Optimal bidding strategies for digital advertising with social interactions

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FINANCE FOR ENERGY
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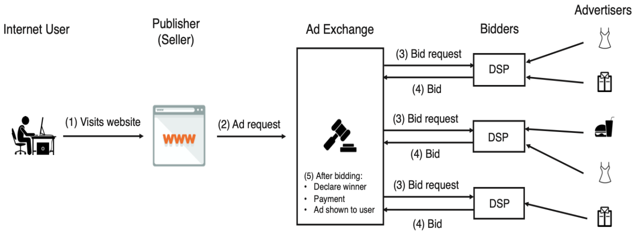
Digital advertising and real-time bidding

- Targeted advertising vs traditional advertising (newspapers, TV, billboards, etc)
- Companies/advertisers can minimise ad costs by targeting directly individuals/users potentially interested by their product/service



- Auctions for Ad display (in milliseconds):
  - Ad exchange sends data to advertisers about page content and user's profile
  - Advertisers place bids for ad display (impression) by publisher to a given user
  - The highest bidder wins the ad space

Literature on advertising models

- Classical approach: modelling of macroscopic variables, e.g. sales process, affected by (traditional) advertising expenditure
  - Vidale, Wolfe (57), Nerlove, Arrow (62)
  - Sethi advertising models: Sethi and collaborators (1973-2020)
- Auction for digital advertising: Levin, Milgrom (10), Goke et al (21)
- Optimisation in digital advertising:
  - Supply side (publisher) perspective: Balseiro et al. (14,15), Yuan (14,15)
  - Demand side (bidders) perspective:
    - discrete time and MDP models: Amin et al. (12), Tillberg et al. (20)
    - stochastic control in continuous-time and HJB equation: Fernandez-Tapia, Guéant, Lasry (17): maximisation of number of banners displayed

Our purpose and basic framework

We address the following problem from the demand-side perspective:

- Agent **A** (company/association) willing to spread Information **I** to Users, e.g.
  - the existence of a new product, a new service: **commercial advertising**
  - the danger of some behaviour (drug, virus, etc): **social marketing**

- **Impression** → **Click/Conversion**: Once they get the information **I**, Users can decide to make an action, e.g.
  - purchase of the new product, subscribe to the new service
  - stop behaving unsafely

▸ **Attribution problem**: how to efficiently diffuse **I** by means of "modern" online channels (digital ad, social networks, etc) in order to generate conversion?

- We propose a continuous-time model for optimal digital advertising strategies:
  - online behaviours of users, social interactions: microscopic modelling of the population
  - advertising auctions

Main findings

- Semi-explicit formulae for optimal bidding:
  - Understand and measure impact of online behaviours and information channels (in addition to auction bids) on ad strategies
  - Role of social interactions among the population of users
  - Quantitative comparison between targeted vs non-targeted advertising

Outline

Online behaviour of User

The User can connect at any (random) time to:

- Website providing the Information (e.g. company own website):

$$N^{\mathbf{I}} \text{ Poisson process with intensity } \eta^{\mathbf{I}}: \text{ number of connections}$$

- Publisher **T** (social networks, search engine) not containing a priori **I** but displaying Targeted ad

$$N^{\mathbf{T}} \text{ Poisson process with intensity } \eta^{\mathbf{T}}: \text{ number of connections.}$$

→ Data collected by Ad exchange and sent to Advertisers

Targeted ad auctions and bidding strategies

• Targeted ad auction: each time the User connects to a Publisher displaying targeted ads, advertisers compete to win the right to display their ad to him.

▸ Model the maximal bid made by other bidders (other than the agent **A**):

$$B_k^{\mathbf{T}} : \text{ maximal bid of other bidders during the } k\text{-th ad auction}$$

We assume that $B_k^{\mathbf{T}}, k \in \mathbb{N}$ are i.i.d. nonnegative r.v.

• Bidding strategies for Agent **A**. Non anticipative $\mathbb{R}_+$-valued process $\beta = (\beta_t)_{t\geq 0}$:

$$\beta_t : \text{ bid that } \mathbf{A} \text{ makes if user is connecting to a Publisher at time } t$$

predictable w.r.t. to data information: $\sigma\{N_s^{\mathbf{I}}, N_s^{\mathbf{T}}, B_{N_s^{\mathbf{T}}}^{\mathbf{T}}, s \leq t\}$.

▸ Agent **A** wins bid at time $t \leftrightarrow N_t^{\mathbf{T}}$-ad auction, if:

$$\beta_t \quad \geq \quad B_{N_t^{\mathbf{T}}}^{\mathbf{T}}.$$

Conversion dynamic of the User

- Conversion state in $\{0,1\}$:
  - $x = 0$: user not aware of $\mathbf{I}$
  - $x = 1$: user aware of $\mathbf{I}$ and clicks for conversion

His conversion state $X = X^\beta$ is affected by the bidding strategy of Agent:

$$\begin{aligned} X^\beta_{0^-} &= 0, \\ dX^\beta_t &= (1 - X^\beta_{t^-})\left(dN^{\mathbf{I}}_t + \mathbf{1}_{\beta_t \geq B_{N^{\mathbf{T}}_t}} dN^{\mathbf{T}}_t\right), \quad t \geq 0. \end{aligned}$$

Conversion dynamic of the User

- Conversion state in $\{0, 1\}$:
  - $x = 0$: user not aware of $\mathbf{I}$
  - $x = 1$: user aware of $\mathbf{I}$ and clicks for conversion

His conversion state $X = X^\beta$ is affected by the bidding strategy of Agent:

$$
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\end{aligned}
$$

- Assumption of spontaneous **click/conversion**: once the user gets information, he purchases the product
  - In reality, multi-stage process before a purchase decision: **conversion funnel**, see Abhisek et al (12), Jordan et al. (12), Berman (18)
  - A simplified modeling of conversion funnel can be considered here by replacing $\eta^{\mathbf{T}}$ by $\eta^{\mathbf{T}} \times q^{\mathbf{T}}$, where $q^{\mathbf{T}}$ is the probability of conversion when seeing the ad (idem for $\eta^{\mathbf{I}} \leftrightarrow \eta^{\mathbf{I}} \times q^{\mathbf{I}}$)

Optimal bidding problem for pay-per-conversion Agent

Maximise over bidding strategies β the **purchase-based gain function**:

$$V(\beta) \quad = \quad \mathbb{E}\Big[\int_0^\infty e^{-\rho t} K dX_t^\beta \Big] - C(\beta),$$

where $\rho \geq 0$ is a discount rate, K is the punctual payment from the User to the Agent when he gets informed and clicks/converts, and $C(\beta)$ is the **ad cost**:

$$C(\beta) = \mathbb{E}\Big[\int_0^\infty e^{-\rho t} \mathbf{1}_{\beta_t \geq B^{\mathbf{T}}_{N_t^{\mathbf{T}}}} \mathbf{c}(\beta_t, B^{\mathbf{T}}_{N_t^{\mathbf{T}}}) dX_t^\beta dN_t^{\mathbf{T}}) \Big].$$

where $\mathbf{c}$ is the **paying rule of the auction**:

- *First-price auction*: $\mathbf{c}(b, B) = b$
- *Second-price (Vickrey) auction*: $\mathbf{c}(b, B) = B$.

Explicit solution

$$V^\star := \sup_\beta V(\beta) \quad = \quad \sup_{b\in\mathbb{R}_+} V(\beta^b),$$

where β^b is the constant bidding strategy: $\beta^b_t = (1 - X^{\beta^b}_{t^-})b$, with gain function:

$$V(\beta^b) \quad = \quad \frac{\eta^{\mathbf{I}} K + \eta^{\mathbf{T}}\mathbb{E}\Big[\big(K - \mathbf{c}(b, B^{\mathbf{T}}_1)\big)\mathbf{1}_{b\geq B^{\mathbf{T}}_1}\Big]}{\eta^{\mathbf{I}} + \rho + \eta^{\mathbf{T}}\mathbb{P}[b \geq B^{\mathbf{T}}_1]}, \quad b \in \mathbb{R}_+.$$

Furthermore, any $b_\star \in \operatorname*{argmax}_{b\in\mathbb{R}_+} V(\beta^b)$ yields an optimal constant bidding strategy $\beta^{b_\star}$.

Properties of the solution

Monotonicity w.r.t. parameters

- $V^\star$ is increasing w.r.t. $\eta^{\mathsf{I}}, \eta^{\mathsf{T}}$, and decreasing in ρ
- The smallest optimal bid policy $\underline{\mathrm{b}}_\star = \min \operatorname*{argmax}_{b \in \mathbb{R}_+} V(\beta^b)$ is decreasing w.r.t. $\eta^{\mathsf{I}}, \eta^{\mathsf{T}}$, and increasing in ρ

Upper bound on optimal bids

$$\underline{\mathrm{b}}_\star \quad \leq \quad K - V^\star \leq \frac{\rho K}{\eta^{\mathsf{I}} + \rho}.$$

Outline

Welfare purpose

• Population of M users behaving unsafely (**D** danger!) over time

$N^{m,\mathbf{D}}$ Poisson process with intensity $1:$ counting the times of **D** of $m \in [\![1, M]\!]$

→ This incurs a cost K to Agent **A** (association) as long as population is in **D**

• **A** willing to alert population of users about **D** (and how to protect against) so that once they get the information **I** and are converted:

- stop behaving unsafely (no more in **D**)

→ Cancels the cost for **A**.

Online behaviour of the population → information channels

- Any user $m \in [\![1, M]\!]$ can browse through
  - Website providing the information $\mathbf{I}$:

$$N^{m,\mathbf{I}} \text{ Poisson process with intensity } \eta^{\mathbf{I}} : \text{ number of connections of user } m$$

  - Publisher $\mathbf{T}$ (search engine) displaying targeted ad:

$$N^{m,\mathbf{T}} \text{ Poisson process with intensity } \eta^{\mathbf{T}} : \text{ number of connections of user } m$$

  - Platform $\mathbf{NT}$ displaying non-targeted ad:

$$N^{m,\mathbf{NT}} \text{ Poisson process with intensity } \eta^{\mathbf{NT}} : \text{ number of connections of user } m$$

$$N^{\mathbf{NT}} := \sum_{m=1}^{M} N^{m,\mathbf{NT}} : \text{ total number of connections to } \mathbf{NT} \text{ of the population}$$

- Social interactions. $N^{m,i,\mathbf{S}}$ Poisson process with intensity $\eta^{\mathbf{S}}$: counting the social interactions between users m and i.

- $(N^{m,\mathbf{D}}, N^{m,\mathbf{I}}, N^{m,\mathbf{T}}, N^{m,\mathbf{NT}}, N^{m,i,\mathbf{S}})$, $m, i = 1, \ldots, M$, are independent

Targeted and non-targeted ad auctions

• Targeted ad auction: each time User m connects to a Publisher displaying targeted ads, advertisers compete to win the right to display their ad.

▸ Model the maximal bid made by other bidders (other than the agent **A**):

$B_k^{m,\mathbf{T}}$: maximal bid of other bidders during the k-th ad auction for user m

We assume that $B_k^{m,\mathbf{T}}$, $k \in \mathbb{N}, m \in [\![1, M]\!]$, are i.i.d. nonnegative r.v.

• Non-Targeted ad auction: Bids are indifferent w.r.t. users of the population

▸ Model the maximal bid made by other bidders (other than the agent **A**):

$B_k^{\mathbf{NT}}$: maximal bid of other bidders during the k-th ad auction for any user

We assume that $B_k^{\mathbf{NT}}$, $k \in \mathbb{N}$ are i.i.d. nonnegative r.v., and independent of $(B_k^{m,\mathbf{T}})_{k,m}$.

Advertising bidding map strategies

Non-anticipative process $\beta = \{(\beta_t^m)_{m=0,\dots,M}, t \geq 0\}$ valued in $\mathbb{R}_+^{M+1}$:

- β_t^0 is the bid that **A** makes when any user is connecting to the Platform **NT**
- β_t^m, $m = 1, \dots, M$, is the bid that **A** makes if user m is connecting to a Publisher **T** at time t

Conversion dynamic of the population of users

Conversion state $X^{m,\beta}$ in $\{0,1\}$ of user $m \in [\![1, M]\!]$ influenced by the bidding map strategy of Agent, and the other users (social interaction):

$$\begin{cases} X_{0^-}^{m,\beta} & = \; 0, \\ dX_t^{m,\beta} & = \; (1 - X_{t-}^{m,\beta})\Big[dN_t^{m,\mathbf{I}} \; + \; \mathbf{1}_{\beta_t^m \geq B^{m,\mathbf{T}}_{N_t^{m,\mathbf{T}}}} dN_t^{m,\mathbf{T}} \\ & \qquad + \; \mathbf{1}_{\beta_t^0 \geq B^{\mathbf{NT}}_{N_t^{\mathbf{NT}}}} dN_t^{m,\mathbf{NT}} \; + \; \sum_{i \neq m} X_{t-}^{i,\beta} dN_t^{m,i,\mathbf{S}} \Big], \quad t \geq 0. \end{cases}$$

Optimal bidding problem for Agent

Minimize over bidding map strategies $\beta = (\beta^m)_{m \in [\![0,M]\!]}$ the cost function:

$$V(\beta) = \sum_{m=1}^{M} \mathbb{E}\Big[\int_0^\infty K(1 - X_{t-}^{m,\beta}) dN_t^{m,\mathbf{D}} + \int_0^\infty \mathbf{1}_{\beta_t^m \geq B^{m,\mathbf{T}}_{N_t^{i},\mathbf{T}}} \mathbf{c}(\beta_t^m, B^{m,\mathbf{T}}_{N_t^{m},\mathbf{T}}) dN_t^{m,\mathbf{T}}$$
$$+ \int_0^\infty \mathbf{1}_{\beta_t^0 \geq B^{\mathbf{NT}}_{N_t^{\mathbf{NT}}}} \mathbf{c}(\beta_t^0, B^{\mathbf{NT}}_{N_t^{\mathbf{NT}}}) dN_t^{m,\mathbf{NT}} \Big].$$

where $\mathbf{c}$ is the paying rule of the auctions:

- *First-price auction*: $\mathbf{c}(b, B) = b$
- *Second-price (Vickrey) auction*: $\mathbf{c}(b, B) = B$.

(*For simplicity of notations, we assume here the same auction rule* $\mathbf{c}$ *on* $\mathbf{T}$ *and* $\mathbf{NT}$ *but they can differ)*

Explicit solution

- Minimal cost

$$V^\star := \inf_\beta V(\beta) \quad = \quad \sum_{p \in \frac{[\![0,M[\![}{M}} v(p)$$

(here $\frac{[\![0,M[\![}{M} = \{\frac{k}{M} : k = 0, \ldots, M-1\}$), where $v(p) = \inf_{b^{\mathbf{T}}, b^{\mathbf{NT}} \in \mathbb{R}_+} v^{b^{\mathbf{T}}, b^{\mathbf{NT}}}(p)$, with

$$v^{b^{\mathbf{T}}, b^{\mathbf{NT}}}(p) \quad = \quad \frac{K + \eta^{\mathbf{T}} \mathbb{E}\Big[\mathbf{c}(b^{\mathbf{T}}, B_1^{\mathbf{T}}) \mathbf{1}_{b^{\mathbf{T}} \geq B_1^{1,\mathbf{T}}}\Big] + \eta^{\mathbf{NT}} \mathbb{E}\Big[\frac{\mathbf{c}(b^{\mathbf{NT}}, B_1^{\mathbf{NT}})}{1-p} \mathbf{1}_{b^{\mathbf{NT}} \geq \frac{B_1^{\mathbf{NT}}}{1-p}}\Big]}{\eta^{\mathbf{I}} + \eta^{\mathbf{T}} \mathbb{P}\Big[b^{\mathbf{T}} \geq B_1^{1,\mathbf{T}}\Big] + \eta^{\mathbf{NT}} \mathbb{P}\Big[b^{\mathbf{NT}} \geq B_1^{\mathbf{NT}}\Big] + p\eta^{\mathbf{S}}}.$$

Explicit solution

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- Optimal bidding map policies based on proportion of informed users:

$$\big(\mathrm{b}_\star^{\mathbf{T}}(p), \mathrm{b}_\star^{\mathbf{NT}}(p)\big) \quad \in \quad \operatorname*{argmin}_{b^{\mathbf{T}},b^{\mathbf{NT}} \in \mathbb{R}_+} v^{b^{\mathbf{T}},b^{\mathbf{NT}}}(p), \qquad p \in \frac{[\![0,M[\![}{M},$$

$\to$ optimal bidding map strategy $\beta^* = (\beta^{*,m})_{m \in [\![0,M]\!]}$ with $p_t^\beta := \frac{1}{M}\sum_{i=1}^M X_t^{i,\beta^*}$,

$$\begin{cases} \beta_t^{\star,m} & = & \mathrm{b}_\star^{\mathbf{T}}(p_{t^-}^{\beta^*})(1 - X_{t^-}^{m,\beta^*}), \quad m = 1,\dots,M, \\ \beta_t^{\star,0} & = & \mathrm{b}_\star^{\mathbf{NT}}(p_{t^-}^{\beta^*})1_{p_{t^-}^{\beta^*} < 1}, \quad t \geq 0. \end{cases}$$

Remarks on proof

- Direct arguments: do not rely on dynamic programming or maximum principle methods

- Change of variable: reformulate the problem $V(\beta)$ defined as a sum over the Poisson processes to an integral over proportion of converted users p^{β}
  - martingale tools using intensity process of Point process

- Bound from below minimal cost

- Achieve the lower bound with a suitable bidding policy.

Properties of the solution

Monotonicity w.r.t. intensity parameters $\eta = (\eta^{\mathbf{I}}, \eta^{\mathbf{T}}, \eta^{\mathbf{NT}}, \eta^{\mathbf{S}})$

- $V^\star$ is decreasing w.r.t. η
- The smallest optimal bid policies $\underline{\mathrm{b}}^{\mathbf{T}}_\star(p)$, $\underline{\mathrm{b}}^{\mathbf{NT}}_\star(p)$ are decreasing w.r.t. η

Monotonicity w.r.t. proportion of converted users p

- The smallest optimal bid policy for non-targeted ad $\underline{\mathrm{b}}^{\mathbf{NT}}_\star(p)$ is decreasing in p
- The smallest optimal bid policy for targeted ad $\underline{\mathrm{b}}^{\mathbf{T}}_\star(p)$ is
  - decreasing in p when there is no non-targeted ad ($\eta^{\mathbf{NT}} = 0$)
  - increasing in p when there is no social interactions ($\eta^{\mathbf{S}} = 0$)

Upper bound on optimal bids

$$\underline{\mathrm{b}}^{\mathbf{T}}_\star(p), \underline{\mathrm{b}}^{\mathbf{NT}}_\star(p) \le v(p) \le \frac{K}{\eta^{\mathbf{I}} + p\eta^{\mathbf{S}}}.$$

Computational cost of optimal bids

- Algo implementation of optimal bids require to compute:

$$\mathrm{b}_\star^{\mathbf{T}}(p),\ \mathrm{b}_\star^{\mathbf{NT}}(p), \qquad \text{for all } p = \frac{k}{M},\ k = 0, \ldots, M-1.$$

→ This is a priori quite expensive when M is large!

Computational cost of optimal bids

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→ This is a priori quite expensive when M is large!

- But, taking advantage of the monotonicity in p of $\mathrm{b}_\star^{\mathbf{T}}(p)$, $\mathrm{b}_\star^{\mathbf{NT}}(p)$, one can proceed by dichotomy

→ Computational complexity is of order $O\big(\ln_2(M)\big)$

e.g. for $M = 7 \times 10^9$, we have $\ln_2(M) = 30$.

Mean-field problem: $M \to \infty$

The average of the minimal cost $V^\star = V^\star_M$ converges to:

$$\frac{1}{M} V^\star_M \longrightarrow \int_0^1 v(p)dp.$$

This corresponds formally to the **optimal control problem on the proportion of converted users**:

$$\frac{dp_t^\beta}{dt} = (1 - p_t^\beta)\Big(\eta^{\mathbf{I}} + \eta^{\mathbf{T}}\mathbb{P}[\beta_t^{\mathbf{T}} \geq B_1^{\mathbf{T}}] + \eta^{\mathbf{NT}}\mathbb{P}[\beta_t^{\mathbf{NT}} \geq B_1^{\mathbf{NT}}] + \eta^{\mathbf{S}} p_t^\beta\Big), \quad t \geq 0,$$

with deterministic control $\beta = (\beta^{\mathbf{T}}, \beta^{\mathbf{NT}})$ and cost functional

$$\begin{aligned} V(\beta) \quad = \quad & \int_0^\infty \Big\{(1 - p_t^\beta)\Big(K + \eta^{\mathbf{T}}\mathbb{E}[\mathbf{c}^{\mathbf{T}}(\beta_t^{\mathbf{T}}, B_1^{\mathbf{T}})\mathbf{1}_{\beta_t^{\mathbf{T}} \geq B_1^{\mathbf{T}}}]\Big) \\ & \qquad + \eta^{\mathbf{NT}}\mathbb{E}[\mathbf{c}^{\mathbf{NT}}(\beta_t^{\mathbf{NT}}, B_1^{\mathbf{NT}})\mathbf{1}_{\beta_t^{\mathbf{NT}} \geq B_1^{\mathbf{NT}}}]\Big\} dt. \end{aligned}$$

- Formulation and (explicit) resolution of some advertising problems
  - Microscopic modelling of users: online behaviour
  - Digital feature of advertising, auctions for ad display
  - Quantitative comparison between targeted vs non-targeted advertising
  - Role of social interactions between users

- Enrich models for more realism while keeping tractable
  - Conversion funnel for user to be receptive or not to the information:
    - purchase or not a product
    - stop or continue to behave unsafely
  - Some heterogeneity in the population
  - Auctions:
    - maximal bid of others bidders by Markov process
    - several bidding agents in fictitious play to learn the law of the maximal bid

THANK YOU FOR YOUR ATTENTION