A rough volatility tour from market microstructure to VIX options *via* Heston and Zumbach

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Table of contents

Table of contents

Starting point : Volatility is rough !

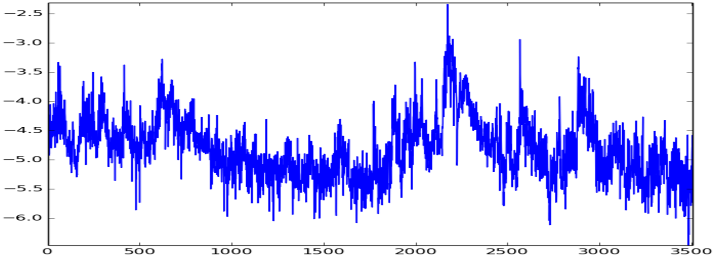

FIGURE – The log volatility of the S&P over about 10 years.

Our goal

Understanding volatility

- It is shown in Gatheral *et al.* that log-volatility time series behave like a fractional Brownian motion, with **Hurst parameter** H **of order** 0.1.
- More precisely, basically all the statistical stylized facts of volatility are retrieved when modeling it by a rough fractional Brownian motion.
- This leads to very accurate volatility forecasts.
- Such models also enable us to reproduce very well the behavior of the implied volatility surface, in particular the ATM skew (without jumps).
- **This phenomenon is universal** : Shown on more than 10.000 assets.
- Microstructural foundations for rough volatility : well understood (see next section). We also want to consider a particularly subtle property of rough volatility : the **Zumbach effect**.
- Rough volatility and SPX options : well understood. Here we focus on **VIX options** and associated conjectures.

Literature

Rough volatility network

- https ://sites.google.com/site/roughvol/
- Forthcoming book

Fractional Brownian motion

Definition

The fractional Brownian motion (fBm) with Hurst parameter H is the only process W^H to satisfy :

- Self-similarity : $(W^H_{at}) \stackrel{\mathcal{L}}{=} a^H (W^H_t)$.
- Stationary increments : $(W^H_{t+h} - W^H_t) \stackrel{\mathcal{L}}{=} (W^H_h)$.
- Gaussian process with $\mathbb{E}[W^H_1] = 0$ and $\mathbb{E}[(W^H_1)^2] = 1$.

Proposition

For all $\varepsilon > 0$, W^H is $(H - \varepsilon)$-Hölder a.s.

Mandelbrot-van Ness representation

$$W^H_t = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

Table of contents

Building the model

Necessary conditions for a good microscopic price model

We want :

- A tick-by-tick model.
- A model reproducing the stylized facts of modern electronic markets in the context of high frequency trading.
- A model helping us to understand the rough dynamic of volatility from the high frequency behavior of market participants.

Building the model

Stylized facts 1-2

- Markets are highly endogenous, meaning that most of the orders have no real economic motivations but are rather sent by algorithms in reaction to other orders, see Bouchaud *et al.*, Filimonov and Sornette.
- Mechanisms preventing statistical arbitrages take place on high frequency markets, meaning that at the high frequency scale, building strategies that are on average profitable is hardly possible.

Building the model

Stylized facts 3-4

- There is some asymmetry in the liquidity on the bid and ask sides of the order book. In particular, a market maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order.
- A large proportion of transactions is due to large orders, called metaorders, which are not executed at once but split in time.

Building the model

Hawkes processes

- Our tick-by-tick price model is based on Hawkes processes in dimension two.
- A two-dimensional Hawkes process is a bivariate point process $(N_t^+, N_t^-)_{t\geq 0}$ taking values in $(\mathbb{R}^+)^2$ and with intensity $(\lambda_t^+, \lambda_t^-)$ of the form :

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \varphi_1(t-s) & \varphi_3(t-s) \\ \varphi_2(t-s) & \varphi_4(t-s) \end{pmatrix} . \begin{pmatrix} dN_s^+ \\ dN_s^- \end{pmatrix} .$$

Building the model

The microscopic price model

- Our model is simply given by

$$P_t = N_t^+ - N_t^- .$$

- N_t^+ corresponds to the number of upward jumps of the asset in the time interval $[0, t]$ and N_t^- to the number of downward jumps. Hence, the instantaneous probability to get an upward (downward) jump depends on the location in time of the past upward and downward jumps.
- By construction, the price process lives on a discrete grid.
- Statistical properties of this model have been studied in details.

Encoding the stylized facts

The right parametrization of the model

- Recall that

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}.$$

- High degree of endogeneity of the market$\rightarrow$ L^1 norm of the largest eigenvalue of the kernel matrix close to one (nearly unstable regime).
- No arbitrage$\rightarrow$ $\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$.
- Liquidity asymmetry$\rightarrow$ $\varphi_3 = \beta\varphi_2$, with $\beta > 1$.
- Metaorders splitting$\rightarrow$ $\varphi_1(x),\ \varphi_2(x) \underset{x\to\infty}{\sim} K/x^{1+\alpha}$, $\alpha \approx 0.6$.

Rough Heston model

Limit theorem

After suitable scaling in time and space, the long term limit of our price model satisfies the following **rough Heston** dynamics :

$$P_t = \int_0^t \sqrt{V_s} dW_s - \frac{1}{2} \int_0^t V_s ds,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda \nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with

$$d\langle W, B \rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}} dt.$$

The Hurst parameter H satisfies $H = \alpha - 1/2$.

An even deeper result

No-arbitrage implies rough volatility and power law market impact

- We have shown that combining typical behaviours of market participants at the high frequency scale automatically generates rough volatility.
- We can actually prove that only assuming no-statistical arbitrage implies rough volatility.
- The key phenomenon to obtain this result is the **market impact**.
- In a perfect market from a statistical arbitrage viewpoint, $H = 0$.
- There is a one to one connection between the value of H and the shape of the market impact curve.
- $H = 0$ corresponds to square-root market impact.

Table of contents

Deriving the characteristic function of the rough Heston model

Strategy

- From our last theorem, we are able to derive the characteristic function of our high frequency Hawkes-based price model.
- We then pass to the limit.

Characteristic function of rough Heston models

We write :

$$I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{f(t)}{(x-t)^\alpha}dt, \;\; D^\alpha f(x) = \frac{d}{dx}I^{1-\alpha}f(x).$$

Theorem

The characteristic function at time t for the rough Heston model is given by

$$\exp\Big(\int_0^t g(a,s)ds + \frac{V_0}{\theta\lambda}I^{1-\alpha}g(a,t)\Big),$$

with $g(a,)$ the unique solution of the fractional Riccati equation :

$$D^\alpha g(a,s) = \frac{\lambda\theta}{2}(-a^2-ia) + \lambda(ia\rho\nu-1)g(a,s) + \frac{\lambda\nu^2}{2\theta}g^2(a,s).$$

Comments on the theorem

The rough Heston formula

- The formula is the very same as the celebrated Heston formula, up to the replacement of a classical time derivative by a fractional derivative.
- This formula allows for fast derivatives pricing and risk management.
- Thanks to this approach, we can derive the infinite dimensional Markovian structure underlying rough Heston models, leading to explicit hedging formulas.

Table of contents

Further aspects of volatility

Super-Heston rough volatility and Zumbach effect

- All the works on microstructural foundations of rough volatility have produced rough Heston type models.
- In the context of rough models, there are other aspects of volatility that one could wish to understand from a microstructural perspective.
- Going beyond the square root associated to the dynamic of the volatility in the rough Heston model $\rightarrow$ additional additive or multiplicative factor leading to fatter volatility tails : *Super-Heston rough volatility*.
- *Zumbach effect*.

Further aspects of volatility

Zumbach effect (Zumbach *et al.*) : description

- Feedback of price returns on volatility.
- Price trends induce an increase of volatility.
- In the literature (notably works by J.P. Bouchaud and co-authors), a way to reinterpret the Zumbach effect is to consider that the predictive power of past squared returns on future volatility is stronger than that of past volatility on future squared returns.
- To check this on data, one typically shows that the covariance between past squared price returns and future realized volatility (over a given duration) is larger than that between past realized volatility and future squared price returns.
- We refer to this version of Zumbach effect as *weak Zumbach effect*.

Further aspects of volatility

Weak and strong Zumbach effect

- It is shown in Gatheral *et al.* that the rough Heston model reproduces the weak form of Zumbach effect.
- However, it is not obtained through feedback effect, which is the motivating phenomenon in the original paper by Zumbach. It is only due to the dependence between price and volatility induced by the correlation of the Brownian motions driving their dynamics.
- In particular in the rough Heston model, the conditional law of the volatility depends on the past dynamic of the price only through the past volatility.
- We speak about *strong Zumbach effect* when the conditional law of future volatility depends not only on past volatility trajectory but also on past returns.

A convenient microscopic model encoding Zumbach effect

Quadratic Hawkes processes

- Inspired by Blanc *et al.*, we model high frequency prices using quadratic Hawkes processes.
- Jump sizes of the price P_t are i.i.d taking values -1 and 1 with probability $1/2$ and jump times are those of a point process N_t with intensity

$$\lambda_t = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s + Z_t^2, \text{ with } Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

- The component Z_t is a moving average of past returns.
- If the price has been trending in the past, Z_t is large leading to high intensity. On the contrary if it has been oscillating, Z_t is close to zero and there is no feedback from the returns on the volatility. So Z_t is a (strong) Zumbach term.

Scaling limits

Purely quadratic case

- When $\phi = 0$ is equal to zero, choosing appropriate scaling parameters, we obtain the following limiting model : $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t$ with

$$V_t = \mu + Z_t^2, \quad Z_t = \sqrt{\gamma}\int_0^t k(t-s)\mathrm{d}\hat{P}_s.$$

- The strong Zumbach effect is naturally encoded since the volatility is a functional of past price returns through Z.
- We can rigorously show that conditional on the history of the market from time 0 to t_0, the law of the volatility for $t \geq t_0$ does depend on past returns and not only through past volatility.

Scaling limits

Purely quadratic case (2)

- The quadratic feedback of price returns on volatility implies that V_t is of super-Heston type (essentially log-normal here).
- This can be seen for example when $\mu = 0$:

$$Z_t = \sqrt{\gamma} \int_0^t k(t-s)|Z_s| \mathrm{d}B_s.$$

- Taking for example k fractional kernel, we get that volatility has Hölder regularity $H - \varepsilon$.
- From a natural microscopic dynamic, we obtain a super-Heston rough volatility model with strong Zumbach effect.
- We obtain more complex log-normal type rough volatility models in the other regimes.

Quadratic rough Heston model

One particular super-Heston rough volatility model

- We consider

$$\mathrm{d}S_t = S_t\sqrt{V_t}dW_t,\ \ V_t = a(Z_t-b)^2+c,$$

where a, b and c some positive constants and Z_t follows

$$Z_t = \int_0^t f^{\alpha,\lambda}(t-s)\theta_0(s)\mathrm{d}s + \int_0^t f^{\alpha,\lambda}(t-s)\sqrt{V_s}\mathrm{d}W_s,$$

with $\alpha \in (1/2, 1)$, $\lambda > 0$ and θ_0 a deterministic function.

- Z_t is *path-dependent* : a weighted average of past returns.
- c : minimal instantaneous variance.
- $b > 0$: asymmetry of the feedback effect.
- a : sensitivity of the volatility feedback.

Table of contents

The Volatility Index

Definition of the VIX

- Introduced in 1993 by the CBOE.
- VIX is the square root of the price of a specific basket of options on the S&P 500 Index (SPX) with maturity $\Delta = 30$ days such that

$$\mathrm{VIX}_t = -\frac{2}{\Delta}\sqrt{\mathbb{E}[\log(S_{t+\Delta}/S_t)|\mathcal{F}_t]} \times 100,$$

  with S the SPX index.
- VIX futures and VIX options exist.

The joint calibration problem

VIX options

- More than 500,000 VIX options traded each day.
- Quite wide spreads for VIX options : non-mature market.
- VIX is by definition a derivative of the SPX, any reasonable methodology must necessarily be consistent with the pricing of SPX options.
- Designing a model that jointly calibrates SPX and VIX options prices is known to be extremely challenging.
- This problem is sometimes considered to be *the holy grail of volatility modeling*.
- We simply refer to it as the *joint calibration problem*.

The joint calibration problem

Attempts to solve the joint calibration problem

- Theoretical approch by J. Guyon : the joint calibration problem is interpreted as a model-free constrained martingale transport problem. Perfect calibration of VIX options smile at time T_1 and SPX options smiles at T_1 and $T_2 = T_1 + 30$ days. Hard to be extended to any set of maturities and high computational cost.
- Models with jumps : most of them fail to reproduce VIX smiles for maturities shorter than one month.
- Continuous models : Unsuccessful so far. Interpretation : the very large negative skew of short-term SPX options, which in continuous models implies a very large volatility of volatility, seems inconsistent with the comparatively low levels of VIX implied volatilities

The VIX conjecture

The joint calibration problem and continuous models

- "So far all the attempts at solving the joint SPX/VIX smile calibration problem [using a continuous time model] only produced imperfect, approximate fits".
- "Joint calibration seems out of the reach of continuous-time models with continuous SPX paths".
- Investigating Guyon's work one can realise the following : a necessary condition for a continuous model to fit simultaneously SPX and VIX smiles is the inversion of convex ordering between volatility and the local volatility implied by option prices.
- The intuition behind this condition could be reinterpreted as some kind of strong Zumbach effect.
- Natural for us to investigate the ability of super-Heston rough volatility models to solve the joint calibration problem.

Calibration for one day in history 19 May 2017

Parameters calibration with Deep Learning

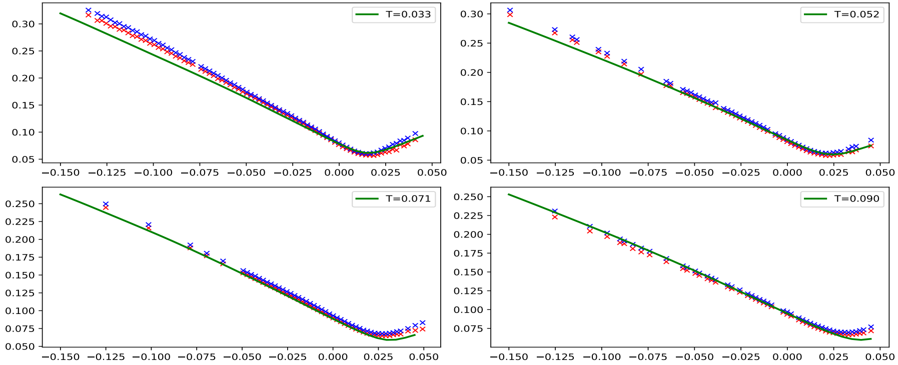

FIGURE – Implied volatility on SPX options for 19 May 2017. Blue and red points are bid and ask of market implied volatilities. Model implied volatility smiles from the model are in green. Strikes are in log-moneyness, maturity in year.

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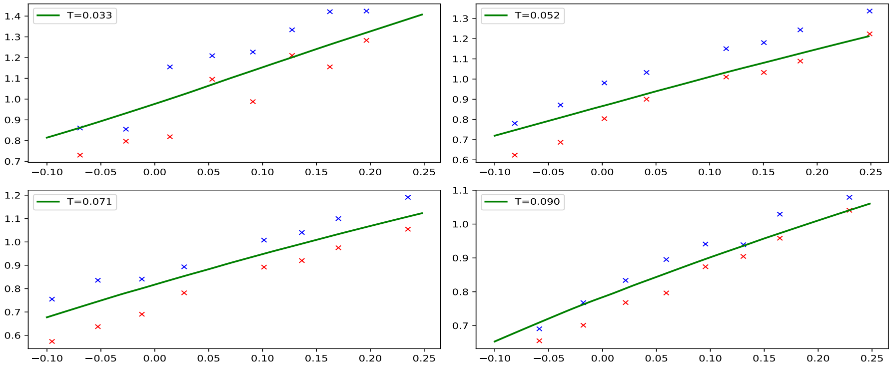

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Take home message for the joint calibration problem

Thanks to the quadratic rough Heston model

- 6 parameters.
- VIX smiles in the bid-ask spread.
- Global shape of the implied volatility surface of the SPX very well reproduced
- Very accurate SPX skews of orders -1.5 (shortest maturites), -1 (longer maturities), as for market data.