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High-dimensional stochastic modelling of power markets

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Fred Espen Benth, Paris-Zoom, June 11, 2021.

A “theorem-proof”-free bird’s eye perspective of……

…..infinite dimensional modelling of forward prices

…..Heath-Jarrow-Morton models in infinite dimensions

Aim for the talk:

- Propose a class of stochastic models for the forward dynamics in power/commodity

$$(t,T) \mapsto F(t,T), \qquad 0 \leq t \leq T < \infty$$

- Markets:
  - "Classical" forward markets (day, month…)
  - Intraday-markets in power (hour, quarter)

- Forward market in power: delivery periods

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, T)\,dT$$

- Intraday power market: hourly delivery

$$\mathrm{ID}(t, h) = \int_{h}^{h+1} F(t, T)\,dT$$

BACKGROUND: FORWARD CURVES IN COMMODITIES AND WHY INFINITE DIMENSIONAL STOCHASTIC MODELS

Two-factor Lucia-Schwartz spot model

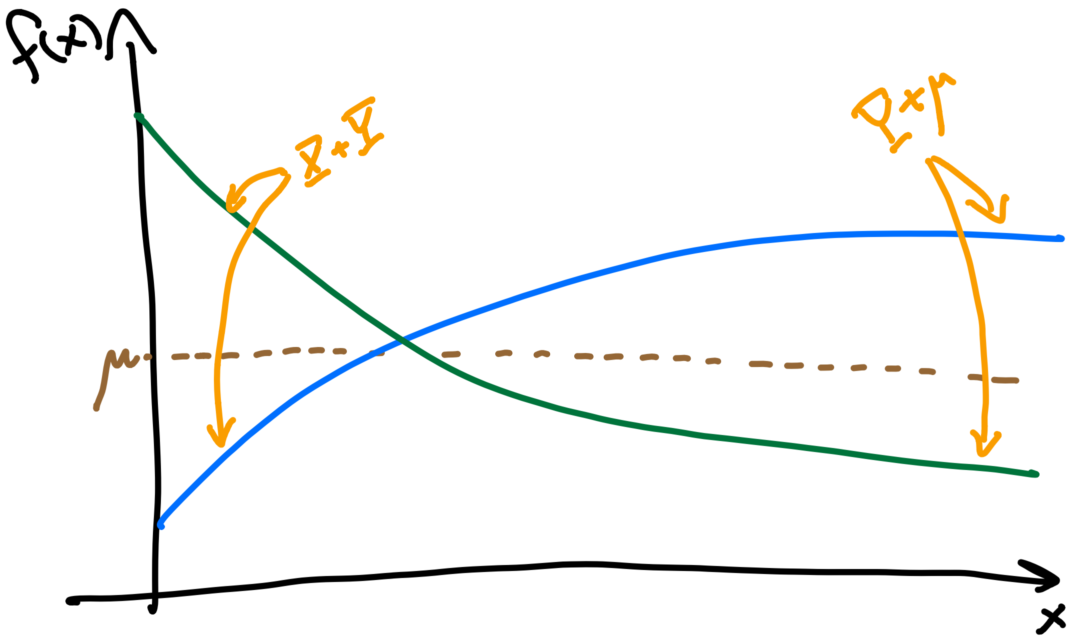
- Two Brownian motion factors

$$dX(t) = \alpha(\mu - X(t))dt + \sigma dB_1(t)$$
$$dY(t) = \eta dB_2(t)$$

- Forward price, $x=T-t$ is time to maturity

$$F(t,T) := f(t,T-t) := \mathbb{E}[X(T) + Y(T) \mid \mathscr{F}_t]$$

$$f(t,x) = \mu + e^{-\alpha x}(X(t) - \mu) + Y(t)$$



- Fix two maturities: Any other maturity can be perfectly hedged by these two!

- Empirical evidence for
  - “High"-dimensionality:
    - many factors across maturities
    - $\sim$5 factors according to Feron & Gruet (2021)
  - Non-Gaussian price changes/returns
    - heavy tailed distributions
  - Stochastic volatility
    - And Samuelson effect

- Risk-neutral forward dynamics Lucia-Schwartz

$$df(t,x) = \partial_x f(t,x)dt + (\sigma e^{-\alpha x}, \eta)\begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

- Many factor model….infinite

$$df(t,x) = \partial_x f(t,x)dt + \sum_{n=1}^{\infty} \sigma_i(t,x)dB_i(t)$$

$$df(t) = \partial_x f(t)dt + \Sigma(t)dB(t) \rightarrow \in H$$

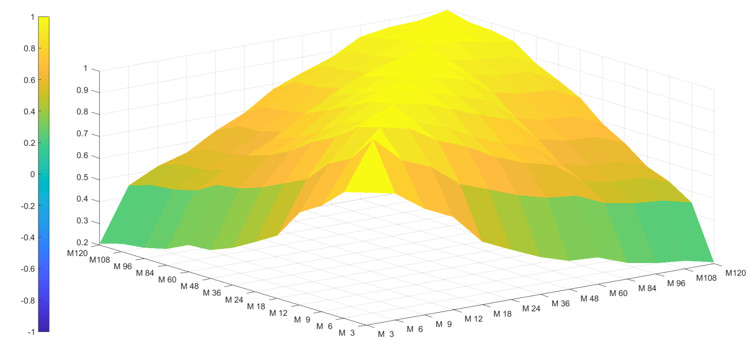
$$\downarrow$$

$$\in L(H)$$

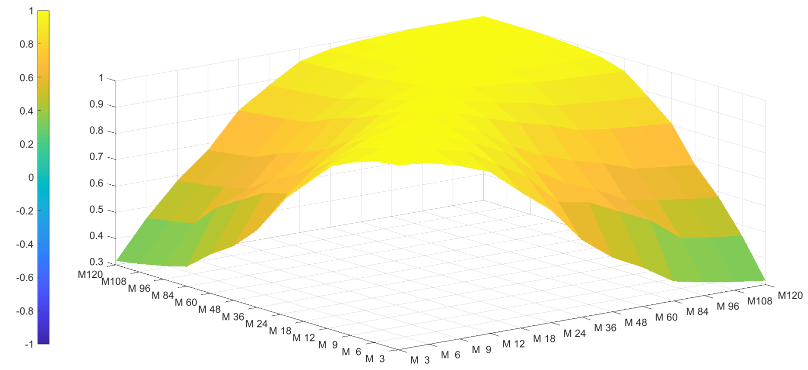
Why infinite-dimensional? A detour to US interest rates

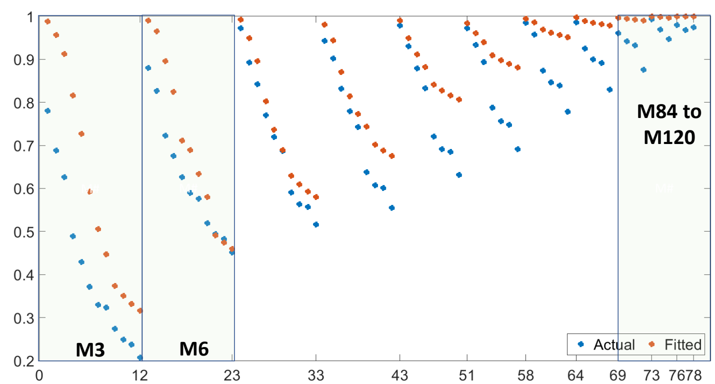
- US yield curves: 2000-2020
- Residuals from dynamic Nelson-Siegel
- Correlation as function rather than high-dimensional matrix

Correlation function



Fitted correlation function





On-going work with Marianna Russo (Trondheim) and Florentina Paraschiv (Zeppelin Uni)

Why infinite-dimensional?

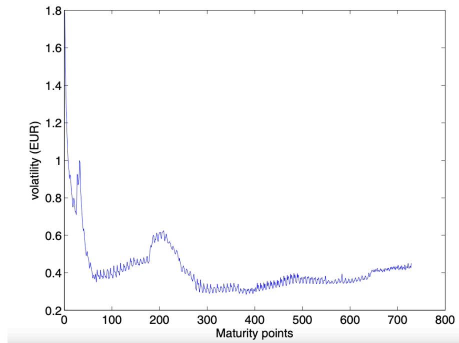
- Avoids perfect hedging of all contracts by a finite selection of maturities

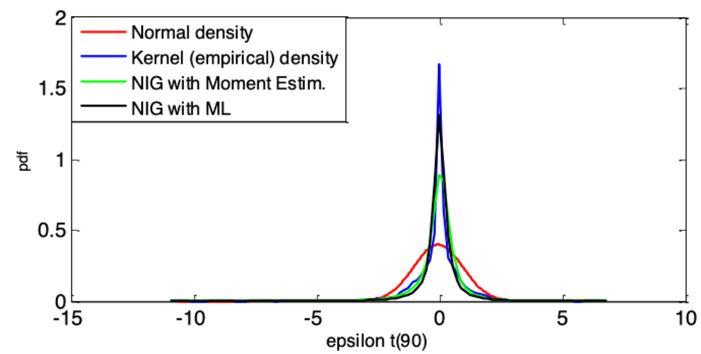
- Many factors require many parameters
  - Correlation functions may be low parametric
  - …previous slide: 2 parameters in fitted correlation function!

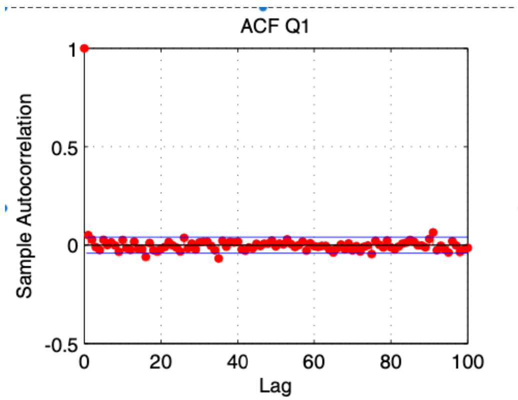
- Opens for flexible correlation modelling across maturities
  - Although maybe non-stationary

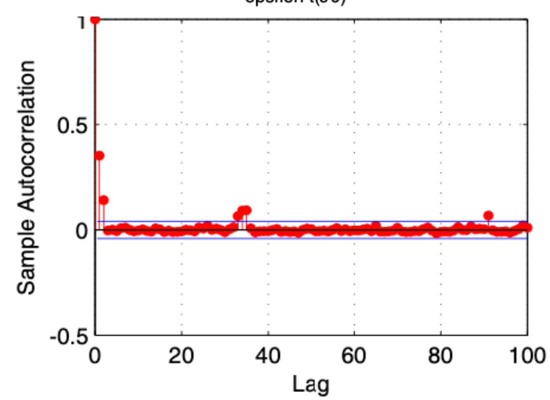
EEX quarterly power futures

- Evidence for Samuelson effect, heavy tails and stochastic volatility









From B., Paraschiv: JBF 2018

Two questions

- Q1: Specification of state space and correlation function?
  - …for the driving noise B
  - The Hilbert space determines the trace class/nuclear operators
  - It is expected that correlations are non-stationary in maturity
- Q2: How to define (stochastic) volatility operator?
  - Samuelson effect
  - Distributional properties of the returns

Q1: STATE SPACE AND COVARIANCE OPERATORS

Filipovic space

- H is space of absolutely continuous real-valued function on $\mathbb{R}_+$:

$$g : \mathbb{R}_+ \to \mathbb{R}$$

- Differentiable (weakly), finite in norm

$$|g|^2 = g(0)^2 + \int_0^\infty w(x) g'(x)^2 dx, \qquad w(0) = 1, \quad \text{increasing}$$

- Properties
  - Flat forward curves in long end: $g'(\infty) = 0$
  - Separable Hilbert space
  - Evaluation operator continuous linear functional: $e_x(g) = g(x)$
  - Shift semigroup is strongly continuous, generator is ∂_x

Covariance operator

- B is a Wiener process in H, with covariance operator Q

$$\mathbb{E}[\langle B_t, g\rangle\langle B_s, h\rangle] = (s \wedge t)\langle Qg, h\rangle$$

- Q is positive definite and of trace class

$$\langle Qg, g\rangle \geq 0, \qquad \mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \langle Qe_n, e_n\rangle = \sum_{n=1}^{\infty} \lambda_n < \infty$$

- Q can be represented by a covariance function

$$\mathbb{E}[e_x(B_t)e_y(B_t)] = t \cdot q(x, y)$$

Covariance function (or *correlation* by standardization)

- q is symmetric, but not stationary in its arguments

$$q(x,y) \neq \tilde{q}(|x-y|)$$

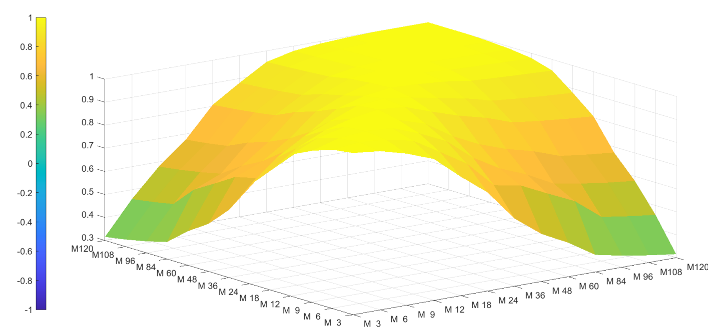
- Limit behaviour (fundamental theorem of calculus)

$$\lim_{x\to\infty} q(x+\Delta, x) = 1, \qquad \lim_{\Delta\to\infty} q(x+\Delta, x) > 0$$

- Example (US yields again…)

$$q(x,y) = \frac{c^2 + (1-e^{-\gamma x})(1-e^{-\gamma y})}{\sqrt{c^2 + (1-e^{-\gamma y})^2}\sqrt{c^2 + (1-e^{-\gamma y})^2}}$$

Fitted correlation function



Q2: ROUGH VOLATILITY IN INFINITE DIMENSIONS

- Recall set-up for forward dynamics

$$df(t) = \partial_x f(t)dt + \Sigma(t)dB(t)$$

- $f(t)$ is H-valued stochastic process, $B(t)$ is Wiener process in H with covariance operator Q_B

- Stochastic volatility process $\Sigma(t)$ with values in $L(H)$
  - Goal: to propose a “rough model” for it!

- Basic tool is the tensor product $\otimes$ (the matrix outer product in infinite dimensions):

$$f \otimes g \in L(H), \qquad (f \otimes g)(h) = \langle f, h \rangle g$$

Heston-type rough volatility model

- Fix a $z \in H, |z| = 1$

$$\Sigma(t) := Y(t) \otimes z, \qquad \Sigma(t)^*\Sigma(t) = Y^{\otimes 2}(t)$$

- Defined via Gaussian process Y with covariance operator $Q_Y(t, s)$

$$Q_Y(t, s) = \mathbb{E}[Y(t) \otimes Y(s)],$$
$$\mathbb{E}[\langle Y(t), g\rangle\langle Y(s), h\rangle] = \langle Q_Y(t, s)g, h\rangle = \langle g, Q_Y(s, t)h\rangle$$

Volatility scaling at each maturity

- Evaluation at maturity x:

$$U_t(x) := e_x \int_0^t \Sigma(s) dB(s)$$

- Instantaneous "volatility" from quadratic variation

$$\sigma_t^2(x) := \frac{d}{dt} \langle\langle U(x), U(x) \rangle\rangle_t = |Y(t,x)|^2 |Q_B^{1/2} z|_H^2$$

- Power scaling of the volatility determined by the regularity of Q_Y

$$\mathbb{E}[\sigma_t^{2k}(x)] \sim (e_x Q_Y(t,t) e_x^* 1)^k$$

Y-process: rough Ornstein-Uhlenbeck-process

- A fractional OU with fractional time derivative with $\alpha \in (0,1)$

$$D^{\alpha}(Y(t) - y) = AY(t) + W'(t)$$

- $A \in L(H)$, W is Gaussian process with γ-regular paths, $\gamma \in (0,1)$ and $\alpha + \gamma > 1$

$$Y(t) = y + I^{\alpha}(AY)(t) + \int_0^t (t-s)^{\alpha-1} dW(s)$$

- Riemann-Liouville fractional integration and differentiation
  - time-derivative in Frechet sense

$$D^{\alpha}(g)(t) = \frac{d}{dt} I^{\alpha-1}(g)(t), \qquad I^{\beta}(g)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \beta > 0$$

- Explicit $\rho-$regular solution, $\rho < \gamma + \alpha - 1$

$$Y(t) = E_{\alpha,1}(At^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha})dW(s)$$

- Mittag-Leffler operator

$$E_{\alpha,\beta}(Ar) := \sum_{k=0}^{\infty} \frac{A^k r^k}{\Gamma(\alpha k + \beta)}$$

- Covariance operator $Q_Y(t,s)$ explicitly known
  - Requires rough path integration (see B., Harang (2020))
  - Regularity inherited from the fractional derivative α and $Q_W(t,s)$

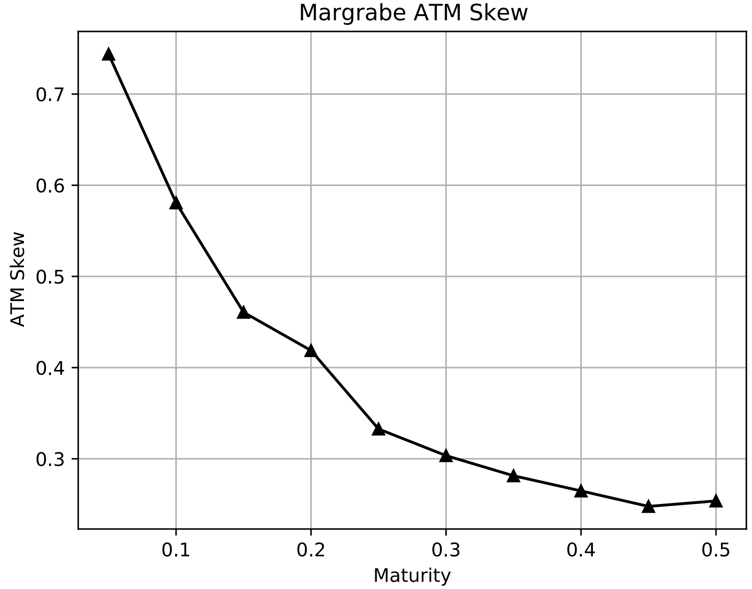
- If $Q_W(t,s)$ is $\beta \in (0,1)$-regular and $\beta + \alpha > 1$, then $Q_Y(t,s)$ is $\eta-$regular, $\eta < \beta + \alpha - 1$

- Regularity of covariance operator gives regularity of moments of vol!

$$\mathbb{E}[\sigma_t^{2k}(x)] \sim (e_x Q_Y(t,t) e_x^* 1)^k$$

- I.e., mix of regularity from
  - fractional differentiation α
  - ..and β from covariance of driving noise W (fractional Brownian motion, say)

Simulated ATM-skew for spread options (bivariate example)



Plot prepared by Alexander Lobbe (Oslo)

- Payoff $\max(S_1(T) - S_2(T), 0)$, GBMs with rough Bergomi model
  - All noises correlated
  - Hurst parameter 0.1 for both

Extension 1: Samuelson effect

- Multiplication operator on the Banach algebra H_w

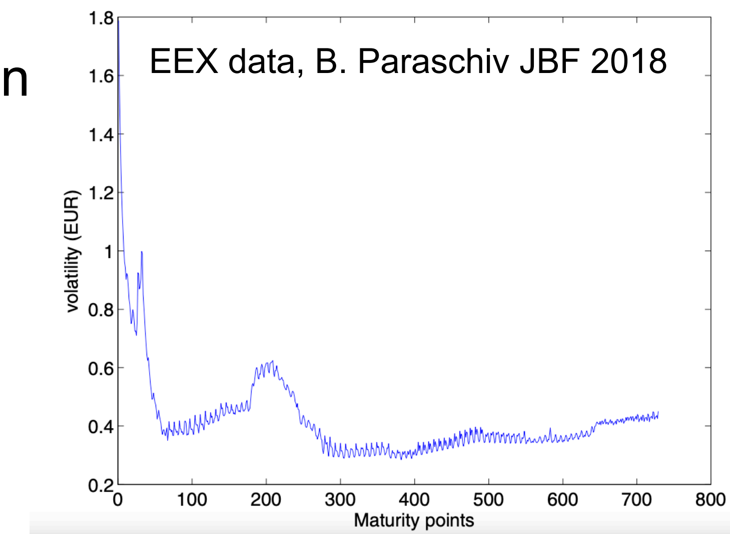
$$\mathcal{M}_g \in L(H_w), \quad \mathcal{M}_g h = g \cdot h$$

- Volatility term in forward dynamics

$$df(t) = \partial_x f(t)\, dt + \mathcal{M}_g \Sigma(t) dW(t)$$

- g can be an exponential function

$$g(x) = \exp(-\rho x)$$

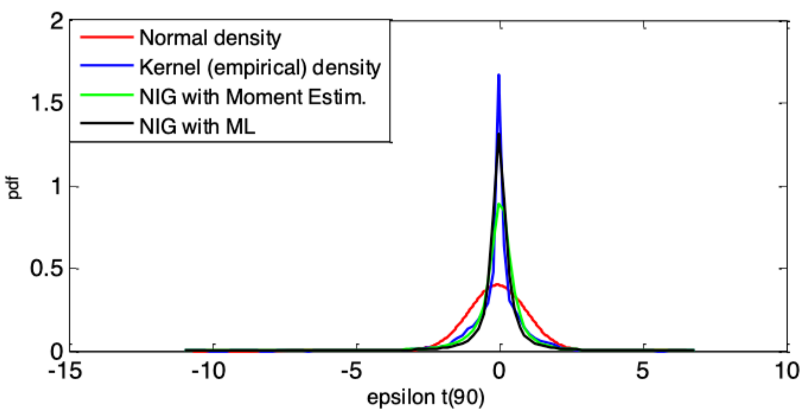


Extension 2: Heavy-tailed “returns” and NIG

- Change Gaussian assumption
  - Subordination of Wiener process W
  - Preserves the covariance operator of W

$$L(t) = W(U(t))$$

- Define normal inverse Gaussian (NIG) “returns”
  - Use inverse Gaussian subordinator



EEX data, B. Paraschiv JBF 2018

Conclusions

- Infinite-dimensional modeling of forward curves
  - Argued for high-dimensionality in HJM-models
  - State space being Filipovic space
  - Non-stationary covariance operator

- Rough volatility as fractional OU process in infinite dimensions
  - Samuelson effect
  - NIG

THANK YOU FOR LISTENING!

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