xi))\right) ds$$
$$- \int_t^T Z_s^\xi \cdot \Sigma dW_s. \tag{2}$$

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Proposition

There exists (if ξ is regular enough) a unique solution to BSDE (2) and we have

$$U_0^A(\xi) = -e^{-R_A Y_0^\xi}, \ \nu^\star(\xi) = \nu^\star(Z^\xi).$$

Problem of the Principals and main difficulty

Problem of Principal 1 given that Principal 2 gives the contract ξ^2

$$U_0^1 := \sup_{\xi^1} \mathbb{E}^{\mathbb{P}^{\nu^\star}} \left[U_1(\ell_1(X_T) - \xi^1) \right].$$

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We need a *good* decomposition for ξ^1. This suggests to rewrite the problem

$$U_0^1 = \sup_{\xi^1} \mathbb{E}^{\mathbb{P}^{\nu^\star}} \left[U_1 \left(\ell_1(X_T) - \xi \cdot \mathbf{1}_2 + \xi^2 \right) \right].$$

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- Assume that Principal 2 gives a contract with the following form

$$\xi^2 = y_2 + \int_0^T \alpha_s^2 ds + \int_0^T \beta_s^2 \cdot dX_s,$$

  therefore the contract ξ^1 has also a semi-martingale decomposition.

$\hookrightarrow$ HJB equation for each Principal.

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- We naturally obtained a system of coupled HJB equations.
- Nash equilibrium in the following class of contracts

$$\mathcal{C}_2 := \{\xi^i = y_i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^i \cdot dX_s,\ y_1 + y_2 \geqslant R_0\}.$$

Equilibrium conditions

Lemma (Robustness of $\mathcal{C}_2$)

If ξ^2 has a semi-martingale decomposition then

$$\xi^1 = y_1 + \int_0^T \alpha_s^1 ds + \int_0^T \beta_s^1 \cdot dX_s,$$

with

$$\begin{cases} y_1 := Y_0^\xi - y_2, \\ \alpha_s^1 := G(s, X_s, Z_s^\xi) - \alpha_s^2, \\ \beta_s^1 := Z_s^\xi - \beta_s^2, \end{cases}$$

where $G(s,x,z) = \frac{R_A}{2}\|\Sigma^\top z\|^2 - b(\nu^\star(z)) \cdot z + c(\nu^\star(z))$.

Equilibrium conditions

Proposition (Necessary condition for Nash equilibrium, M., Ren (17'))

Let $\xi \in \mathcal{C}_2$ be a Nash equilibrium, then the following equilibrium conditions are satisfied

$$(\textbf{\textit{Eq-Na}}) \begin{cases} y_1 + y_2 = Y_0^{\xi}, \\ \alpha_s^1 + \alpha_s^2 = G(s, X_s, \beta_s^1 + \beta_s^2), \\ \beta_s^1 + \beta_s^2 = Z_s^{\xi}. \end{cases}$$

The problem of Principal 1

Let now $\xi^2 \equiv (y_2, \alpha^2, \beta^2)$ be fixed such that

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Then, we get

$$U_0^1 = \sup_{y_1 \geqslant R_0 - y_2} \sup_{\substack{(\alpha_1, \beta_1) \\ \alpha_s^1 + \alpha_s^2 = G(s, X_s, \beta_s^1 + \beta_s^2)}} \mathbb{E}^{\mathbb{P}^{\nu^\star(\beta^1 + \beta^2)}} \left[U_1 \left(\ell_1(X_T) - Y_T^{y_1, \alpha^1, \beta^1} \right) \right],$$

with

$$Y_T^{y_1^\star, \alpha^1, \beta^1} = y_1^\star + \int_0^T (G(s, X_s, \beta_s^1 + \beta_s^2) - \alpha_s^2) ds + \int_0^T \beta_s^1 \cdot dX_s.$$

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which can be rewritten

$$\boxed{U_0^1 = \sup_{\beta_1} \mathbb{E}^{\mathbb{P}^{\nu^\star(\beta^1+\beta^2)}} \left[U_1 \left(\ell_1(X_T) - Y_T^{y_1^\star, \alpha^1, \beta^1} \right) \right].}$$

The problem of Principal 1 and HJB equation

HJB equation associated with the stochastic control problem

$$\begin{cases} -\partial_t u^1 - \sup_{\beta^1 \in \mathbb{R}^2} H(t,x,y,\nabla_x u^1, \partial_y u^1, \Delta u^1, \partial^2_{yy} u^1, \partial^2_{x,y} u^1, \alpha^2_s, \beta^2_s, \beta^1) = 0, \\ u^1(T,x,y) = U_1(\ell_1(x) - y), \ (t,x,y) \in [0,T) \times \mathbb{R}^2 \times \mathbb{R}. \end{cases} \tag{3}$$

with

$$\begin{aligned} &H(\cdot, p_x, p_y, q_x, q_y, r, a_2, b_2, \beta^1) \\ &= p_x \cdot b(\cdot, \nu^\star) + p_y(G(\cdot, \beta^1 + b_2) - a_2 + \beta^1 \cdot b(\cdot, \nu^\star)) \\ &+ \frac{1}{2}\mathrm{Tr}(\Sigma\Sigma^\top q_x) + \frac{1}{2}\mathrm{Tr}((\beta^1)^\top \Sigma\Sigma^\top \beta^1 q_y) + \mathrm{Tr}(\Sigma\Sigma^\top \beta^1 r) \end{aligned}$$

Theorem (Verification)

*Let $\beta^{1,\star}(t,x,y,p_x,p_y,q_x,q_y,r,a_2,b_2)$ be a maximizer of H (**(FOC)** holds). Then, $U_0^1 = u^1(0,x,R_0 - y_2)$ and optimal contract is given by $\xi^{1,\star} \equiv (R_0 - y_2, \alpha^{1,\star}, \beta^{1,\star})$.*

System of HJB equations and Nash equilibrium

We do the same thing for Principal 2. We get an optimal

$$\beta^{2,\star}(t, x, y, p_x, p_y, q_x, q_y, r, a_1, b_1).$$

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Roughly speaking (M., Ren (17'))

- Fixed point by mixing the **(FOC)** satisfied by $\beta^{1,\star}, \beta^{2,\star}, \alpha^{1,\star}, \alpha^{2,\star}$.
- We consider a system of coupled HJB equations with these optimizers.
- As soon as their exists a solution to this system of fully coupled HJB equation, we have a Nash equilibrium in $\mathcal{C}_2$ (see Dockner, Jorgensen, Long, Sorger, 2000.)

Example in the linear-quadratic case

Assume that

- $b(\nu) = K\nu$, with $K \in M_2(\mathbb{R})$ diagonal with coefficients $k_1, k_2 > 0$.
- $c(\nu) = \frac{\|\nu\|^2}{2}$
- Risk neutral principals with appetence $\gamma_i > 0$ so that

$$\ell_i(x) = (1 + \gamma_i)x_i - \gamma_i x_j, \; 1 \leqslant i \neq j \leqslant 2.$$

In this case,

$$\boxed{\nu^\star(Z^\xi) = KZ^\xi = K(\beta^1 + \beta^2).}$$

Example in the linear-quadratic case

Assumption (A)

Assume that there exists $\beta^{i,\star}, \alpha^{i,\star}, i \in \{1,2\}$ *such that the following system of HJB equations has a solution with* $1 \leqslant i \neq j \leqslant 2$

$$\begin{cases} -\partial_t v^i - h(t, x, \nabla_x v^i, \Delta v^i, \alpha_t^{j,\star}, \beta_t^{j,\star}, \beta_t^{i,\star}) = 0, \\ v^i(T, x) = (1+\gamma_i)x^i - \gamma_i x^j, \ (t,x) \in [0, T) \times \mathbb{R}^2, \\ \beta^{i,\star} = (R_A \Sigma\Sigma^\top + K^2)^{-1}(K^2 \nabla_x v^i - R_A \Sigma\Sigma^\top \beta^{j,\star}), \\ \alpha^{1,\star} + \alpha^{2,\star} = \frac{R_A}{2}\|\Sigma^\top(\beta^{1,\star} + \beta^{2,\star})\| - \frac{\|K(\beta^{1,\star}+\beta^{2,\star})\|^2}{2}. \end{cases}$$

with

$$h(t, x, p, q, a, b, \beta) := p \cdot K^2(b+\beta) + \frac{1}{2} Tr(q\Sigma\Sigma^\top) - \frac{R_A}{2}\|\Sigma^\top(b+\beta)\|^2 - \frac{\|K(b+\beta)\|^2}{2} + a + K(b+\beta) \cdot Kb.$$

Example in the linear-quadratic case

Proposition (M., Ren (17'))

*Under Assumption **(A)**, optimal contracts in $\mathcal{C}_2$ are given by $\xi^{i,\star} \equiv (y_i, \alpha^{i,\star}, \beta^{i,\star})$ with $y_1 + y_2 = R_0$ and*

$$\begin{cases} \beta^{1,\star} + \beta^{2,\star} = M\Gamma, \\ \alpha^{1,\star} + \alpha^{2,\star} = \frac{R_A}{2}\|\Sigma^\top M\Gamma\| - \frac{\|KM\Gamma\|^2}{2}, \end{cases}$$

with $M := (2R_A\Sigma\Sigma^\top + K^2)^{-1}K^2$ and $\Gamma := (1 + \gamma_1 - \gamma_2, 1 + \gamma_2 - \gamma_1)^\top$. Moreover

$$\nu^\star = KM\Gamma.$$

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$$\nu^\star = KM\Gamma.$$

Proof

We set $u^i(t,x,y) := v^i(t,x) - y$. We can (merely) apply the results of Dockner et al. by noticing that $v^1 + v^2$ satisfies an HJB equation with an explicit solution.

Interpretations

- If $k_1 = k_2$ the Agent works more for the more ambitious Principal.
- If $\gamma_1 = \gamma_2$ the Agent works more for the Principal with which he is more efficient.
- Let $k_1 = k_2$ and
$$\Sigma := \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$
Then, the less the projects are correlated, the more the Agent works for the more ambitious Principal.
- Risk-neutral Agent: he works more for P1 if and only if
$$k_1(1+\gamma_1-\gamma_2) > k_2(1+\gamma_2-\gamma_1).$$
A lack of ambition can be balanced by the efficiency. Moreover, a big difference of appetence parameters leads to a leverage effect between an initial ambition parameter and the quantity of work provides by the agent.

Comparison with the aggregated model and link with the literature

Now, we aggregate the Principal so that the Agent is directly hired by the parent firm. We get $\nu^\star_{Pf} = KM_{Pf}\Gamma$, with

$$M_{Pf} := (R_A \Sigma\Sigma^\top + K^2)^{-1} K^2.$$

Proposition (M., Ren (17'))

The optimal effort in the aggregated model coincides with the competitive model if and only if the Agent is risk-neutral.

Same result that Bernheim and Whinston (85') extended to continuous models. This comes from the fact that a risk neutral agent given the first best effort so that there is no welfare looses for the Principals.

Thank you.