Network Dynamics and Strategic Preferential Attachment

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Based on recent work with :

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1 Motivations

Internet of Things/Agents (IoT/A)

- There are 3 characteristics that enable dense networks of connected objects :
  - Cheap network bandwidth
  - Cheap computing power
  - Cheap memory
- In fact, IoT is not possible without cheap computing power and memory $\implies$ connected devices need to be independent and strategic
- This is why cheap network bandwidth is also really important $\implies$ devices need to communicate and exchange data and/or value

$\implies$ What is the resulting dynamics of such graphs ?

The Lightning Network : An extreme example

- Fully decentralized payment network built on top of the Bitcoin blockchain
- Nodes in the network connect as they wish to pass on bitcoins without trusted third-parties
- Opening a payment channel (an edge in the graph) is costly but also rewarding

$\Longrightarrow$ Game theory aspects of network dynamics

Current Network Theory

Random Graph Theory

- Nodes in a graph create edges randomly
- Consider the mean field approximation of graph characteristics (ex: degree)
- Study the limit case $N \to \infty$, the degree distribution can be a power law which is very good for :
  - Social networks
  - WWW
  - Citations in scientific papers

Strategic Network Formation Models

- Study games in which players decide to connect to specific nodes
- Do not use the mean field approximation
- Highly intractable

$\Longrightarrow$ Insights from the MFG literature allow to fill the gap

2 Random Network Theory

The Roots of Random Graph Theory

Erdõs-Rényi (1960)

- Graphs can be understood as a set of nodes (N) randomly creating edges (E)
- Study the properties of the limit graph ($N \to \infty$)
- Probabilistic view : How likely is it that a graph will have a property Q given the edge probability between 2 nodes : p (the same for every edge)
- The degree distribution is a Poisson Law (not realistic for several real-life situations)

Preferential Attachment (1)

$\Longrightarrow$ Barabasi and Albert (1999)

- Random graph generation process
- Model the dynamics of the generation process of a graph by reducing it to the dynamics of a mean-field
- Each unit of time (t), a new node enters the graph and creates m edges
- The likelihood that this new node i connects to node j is proportional to the degree of node j : k_j
- Considering the mean field approximation for the degree, we have

$$\dot{k}_i(t) = c(t)\frac{k_i(t)}{\sum_j k_j(t)}$$

with $c(t)$ an adjustment term.

Preferential Attachment (2)

- Consider the degree distribution $\mu(t, k)$
- Let's define

$$\phi(t, k, \mu) = c(t) \frac{k}{\int_0^\infty k' \mu(t, k') dk'}$$

- We can rewrite the degree dynamics as

$$dk = \phi(t, k, \mu) dt$$

- Which yields the Kolmogorov equation

$$\frac{\partial \mu}{\partial t} = -\frac{\partial}{\partial k} [\phi(t, k, \mu) \mu(t, k)] \text{ in } (0, \infty)^2$$

Preferential Attachment (3)

Constant deterministic case

- Assume we have $\phi(t, k, \mu) = c_0 k$, we obtain

$$dk = c_0 k dt$$

- And the associated Kolmogorov equation

$$\frac{\partial \mu}{\partial t} = -c_0 \frac{\partial}{\partial k}(k\mu)$$

- Pareto distributions are solution of this equation : assume $\mu(t, k) = \frac{a(t)}{k^\alpha}$, we obtain

$$\dot{a}(t) = -c_0(1-\alpha)a(t) \quad \Longrightarrow \quad a(t) = Ce^{-c_0(1-\alpha)t}$$

- This yields a power law distribution (very typical of many networks)

$$\mu(t, k) = \frac{Ce^{-c_0(1-\alpha)t}}{k^\alpha}$$

3 Strategic Preferential Attachment

Strategic Interactions

- Instead of random, the new links are now the result of a strategic behavior
- Flow of future gains is given by

$$\int_0^\infty e^{-rt} F(k_i(t), \mu(t)) dt$$

  with $\mu(t)$ the degree distribution at time t and F a given function that can depend non-locally on $\mu(t)$
- A node can create links at rate q in which case it pays

$$C(q, k, \mu)$$

Degree Dynamics

- Preferential attachment + new strategic behavior

$$dk_t^i = c(t)\frac{k_t^i}{\int_0^\infty k'\mu(t,k')dk'}dt + q_t^i dt$$

- Nodes can create connections so the rate at which other nodes make connections is also going to affect the degree of node i
- Assume all nodes with degree k behave the same way : $q_t(k)$, we have

$$dk_t^i = c(t)k_t^i\frac{\int_0^\infty q_t(k')\mu(t,k')dk'}{\int_0^\infty k'\mu(t,k')dk'}dt + q_t^i dt \qquad (1)$$

Equilibrium (1)

- Let's define the value function as

$$u(t,\kappa^i) = \max_{(q_s^i)} \mathbb{E}\left[\int_t^{\infty} e^{-r(s-t)} F(k_s^i,\mu(s)) - C(q_s^i,k_s^i,\mu(s))ds\right]$$

where $(k_s^i)_{s\geq 0}$ solves (1) subject to $k_t^i = \kappa^i$

- This value function is the solution to the following HJB equation

$$-\partial_t u + ru + H(k,\partial_k u,\mu(t)) - L(k,\mu(t),q_t)\partial_k u = F(k,\mu(t)) \text{ in } (0,\infty)^2$$

with

$$H(k,p,\mu) = \inf_{q^i\geq 0}\left\{C(q^i,k,\mu) - q^i p\right\}$$

$$L(k,\mu,\nu) = k\frac{\int_0^{\infty}\nu(k')\mu(k')dk'}{\int_0^{\infty}k'\mu(k')dk'}$$

Equilibrium (2)

- Assuming there is a function $q^*(t,k,\mu)$ that gives the control of each node as a function of t, k and μ only, the degree distribution evolves according to

$$\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial k}(\mu(t,k)(q^*(t,k,\mu) + L(k,\mu,q^*(t,\cdot,\mu)))) = 0 \text{ in } (0,\infty)^2$$

Nash Equilibria

- Nash equilibria are characterized by finding a couple (u,μ) solution of

$$\begin{aligned}-\partial_t u + ru + H(k, \partial_k u, \mu(t))\\ - L(k, \mu(t), D_p H(\cdot, \partial_k u(t,\cdot), \mu(t)))\partial_k u = F(k,\mu(t)) \text{ in } (0,\infty)^2\end{aligned}$$

$$\begin{aligned}\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial k}((D_p H(k, \partial_k u, \mu(t))\\ + L(k, \mu(t), D_p H(\cdot, \partial_k u(t,\cdot), \mu(t))))\mu) = 0 \text{ in } (0,\infty)^2\end{aligned}$$

4 Conclusion

Conclusion

- Strategic Preferential Attachment bridges the gap between Random Graph Theory and Strategic Network Formation
- We can incorporate many graph characteristics beside degree
- Already some interesting use cases :
  - Internet of Things
  - Decentralized Payment Network
- Computation is challenging but we are also working on extending standard numerical techniques to solve for those equilibria

Thank You !

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