

Air quality, urban economics and atmospheric dispersion models: a mathematical unifying framework

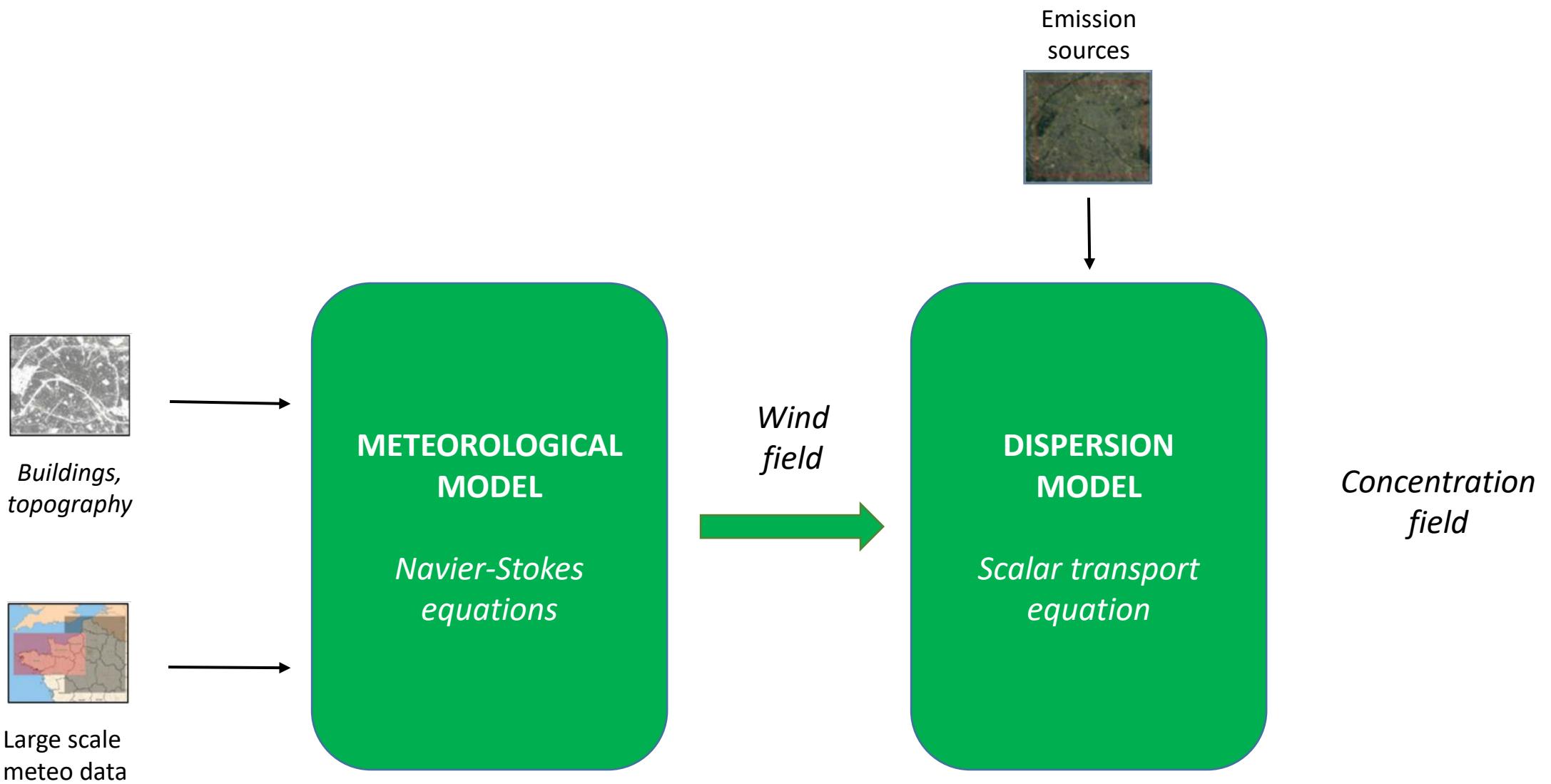
Mohamed BAHLALI (Climate Economics Chair)

Quentin PETIT (CEREMADE, Paris Dauphine University)

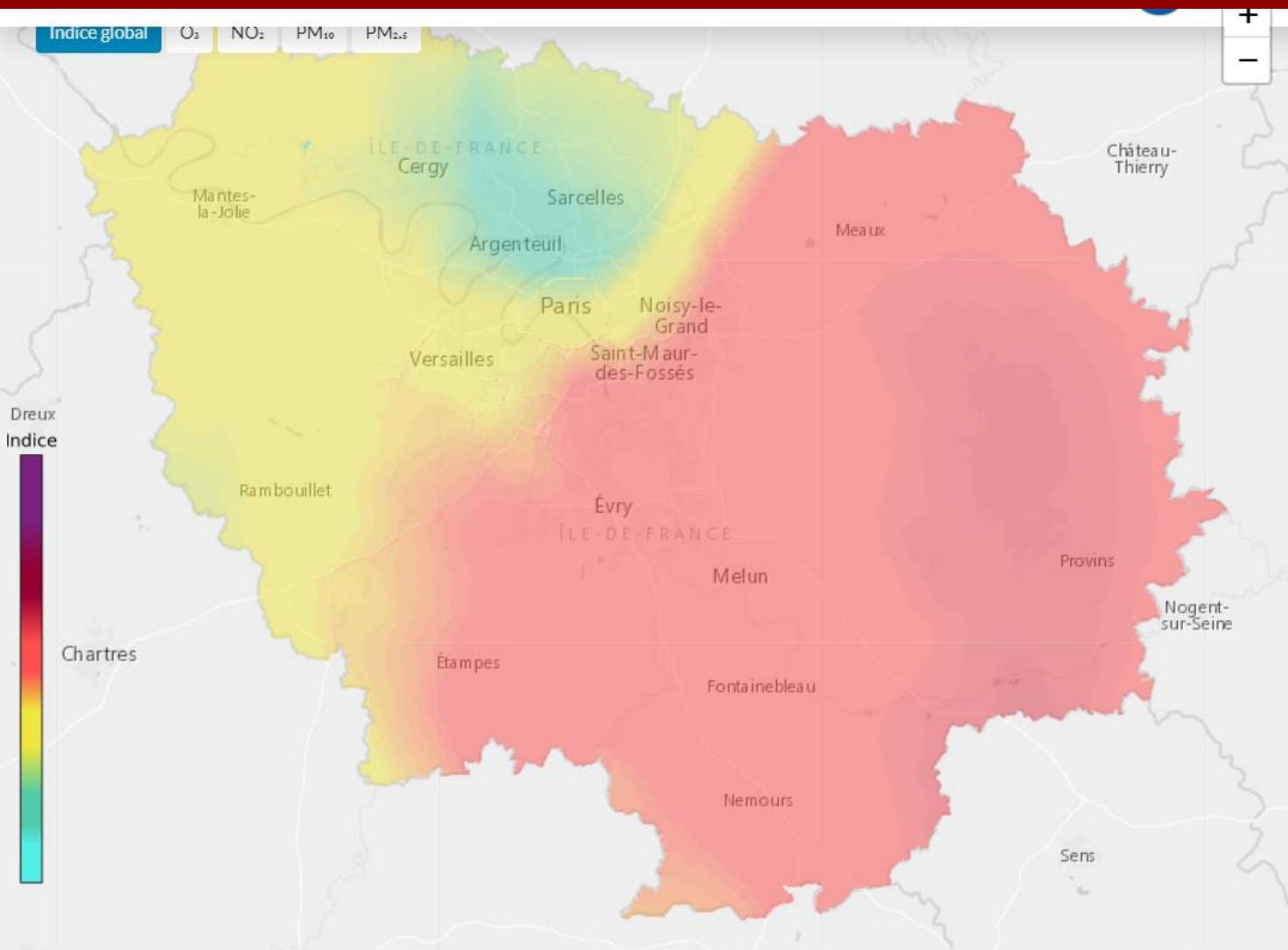


Context

Atmospheric dispersion models



Atmospheric dispersion models



Air quality in Paris region, yesterday at 10 am (AirParif)

La pollution en direct en Île-de-France

11 Rue Pierre et Marie Curie, 75005 Paris, France

31/03/2022

10:00:00

Qualité de l'air

Dégradée

Concentration par polluants :

Ozone (O₃)

40 µg/m³

Dioxyde d'Azote (NO₂)

35 µg/m³

Particules (diamètre inférieur à 10µm) (PM₁₀)

35 µg/m³

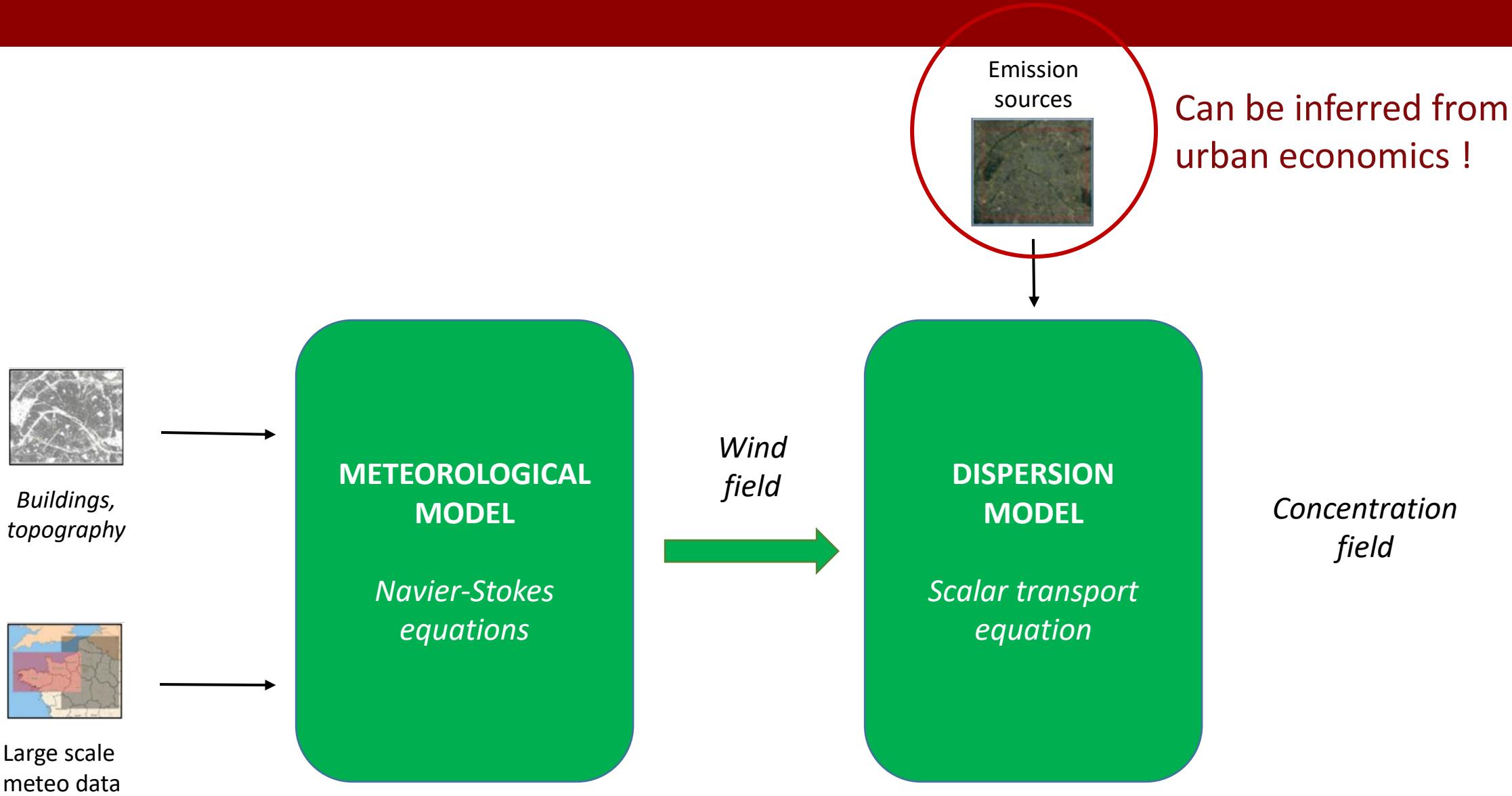
Particules fines (diamètre inférieur à 2,5µm) (PM_{2.5})

23 µg/m³

Qualité de l'air

Bonne Moyenne Dégradée Mauvaise Très mauvaise Extrêmement mauvaise

Atmospheric dispersion models



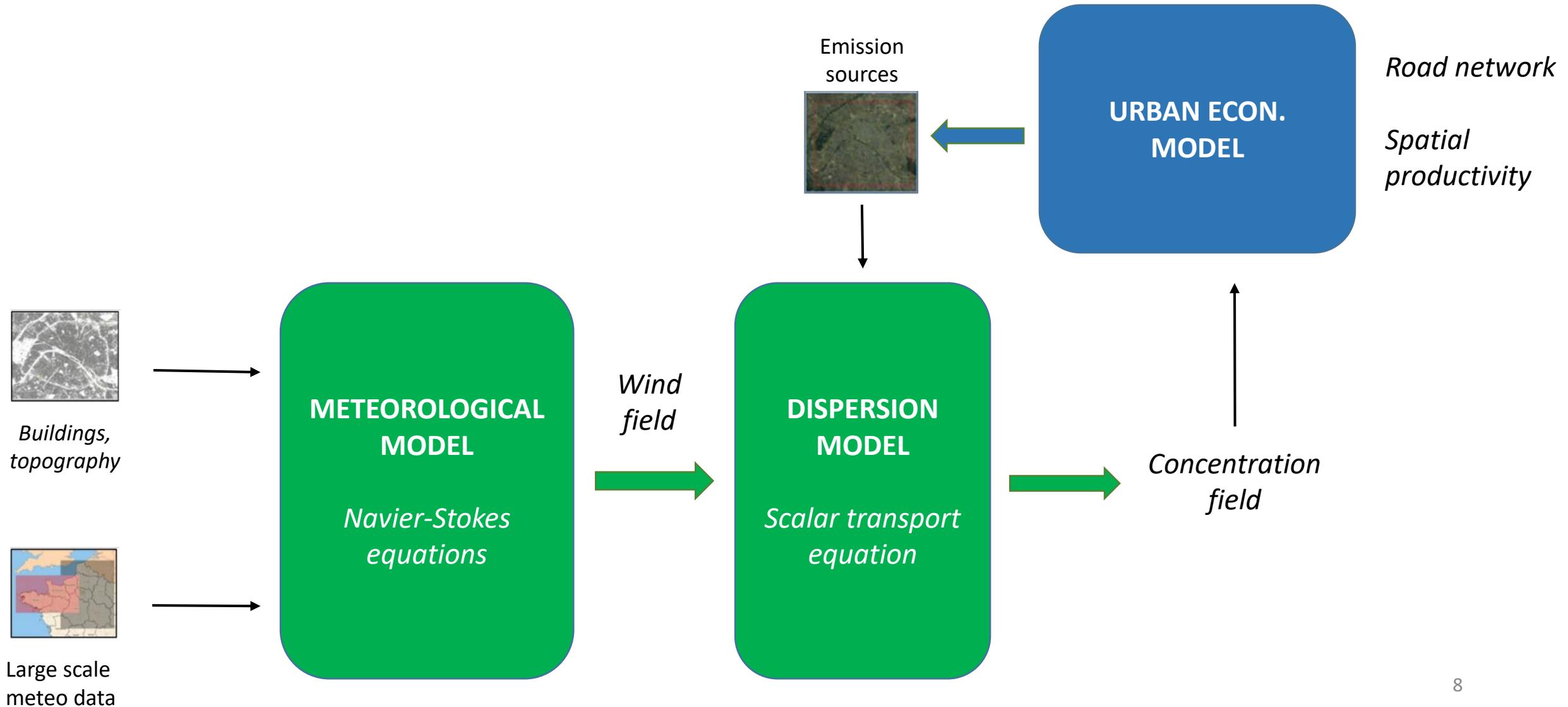
Urban economics models

- A long tradition
 - Precursors: von Thünen (1851)
 - Then came Burgess (1923), Hoytt (1939)...
 - **Microeconomic models: Alonso (1964), Mills (1967), Muth (1969)**
 - *A model of rents and residents density in a monocentric city, based on a microeconomic rationale*
 - Then Fujita and Ogawa (1982) and Lucas and Ross-Hansberg (2002) added **endogenous wages**, and relaxed the assumption on **monocentricity**, in a linear and circular city

Urban economics models

- Several papers applied Alonso (1964) framework to account for **endogenous pollution**
 - Arnott et al. (2008), Schindler et al. (2017), Kyriakopoulou and Picard (2021): optimal structure of the city with endogenous traffic-based air pollution
 - They accounted for pollution in agents' utility function, in line with the recent empirical literature
- But their **description of pollution** is very **simplified**:
 - Space is discrete;
 - City is linear or circular;
 - Pollution is totally immobile, and coincides with emission sources

Our model



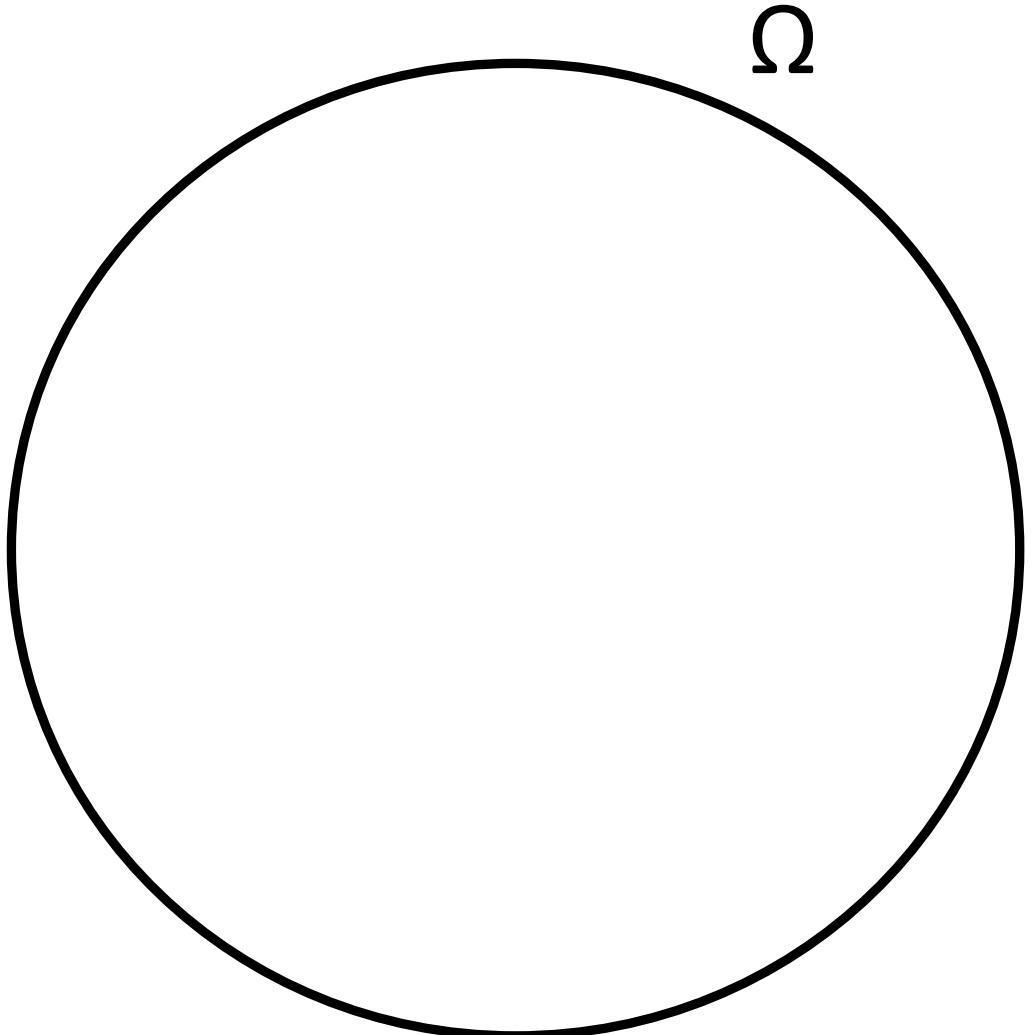
Our model

Model

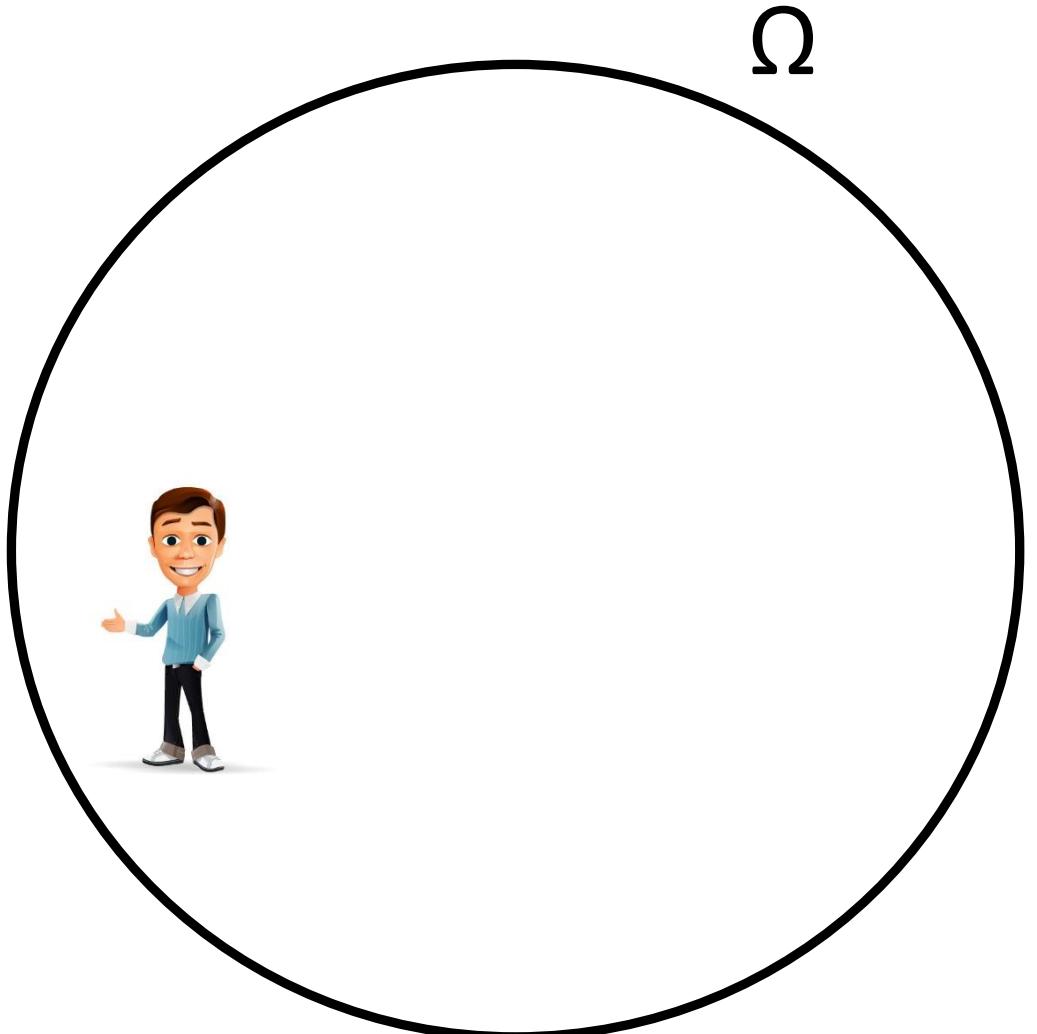
- Derives from a model developed during Q. Petit's PhD (ongoing paper with G. Carlier, Y. Achdou and D. Tonon)
 - An equilibrium model for the spatial distribution of labour and housing
 - The city can be of any shape
 - They prove existence and uniqueness of the equilibrium
 - They obtain an explicit formulation for the distribution of residents
 - A considerable extension of the models cited above
- We added pollution dispersion to their model

Model : main variables

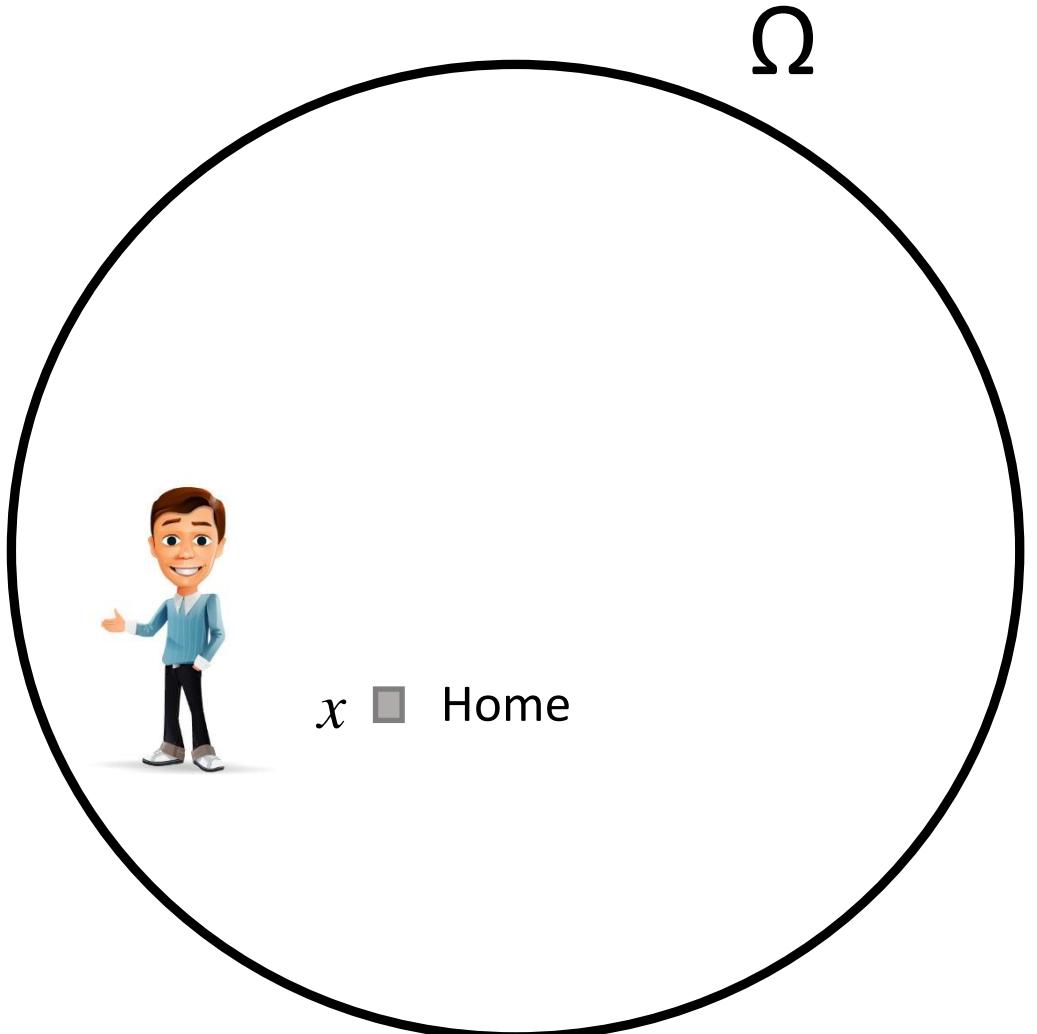
Model : main variables



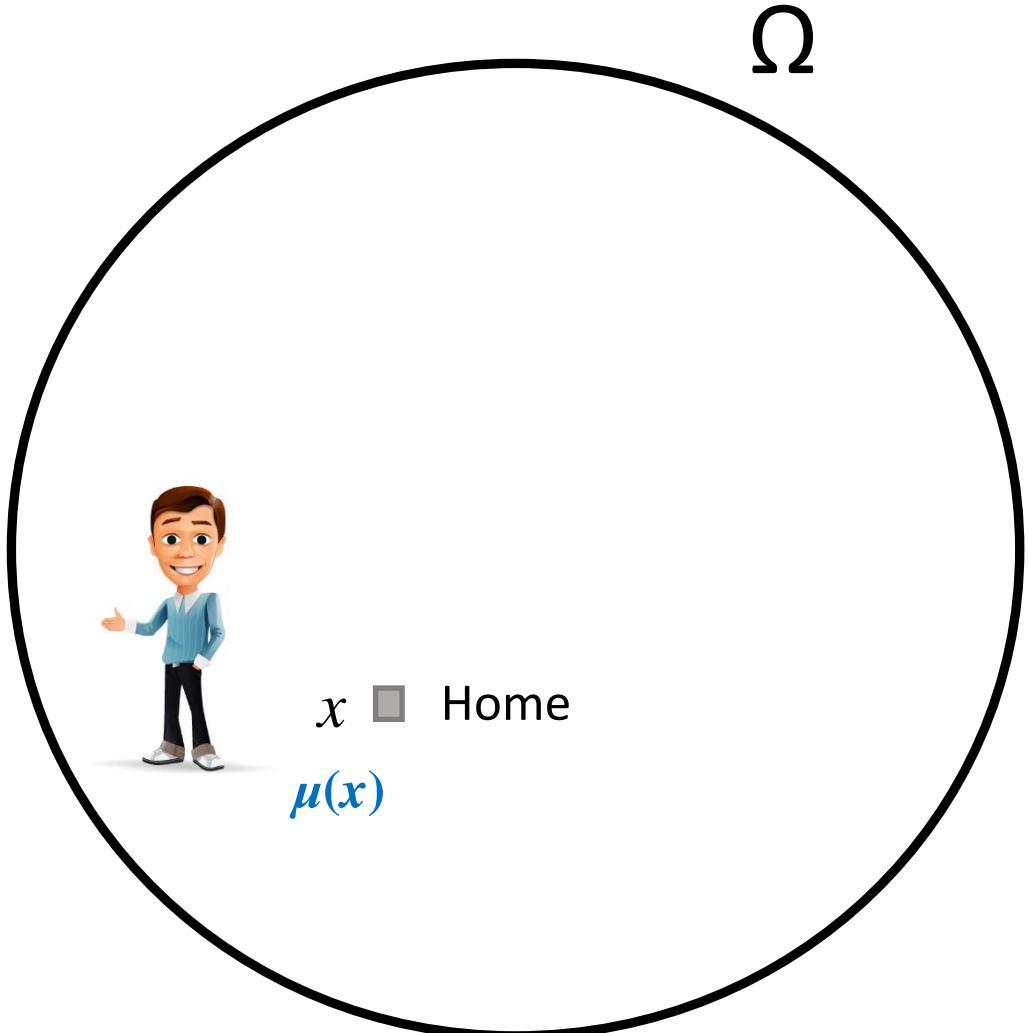
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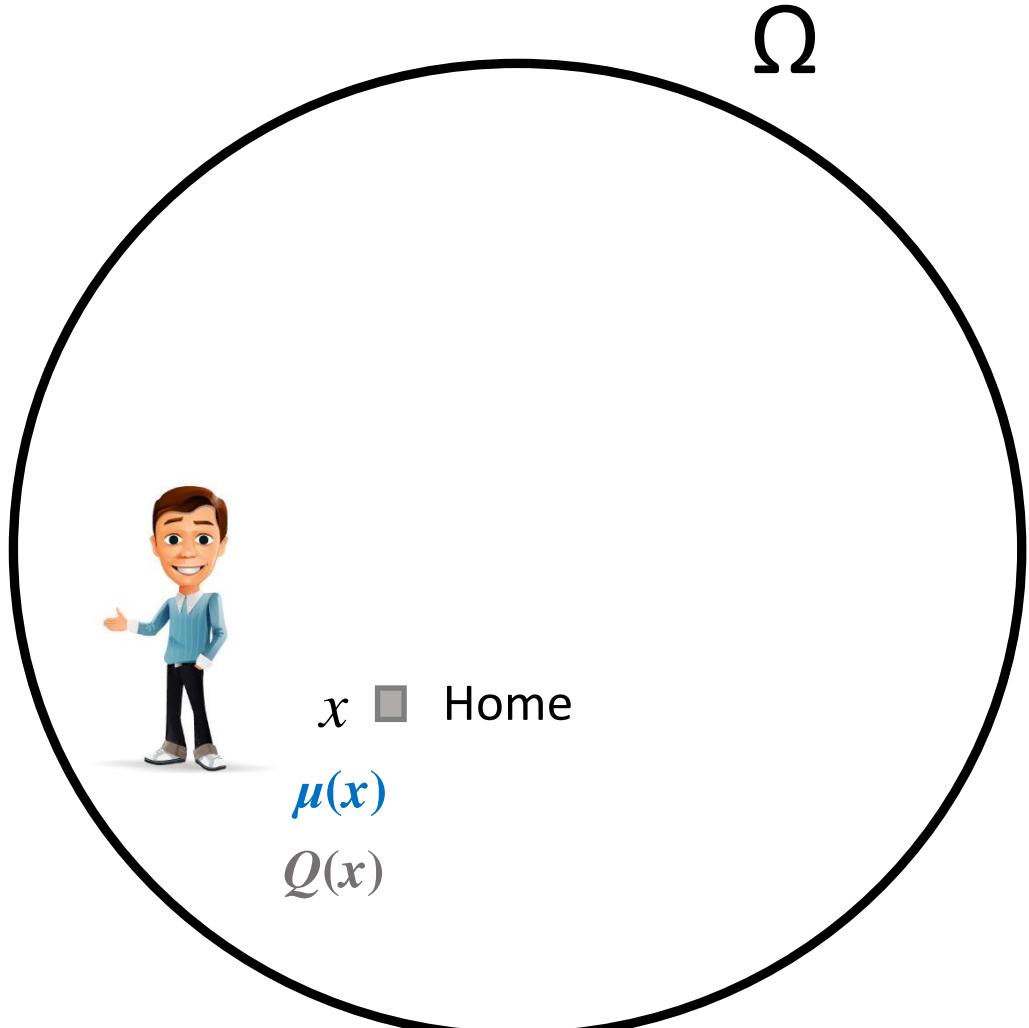
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Model : main variables



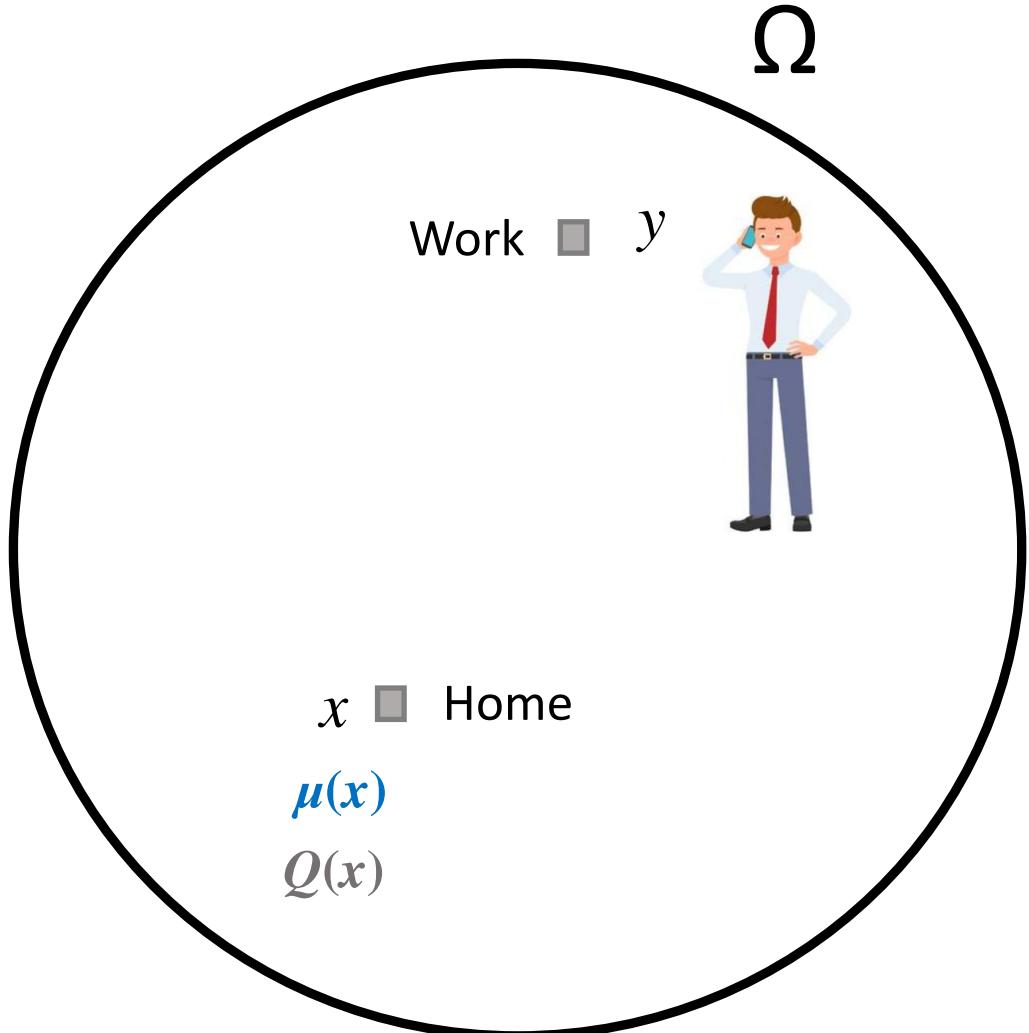
Model : main variables



MAIN VARIABLES

- $\mu \in P_c(\bar{\Omega})$: population distribution
- $Q \in C(\bar{\Omega}, \mathbb{R}_+^*)$: housing price function per square meter

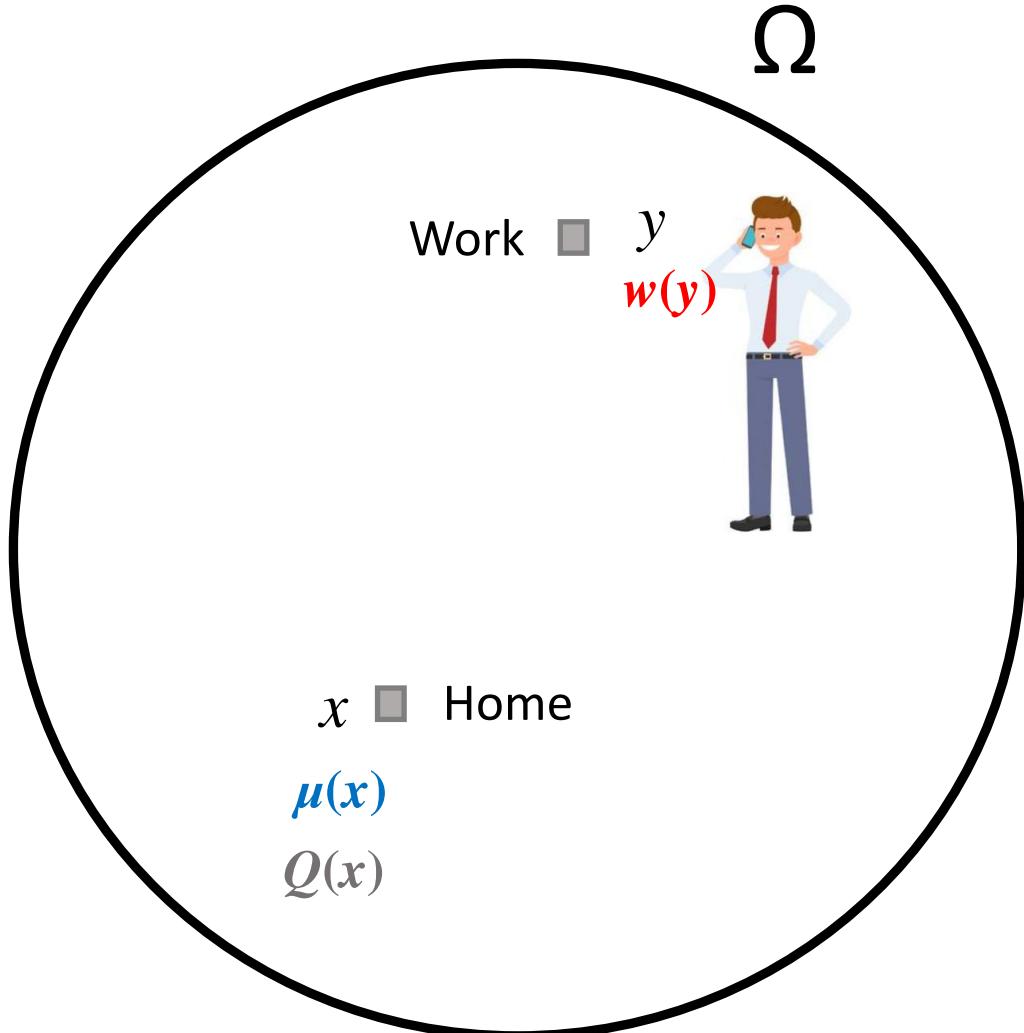
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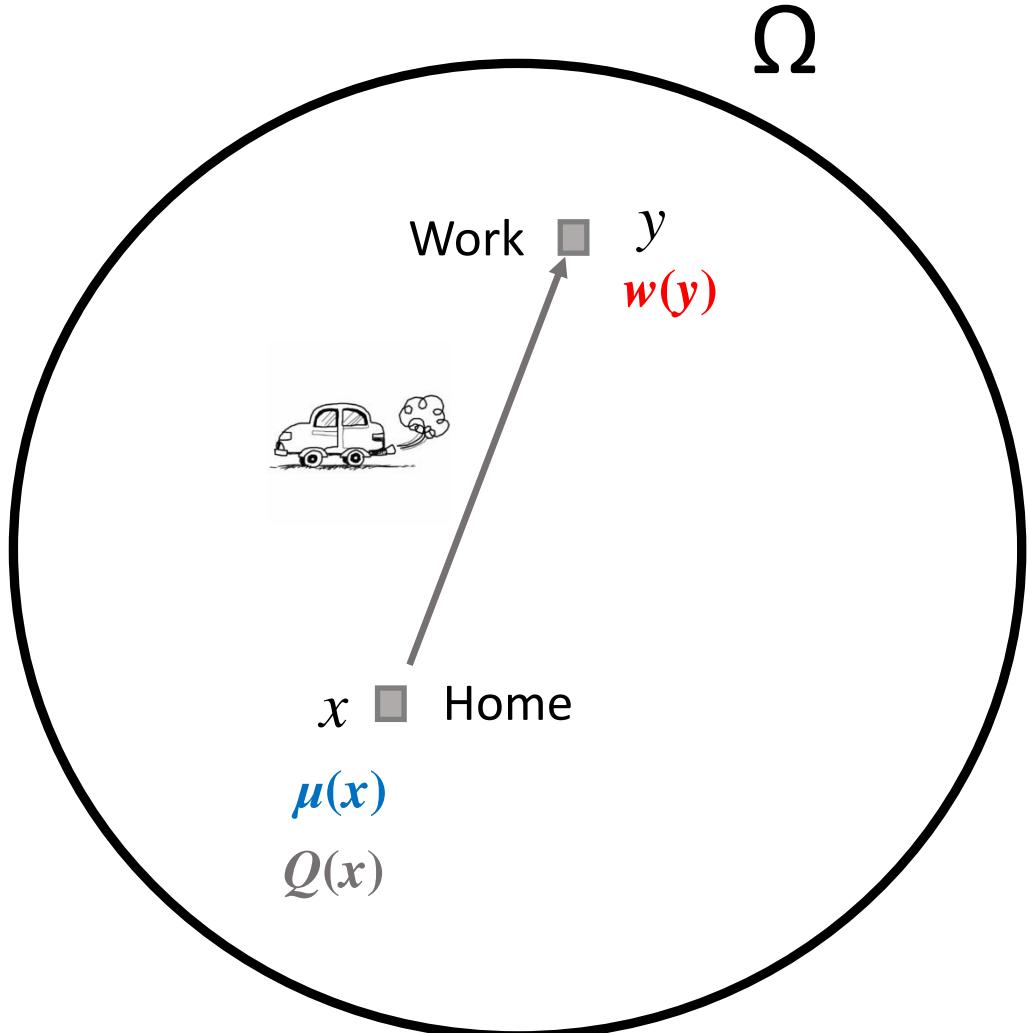
Model : main variables



MAIN VARIABLES

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- $w \in C(\bar{\Omega}, \mathbb{R}_+^*)$: wage function

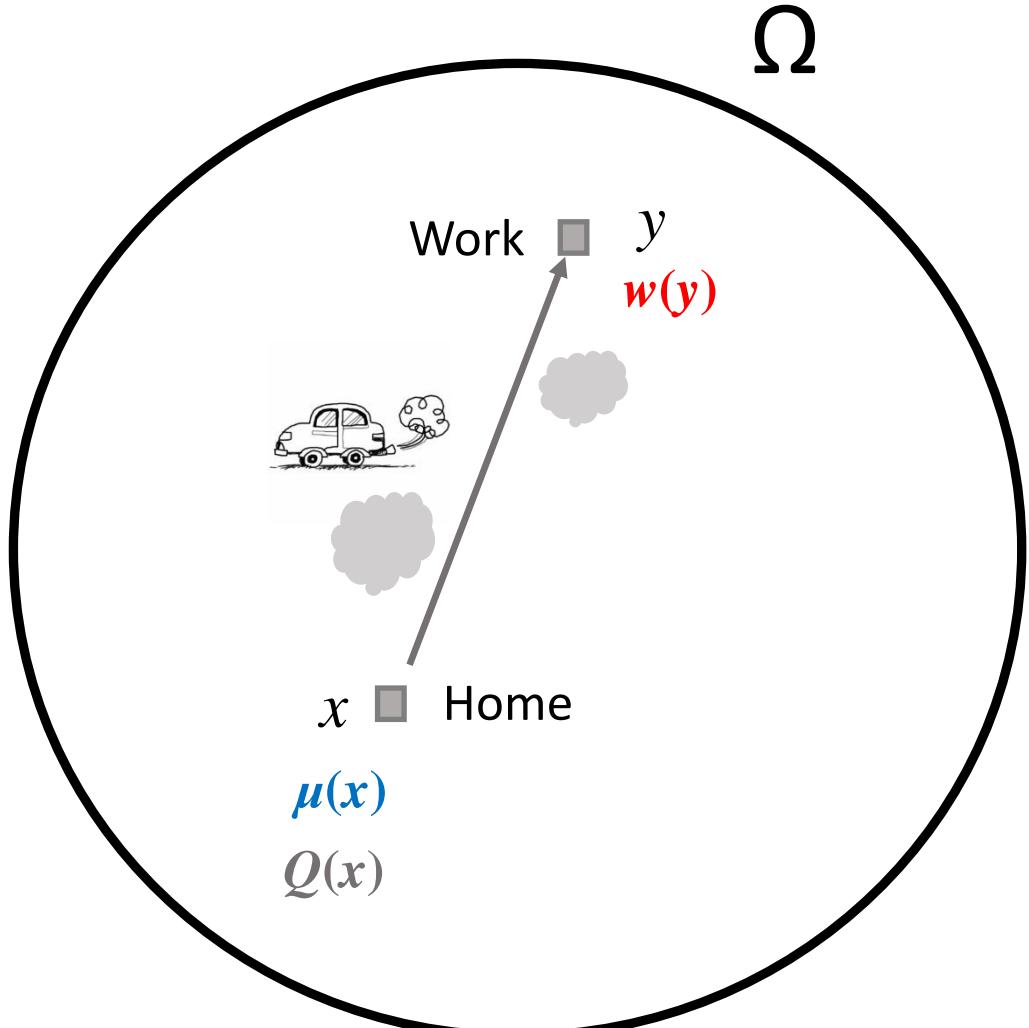
Model : main variables



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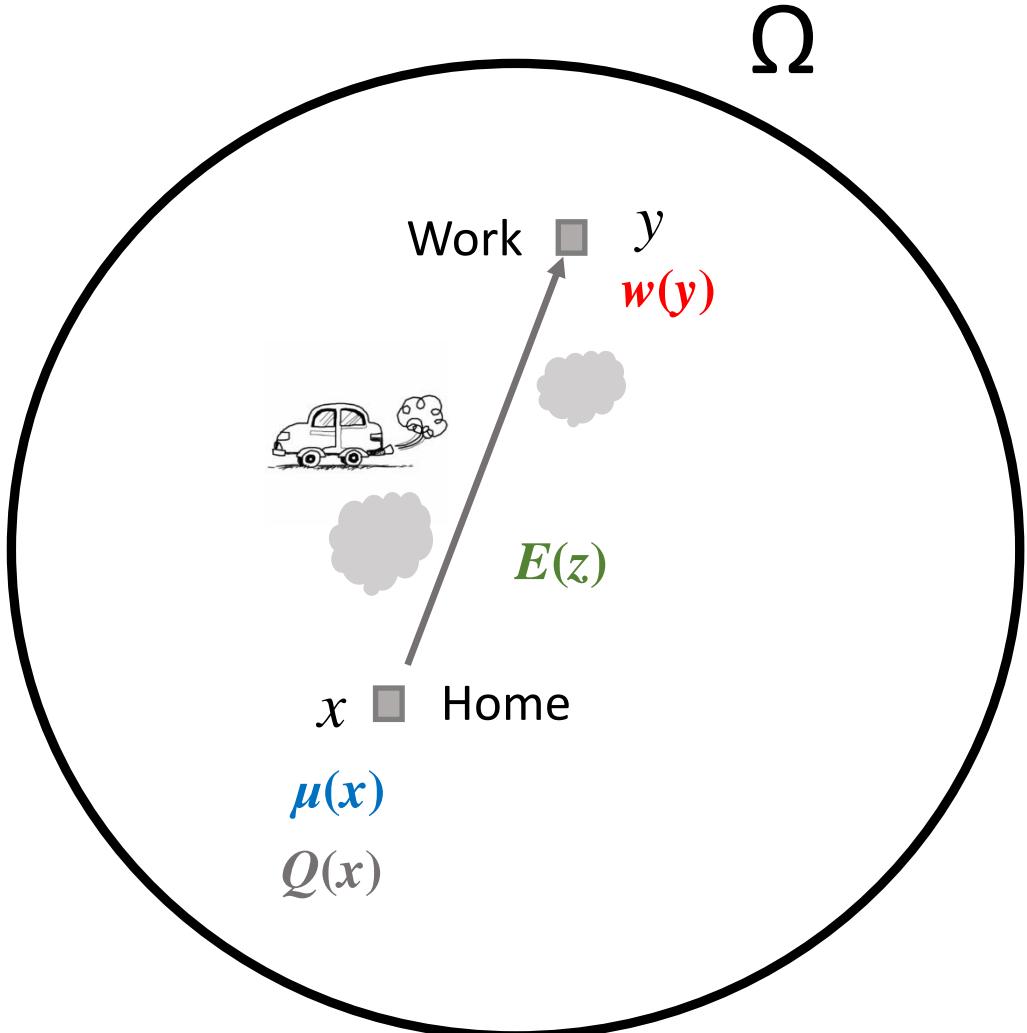
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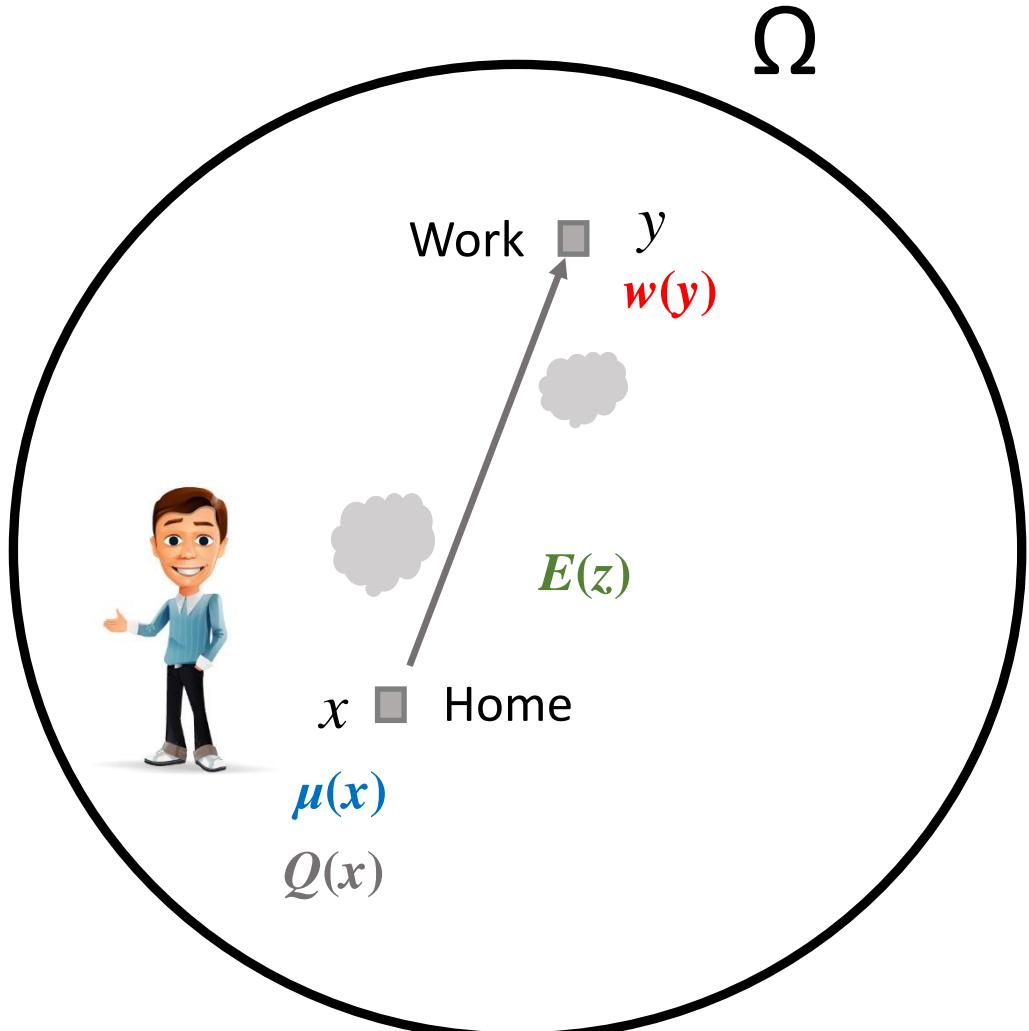
Model : main variables



MAIN VARIABLES

- $\mu \in P_c(\bar{\Omega})$: population distribution
- $Q \in C(\bar{\Omega}, \mathbb{R}_+^*)$: housing price function per square meter
- $w \in C(\bar{\Omega}, \mathbb{R}_+^*)$: wage function
- $E \in C(\bar{\Omega}, \mathbb{R}_+^*)$: pollution concentration

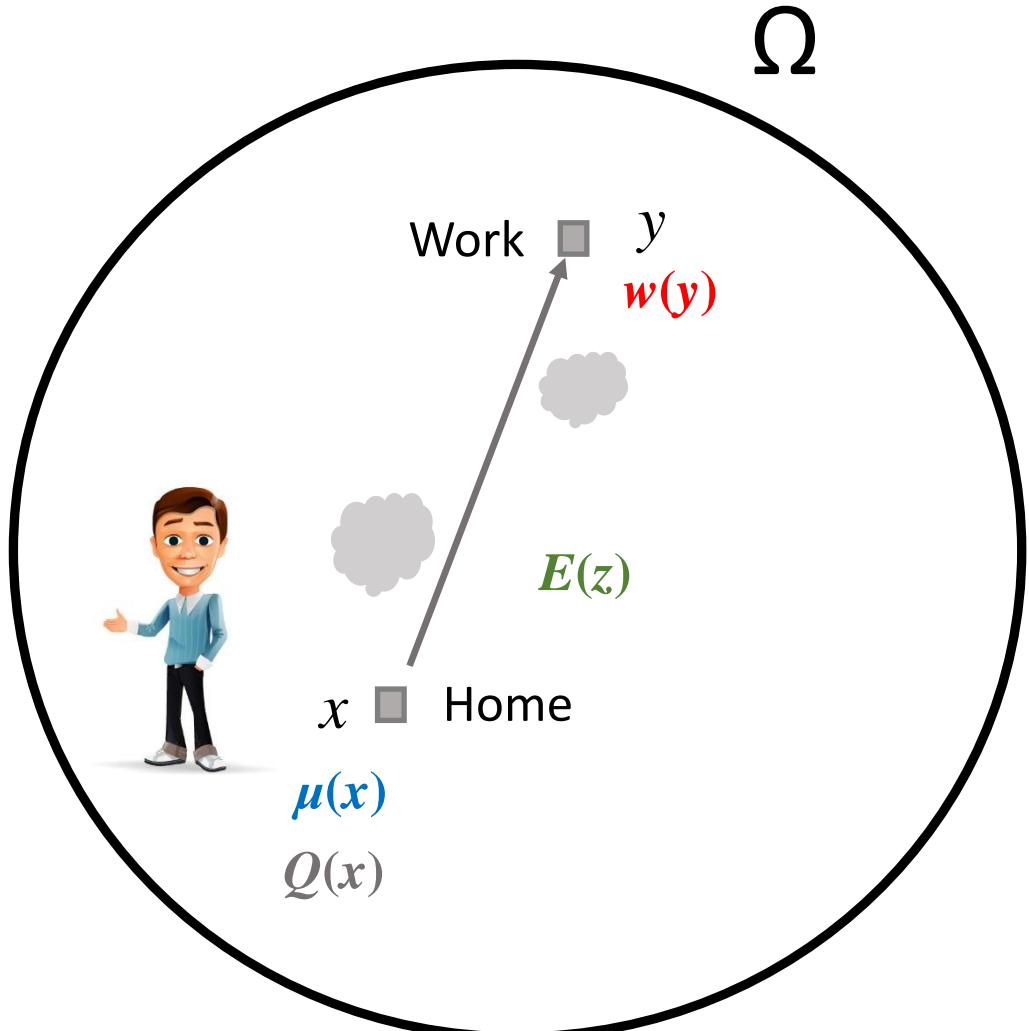
Model : Agents' Utility



INDIRECT UTILITY FUNCTION

$$U_{\theta,\gamma}(R, \mathbf{Q}, \mathbf{E}) = \sup_{C,S \geq 0} \{ C^\theta S^{1-\theta} \mathbf{E}^{-\gamma}, C + QS \leq R \}$$

Model : Agents' Utility

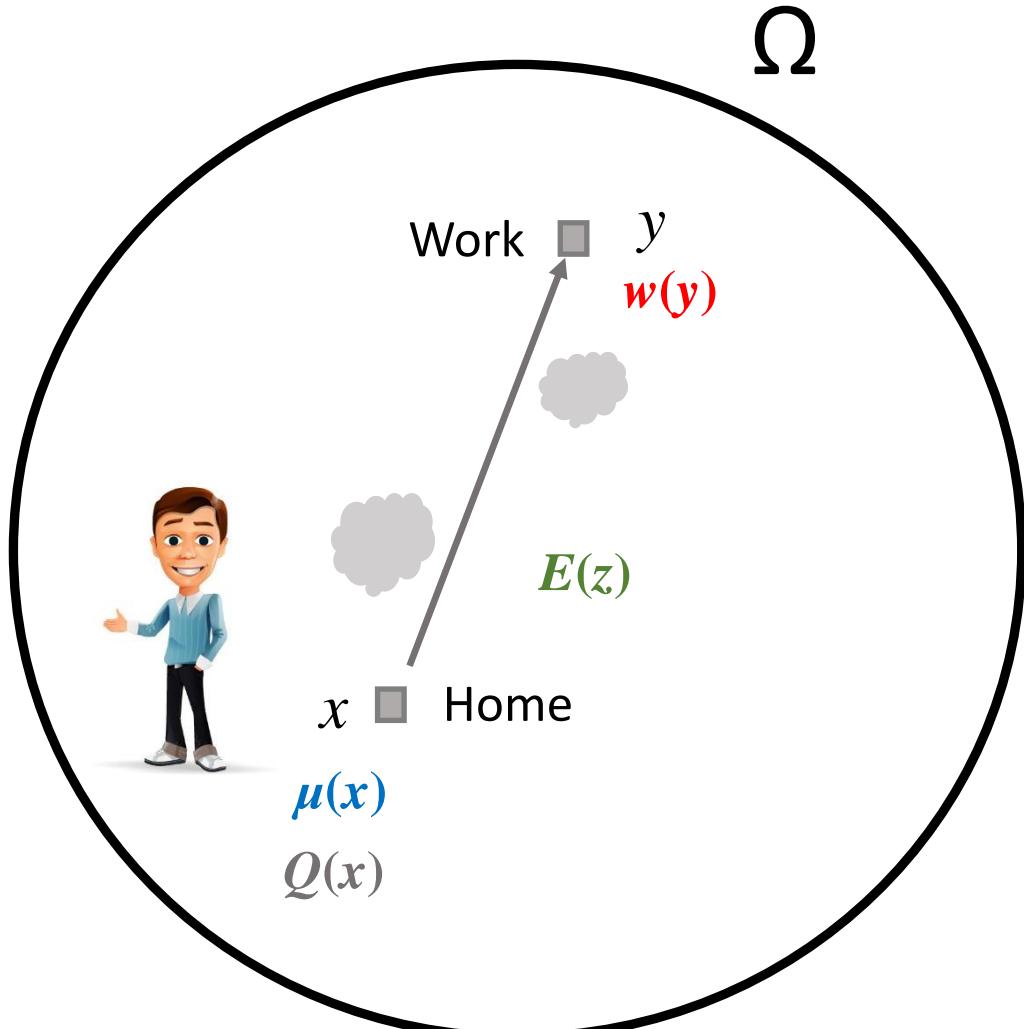


INDIRECT UTILITY FUNCTION

$$U_{\theta,\gamma}(R, Q, E) = \sup_{C,S \geq 0} \{ C^\theta S^{1-\theta} E^{-\gamma}, C + QS \leq R \}$$

$$C_\theta(R) = \theta R$$

Model : Agents' Utility



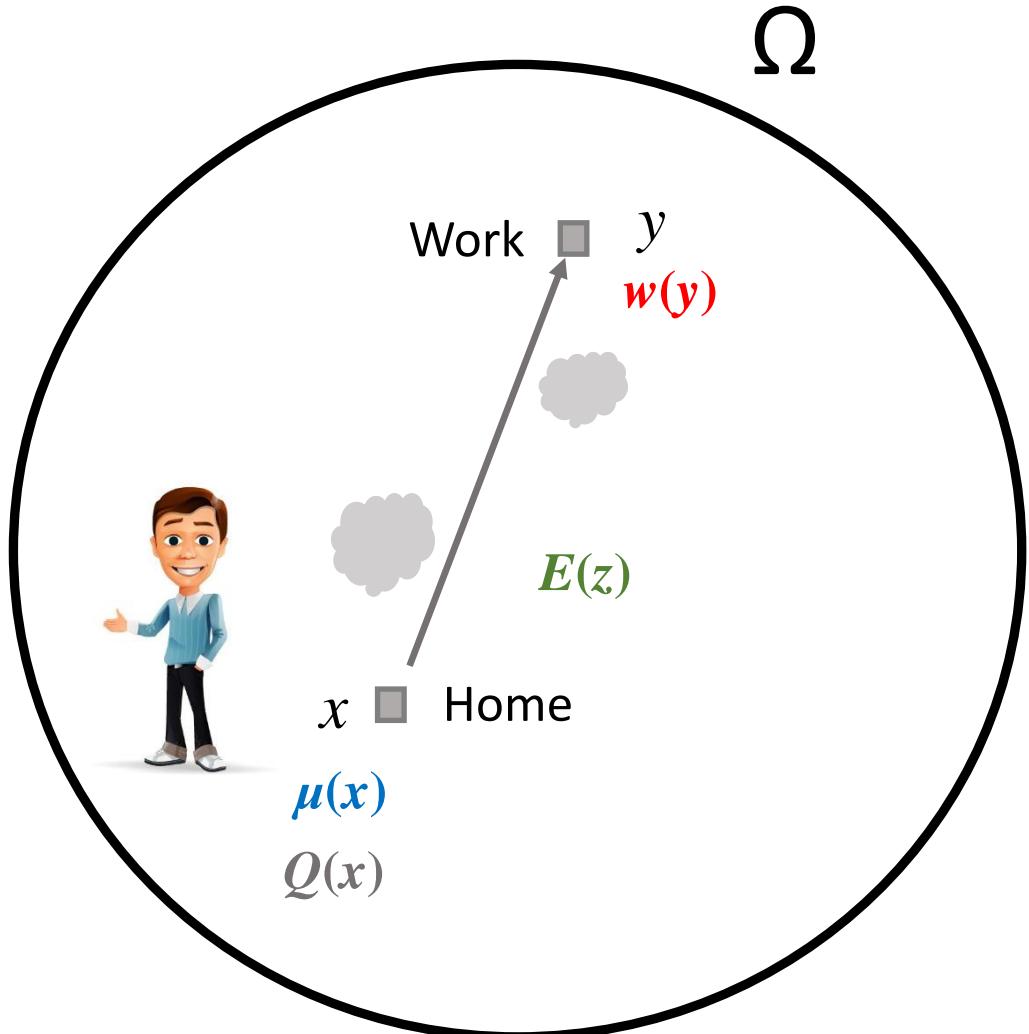
INDIRECT UTILITY FUNCTION

$$U_{\theta,\gamma}(R, Q, E) = \sup_{C,S \geq 0} \{ C^\theta S^{1-\theta} E^{-\gamma}, C + QS \leq R \}$$

$$C_\theta(R) = \theta R$$

$$S_\theta(R, Q) = (1 - \theta)R/Q$$

Model : Agents' Utility



INDIRECT UTILITY FUNCTION

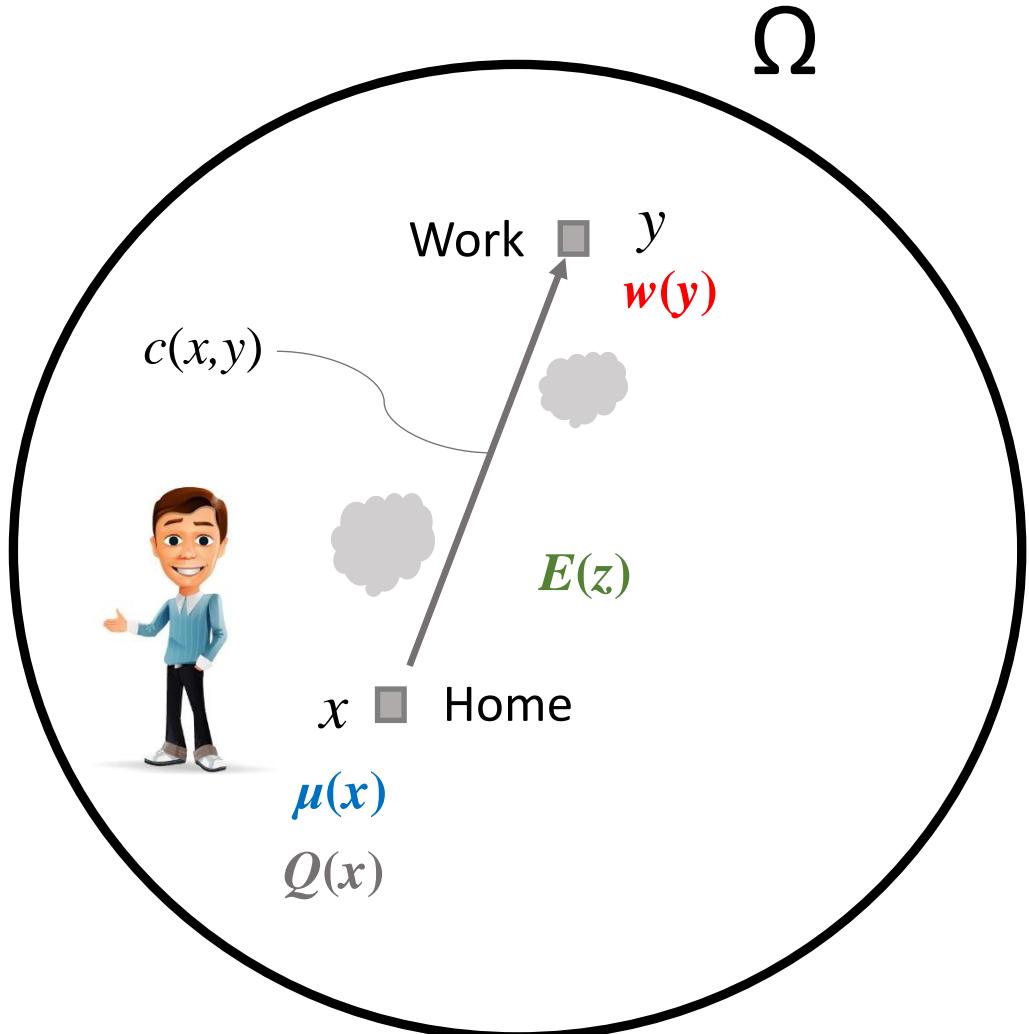
$$U_{\theta,\gamma}(R, Q, E) = \sup_{C,S \geq 0} \{ C^\theta S^{1-\theta} E^{-\gamma}, C + QS \leq R \}$$

$$C_\theta(R) = \theta R$$

$$S_\theta(R, Q) = (1 - \theta)R/Q$$

$$U_{\theta,\gamma}(R, Q, E) = \theta^\theta (1 - \theta)^{1-\theta} \frac{R}{Q^{1-\theta} E^\gamma}$$

Model : Agents' Utility



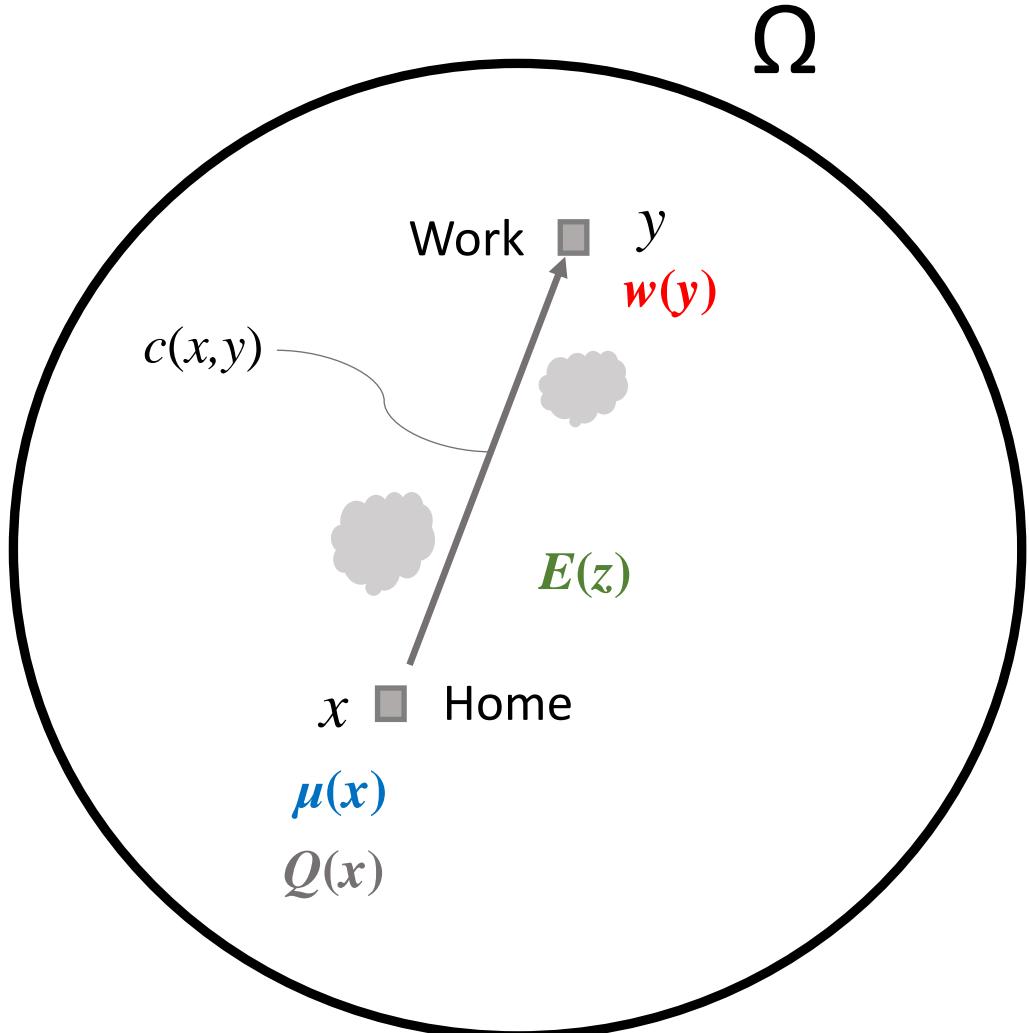
INDIRECT UTILITY FUNCTION

$$U_{\theta,\gamma}(R, Q, \mathbf{E}) = \theta^\theta (1 - \theta)^{1-\theta} \frac{R}{Q^{1-\theta} \mathbf{E}^\gamma}$$

R : Revenue of an agent

$$\begin{aligned} R(x, \mathbf{w}) &= \max_{y \in \Omega} \{ \mathbf{w}(y) - c(x, y) \} \\ &\approx \sigma \ln \left(\int_{\Omega} e^{\frac{\mathbf{w}(y) - c(x, y)}{\sigma}} dy \right) \end{aligned}$$

Model : Labour



LABOUR

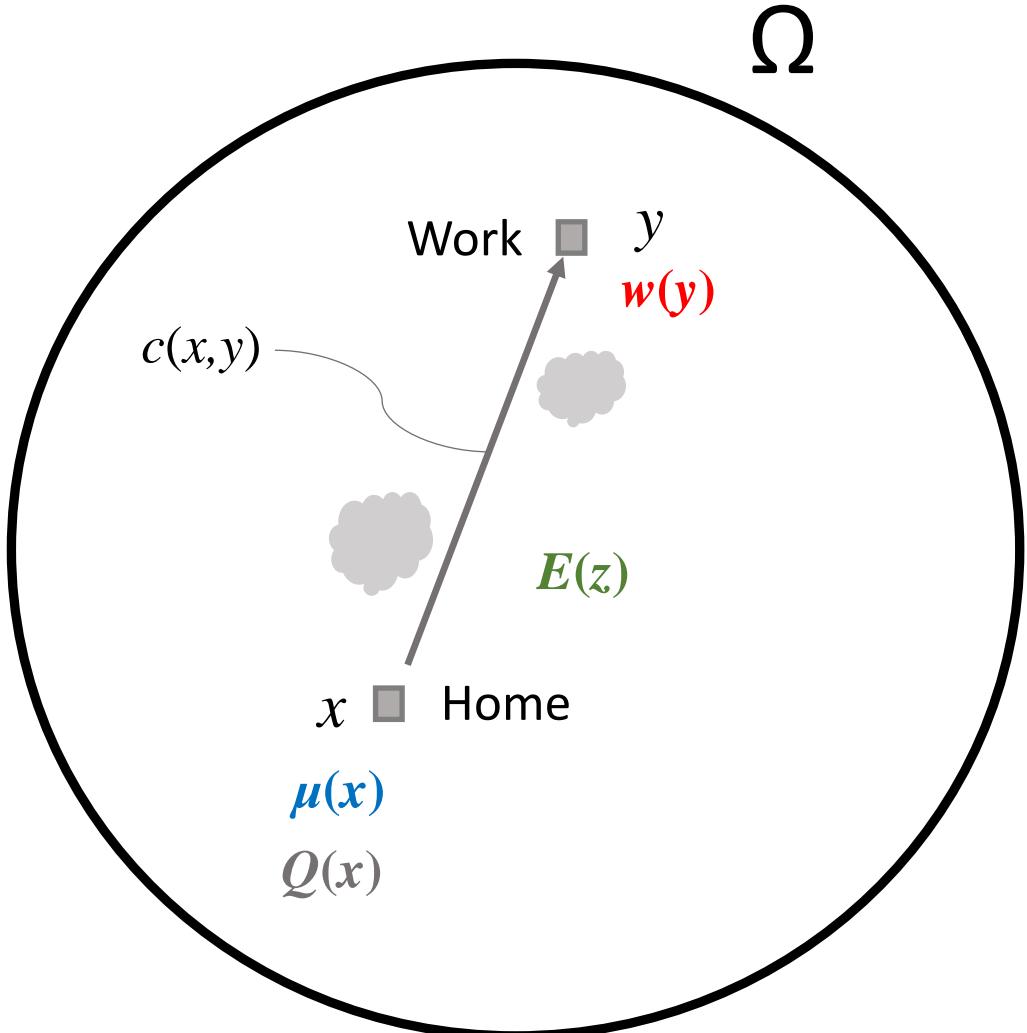
Probability d. f. for an agent at $x \in \bar{\Omega}$ to work at $y \in \bar{\Omega}$:

$$G(x, y, \mathbf{w}) = \frac{e^{\frac{w(y)-c(x,y)}{\sigma}}}{\int_{\Omega} e^{\frac{w(z)-c(x,z)}{\sigma}} dz}$$

Labour offer d. f. at $y \in \bar{\Omega}$:

$$\int_{x \in \Omega} G(x, y, \mathbf{w}) d\mu(x)$$

Model : Labour



LABOUR

Labour demand d. f. at $y \in \bar{\Omega}$:

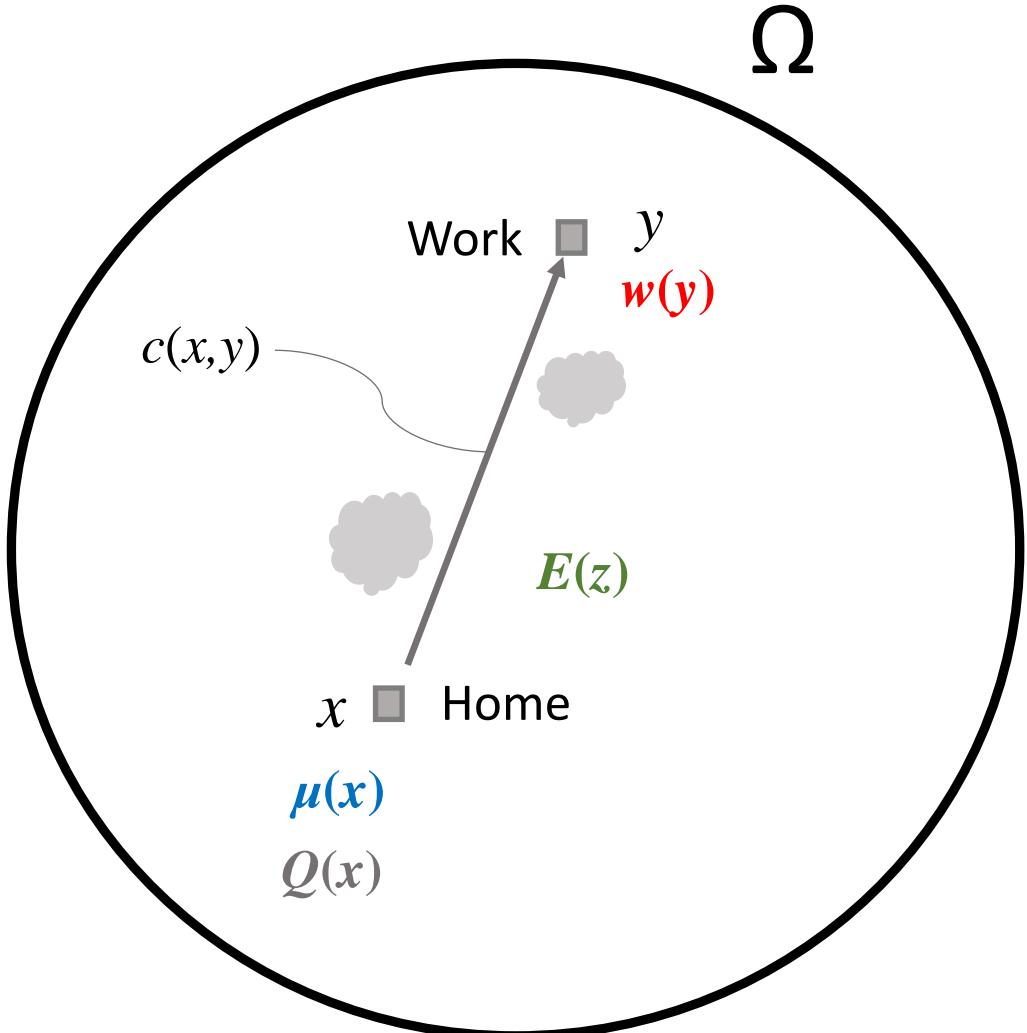
$$L(y, \mathbf{w}(y))$$

(productivity differs exogenously in space)

Labour equilibrium at $y \in \bar{\Omega}$:

$$\int_{x \in \Omega} G(x, y, \mathbf{w}) d\mu(x) = L(y, \mathbf{w}(y))$$

Model : Housing



HOUSING

Housing (surface) demand d. f. at $x \in \bar{\Omega}$:

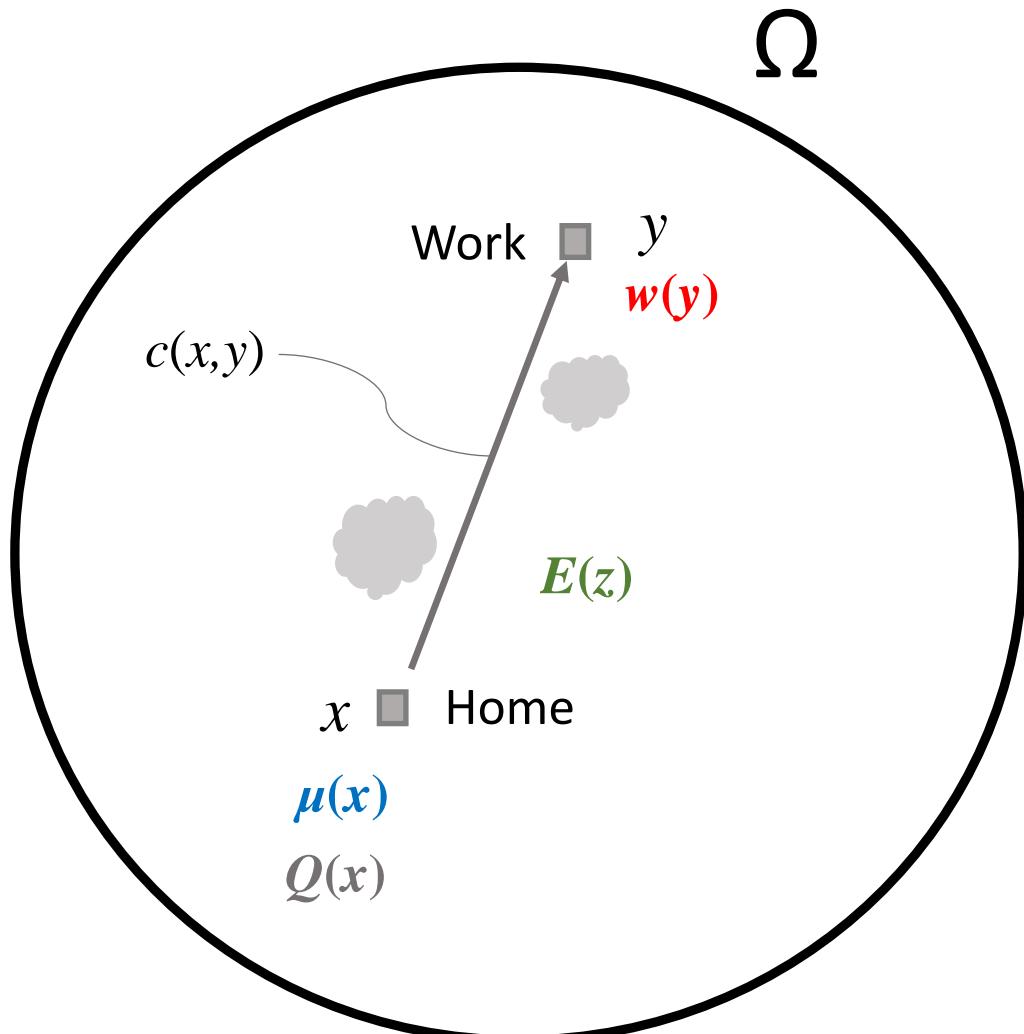
$$S_\theta(R(x, \mathbf{w}), Q(x))\boldsymbol{\mu}(x) := (1 - \theta) \frac{R(x, \mathbf{w})}{Q(x)} \boldsymbol{\mu}(x)$$

Housing (surface) supply d. f. normalized to 1 everywhere

Housing equilibrium at $x \in \bar{\Omega}$:

$$S_\theta(R(x, \mathbf{w}), Q(x))\boldsymbol{\mu}(x) = 1$$

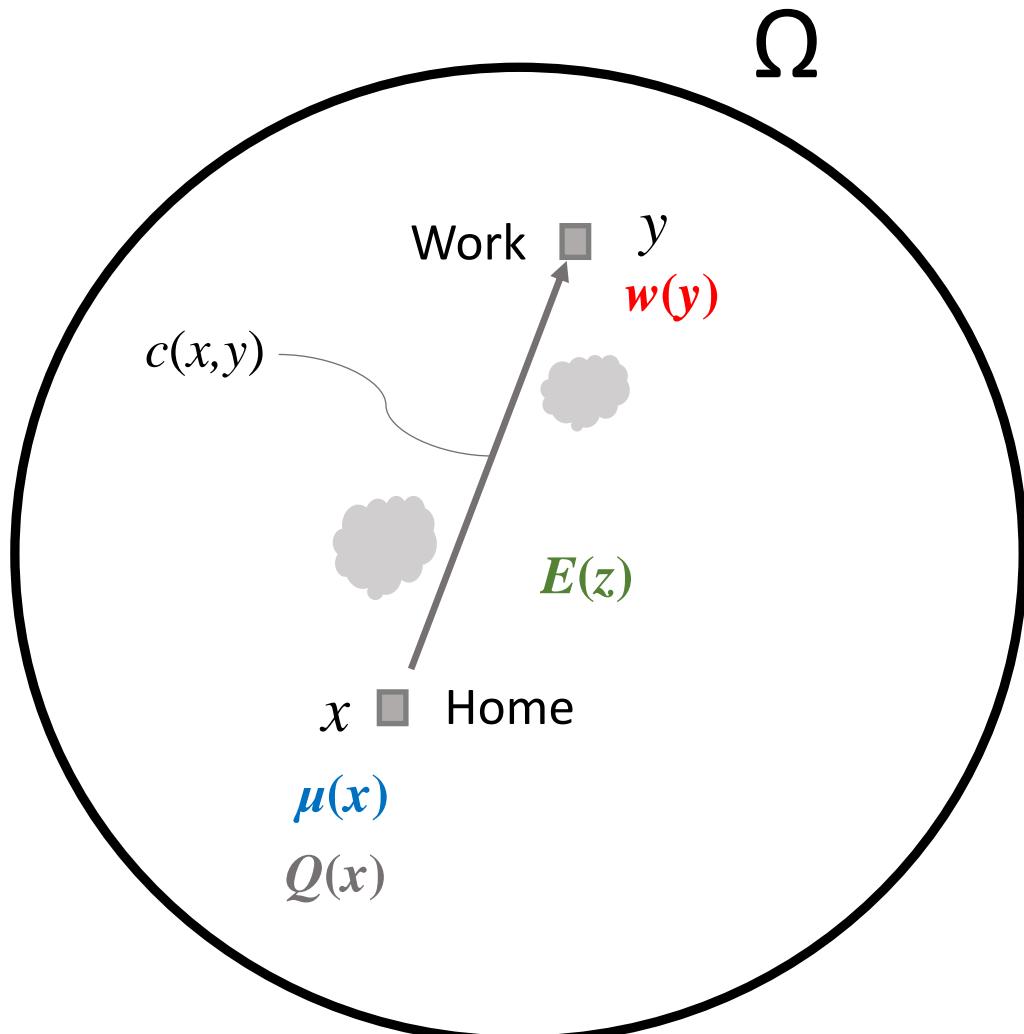
Model : Pollution



POLLUTION DISPERSION : *scalar transport equation*

$$\Delta \mathbf{E}(z) - \operatorname{div}(\mathbf{V}(z)\mathbf{E}(z)) + \chi(p, z) + f(z) - \Lambda \mathbf{E} = 0$$

Model : Pollution



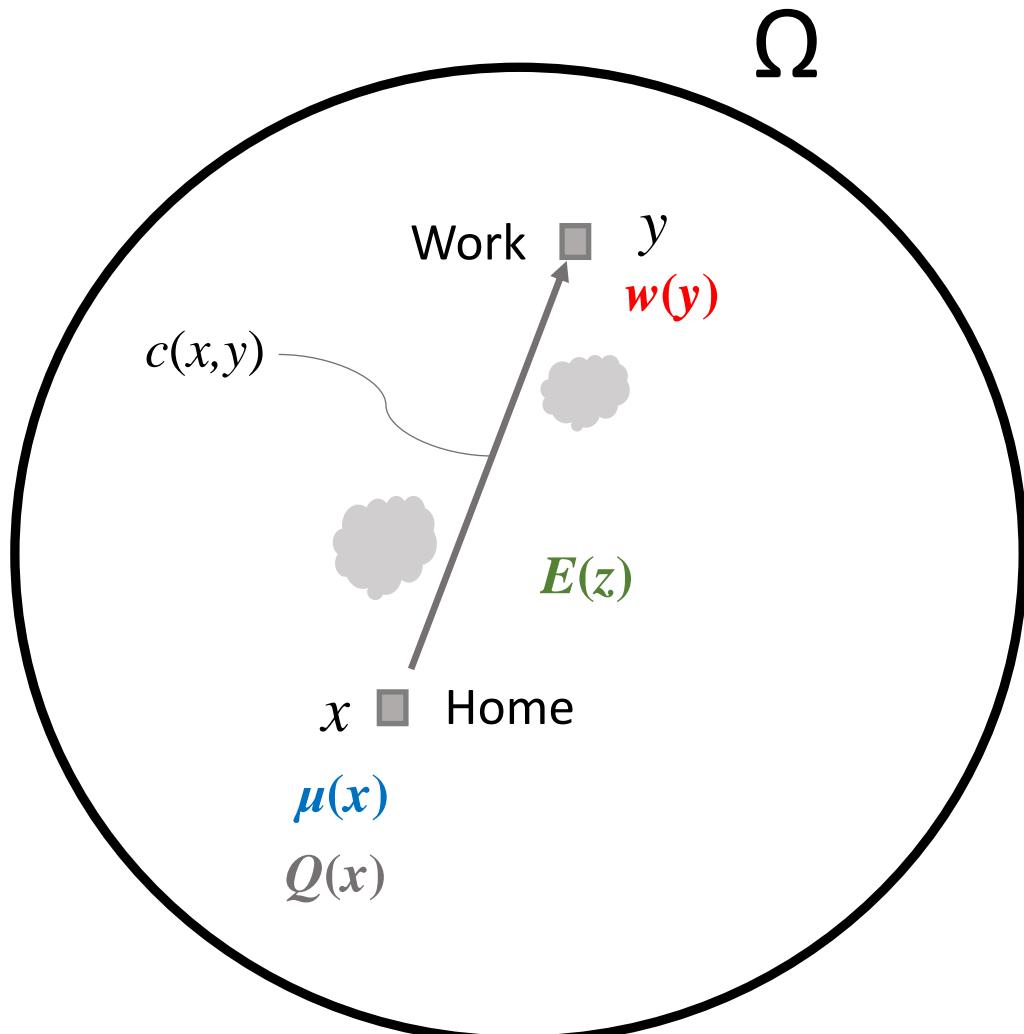
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diffusion

Model : Pollution



POLLUTION DISPERSION : *scalar transport equation*

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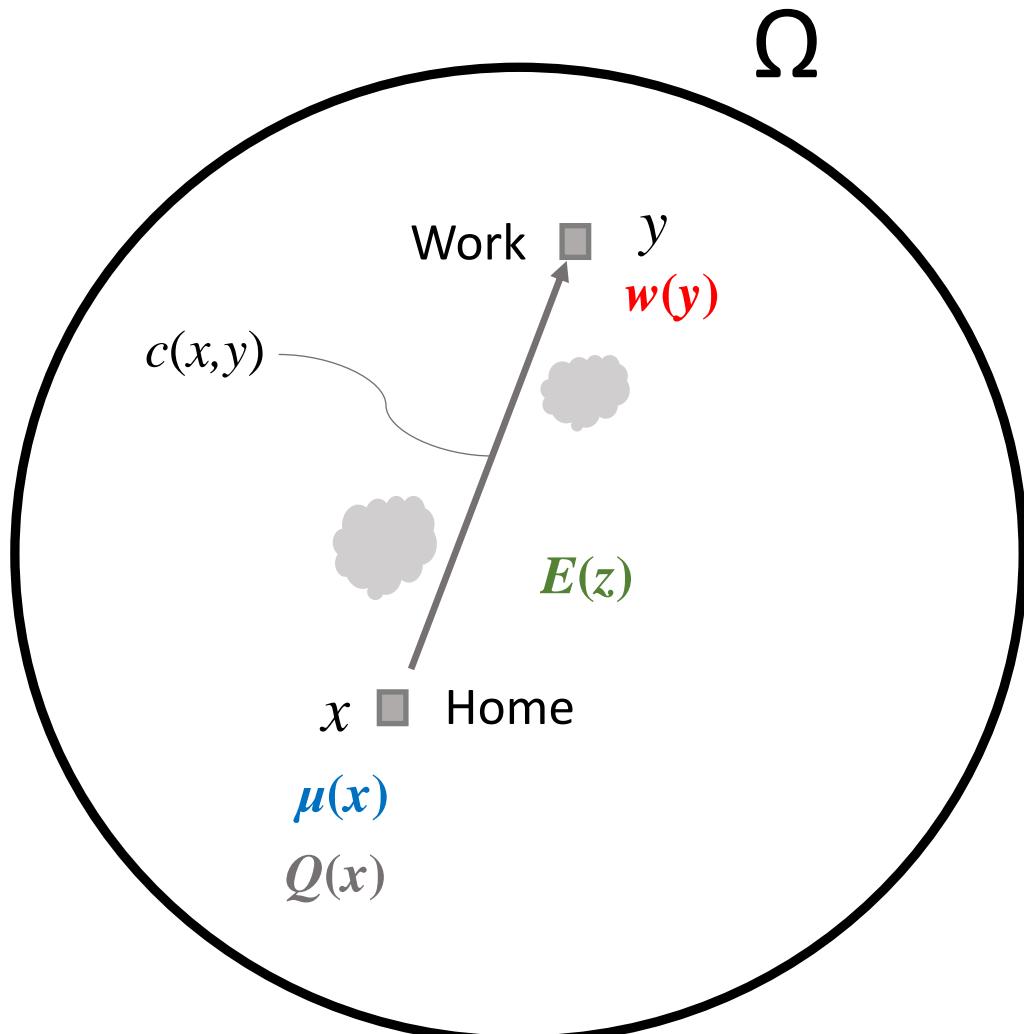


diffusion



advection

Model : Pollution



POLLUTION DISPERSION : *scalar transport equation*

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↑

diffusion

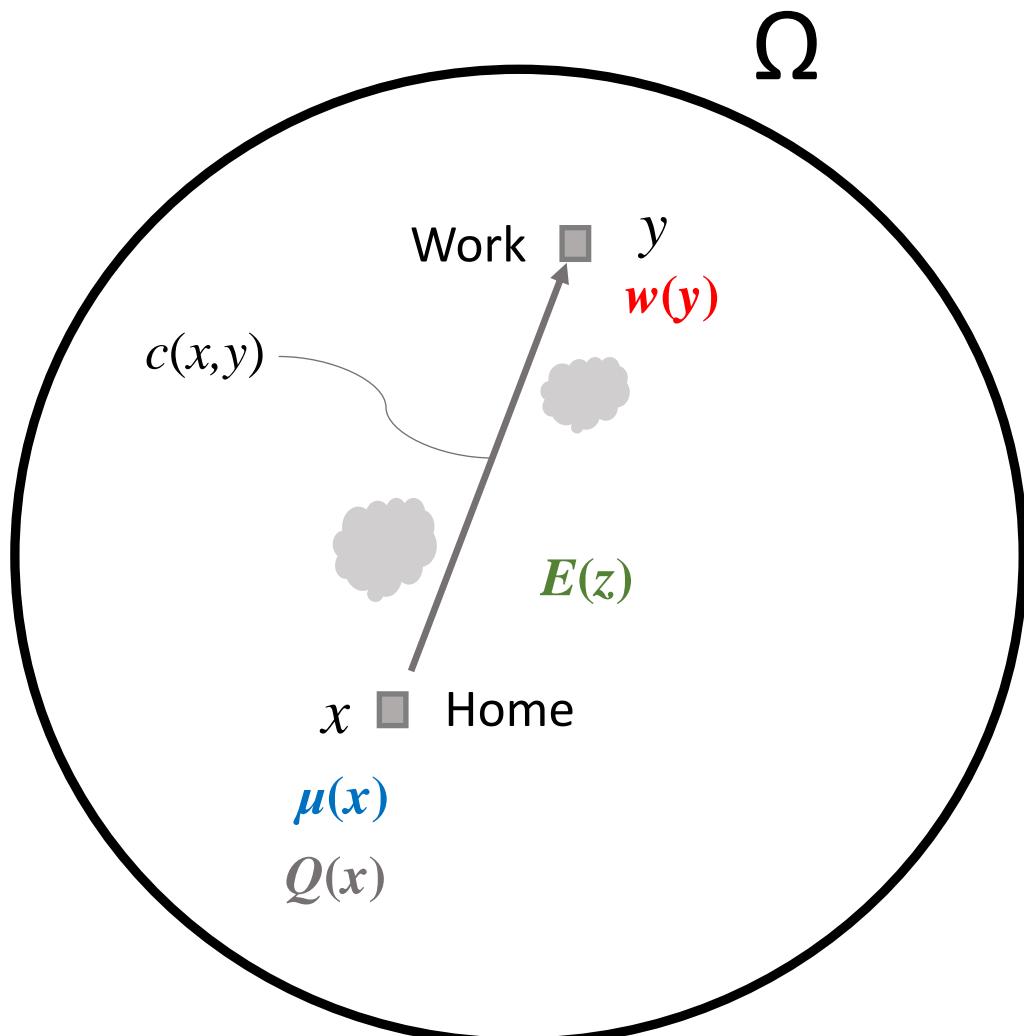
↑

advection

↑

*chemical
interactions*

Model : Pollution

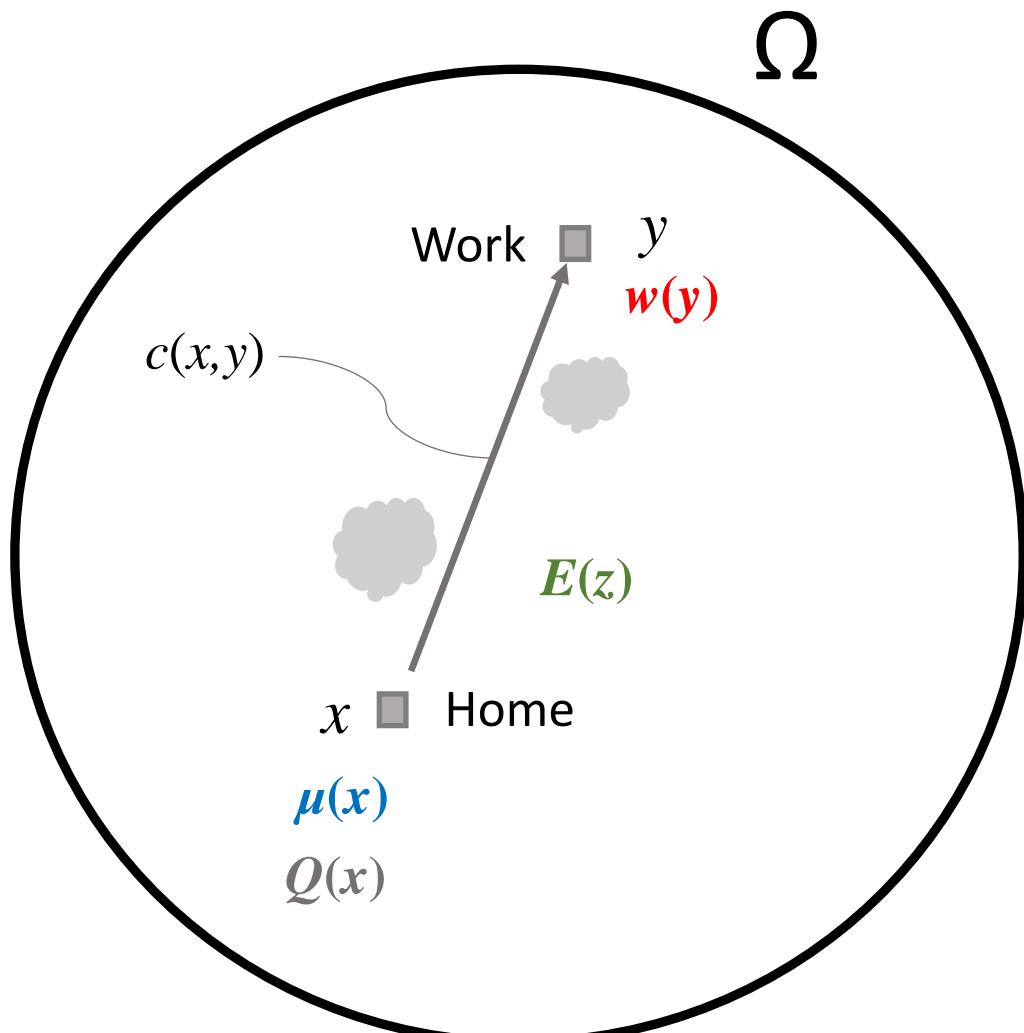


POLLUTION DISPERSION : *scalar transport equation*

$$\Delta \mathbf{E}(z) - \operatorname{div}(\mathbf{V}(z)\mathbf{E}(z)) + \chi(p, z) + f(z) - \Lambda \mathbf{E} = 0$$

↑
diffusion ↑
advection ↑
chemical interactions ↑
source term

Model : Pollution

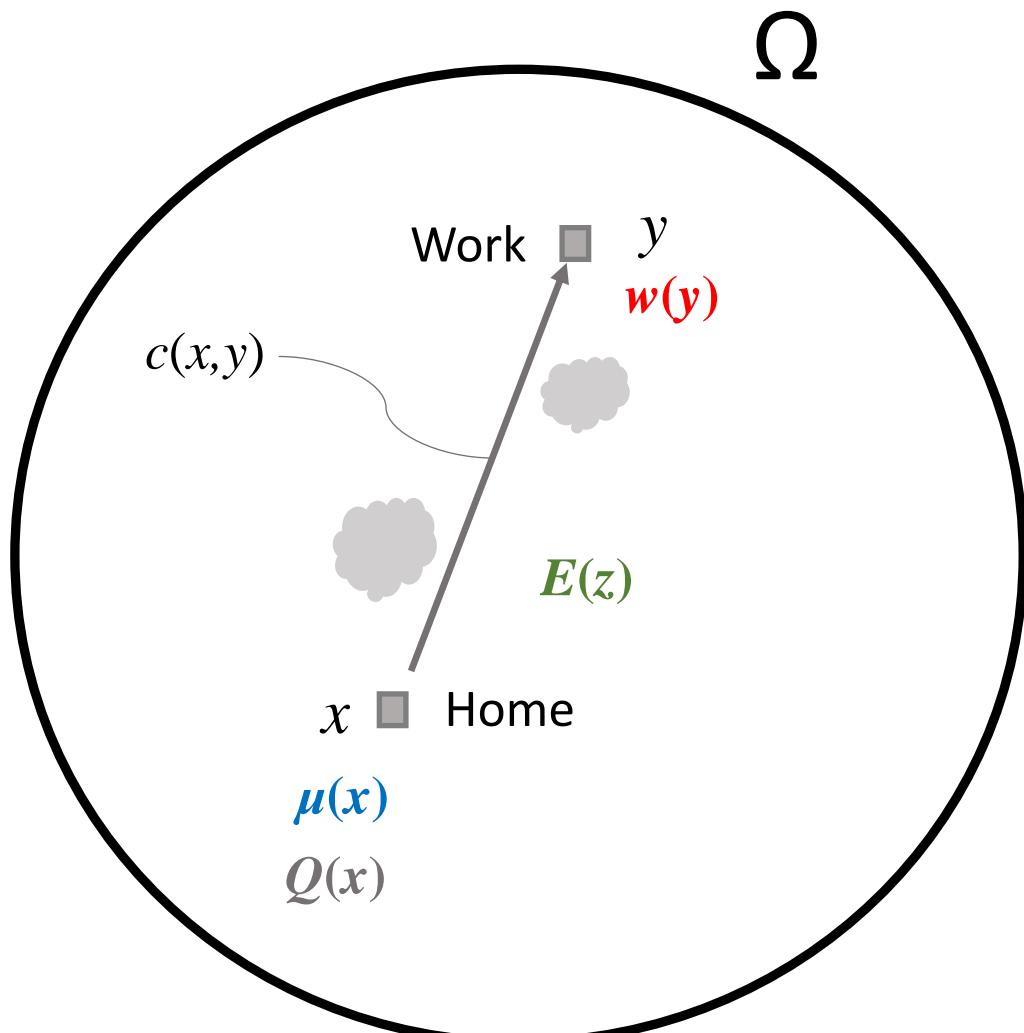


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↑
diffusion ↑
advection ↑
chemical interactions ↑
source term ↑
lessivage

Model : Pollution

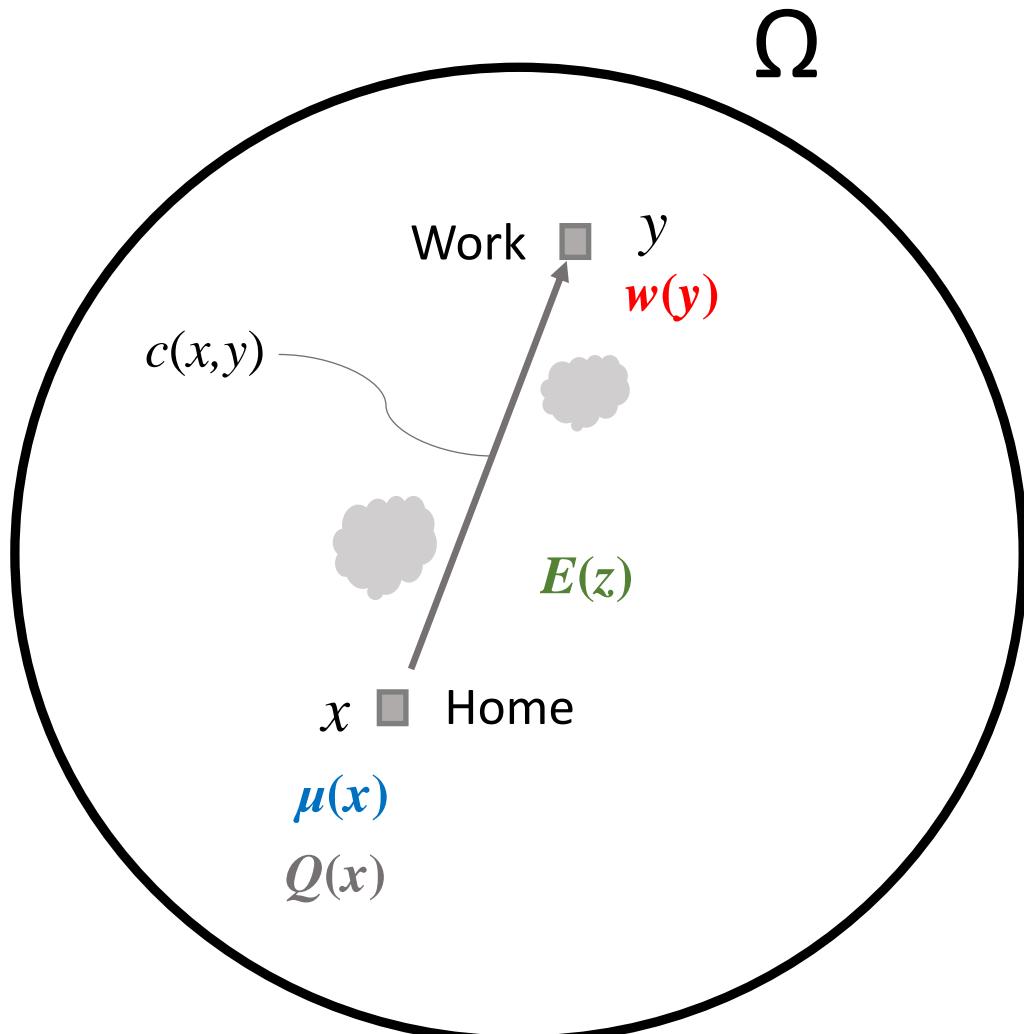


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↑ ↑ ↑ ↑ ↑
diffusion advection chemical interactions source term
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Model : Pollution



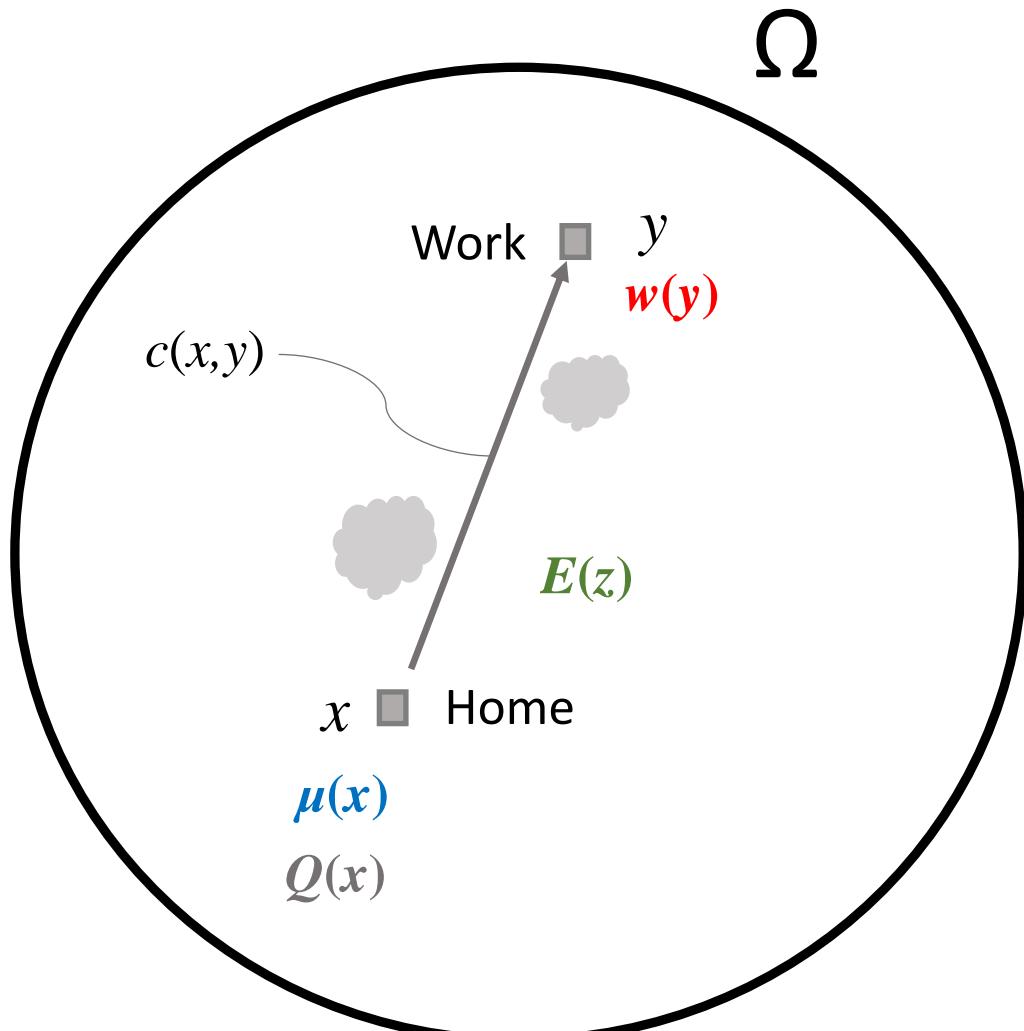
POLLUTION DISPERSION : *scalar transport equation*

$$\Delta \mathbf{E}(z) - \nabla \cdot (\mathbf{V}(z) \mathbf{E}(z)) + f(z) - \Lambda \mathbf{E} = 0$$

The wind field \mathbf{V} solves the 2D stationary Navier-Stokes equations

$$\begin{cases} -\Delta \mathbf{V}(z) + R(z)(\mathbf{V}(z) \cdot \nabla) \mathbf{V}(z) + \nabla p(z) = 0 & (z \in \Omega) \\ \nabla \cdot (\mathbf{V}(z)) = 0 & (z \in \Omega) \\ \mathbf{V}(s) = \xi & (s \in \partial\Omega) \end{cases}$$

Model : Pollution



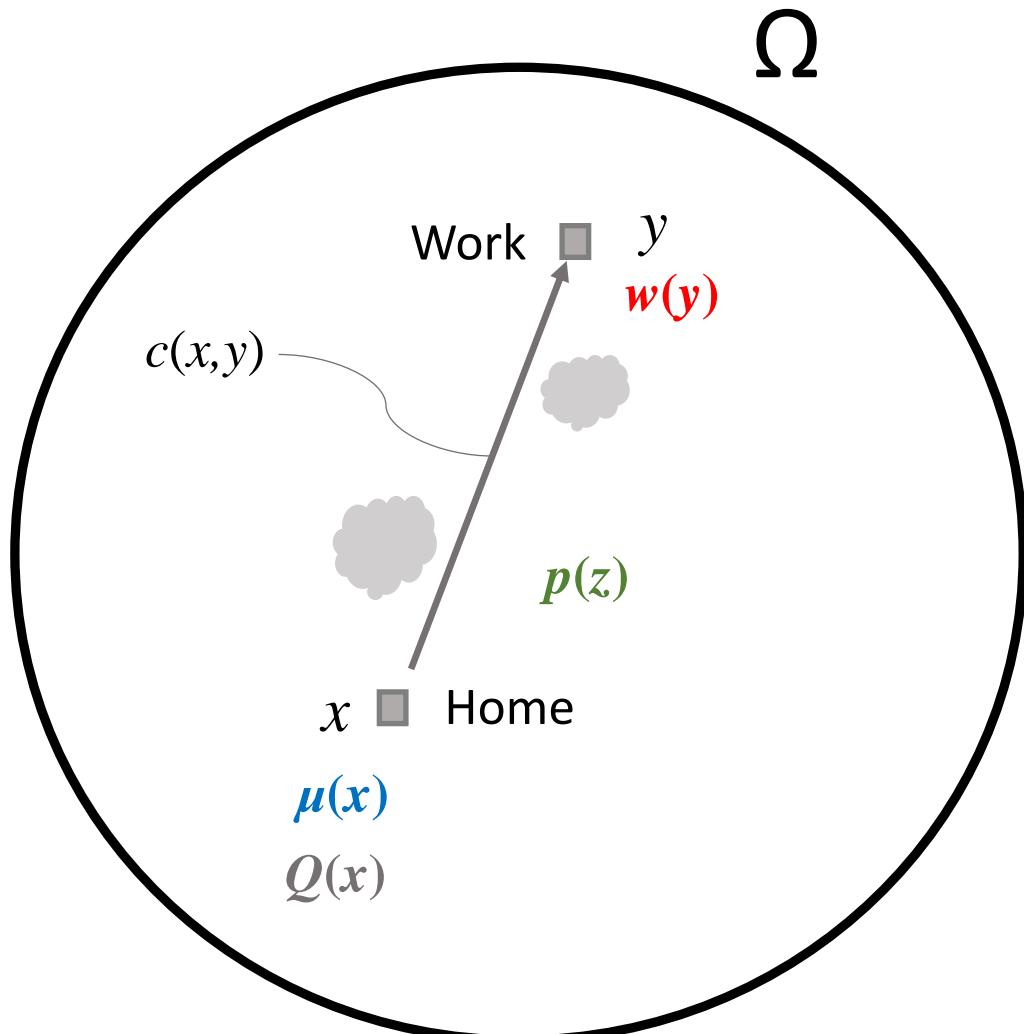
POLLUTION DISPERSION : *scalar transport equation*

$$\Delta \mathbf{E}(z) - \nabla \cdot (\mathbf{V}(z) \mathbf{E}(z)) + f(z) - \Lambda \mathbf{E} = 0$$

The wind field \mathbf{V} solves the 2D stationary Navier-Stokes equations

$$\left\{ \begin{array}{l} -\Delta \mathbf{V}(z) + R(z)(\mathbf{V}(z) \cdot \nabla) \mathbf{V}(z) + \nabla p(z) = 0 \quad (z \in \Omega) \\ \nabla \cdot (\mathbf{V}(z)) = 0 \quad (z \in \Omega) \\ \mathbf{V}(s) = \xi \quad (s \in \partial\Omega) \end{array} \right.$$

Model : Pollution



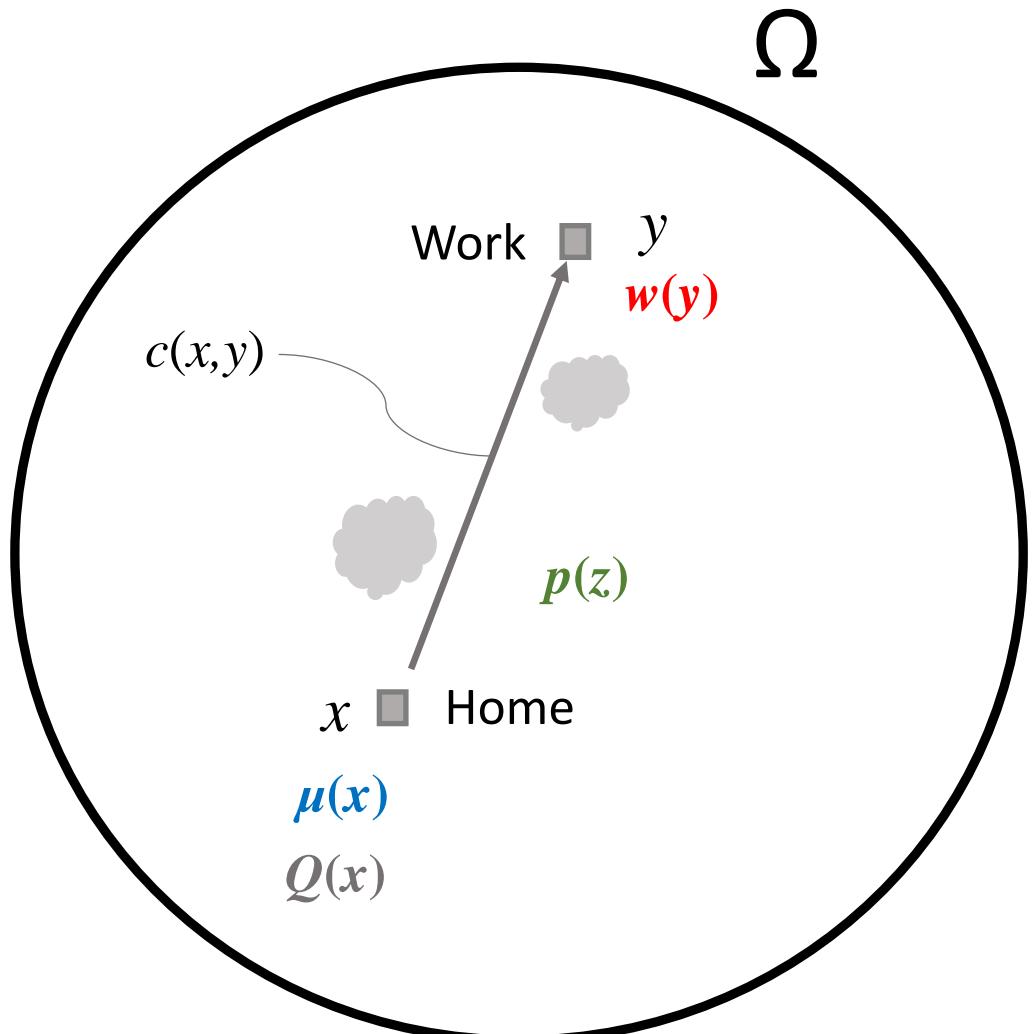
POLLUTION DISPERSION

$$\Delta \mathbf{E}(z) - \nabla \cdot (\mathbf{V}(z) \mathbf{E}(z)) + f(z) - \Lambda \mathbf{E} = 0$$

The automobile pollution source term is given by

$$f(z) = \int_{\Omega^2} \delta^{-1} \mathbf{1}_{z \in \Sigma(x,y)} \boldsymbol{\mu}(x) G(x, y, \mathbf{w}) dx dy \\ := f_{\boldsymbol{\mu}, \mathbf{w}}(z)$$

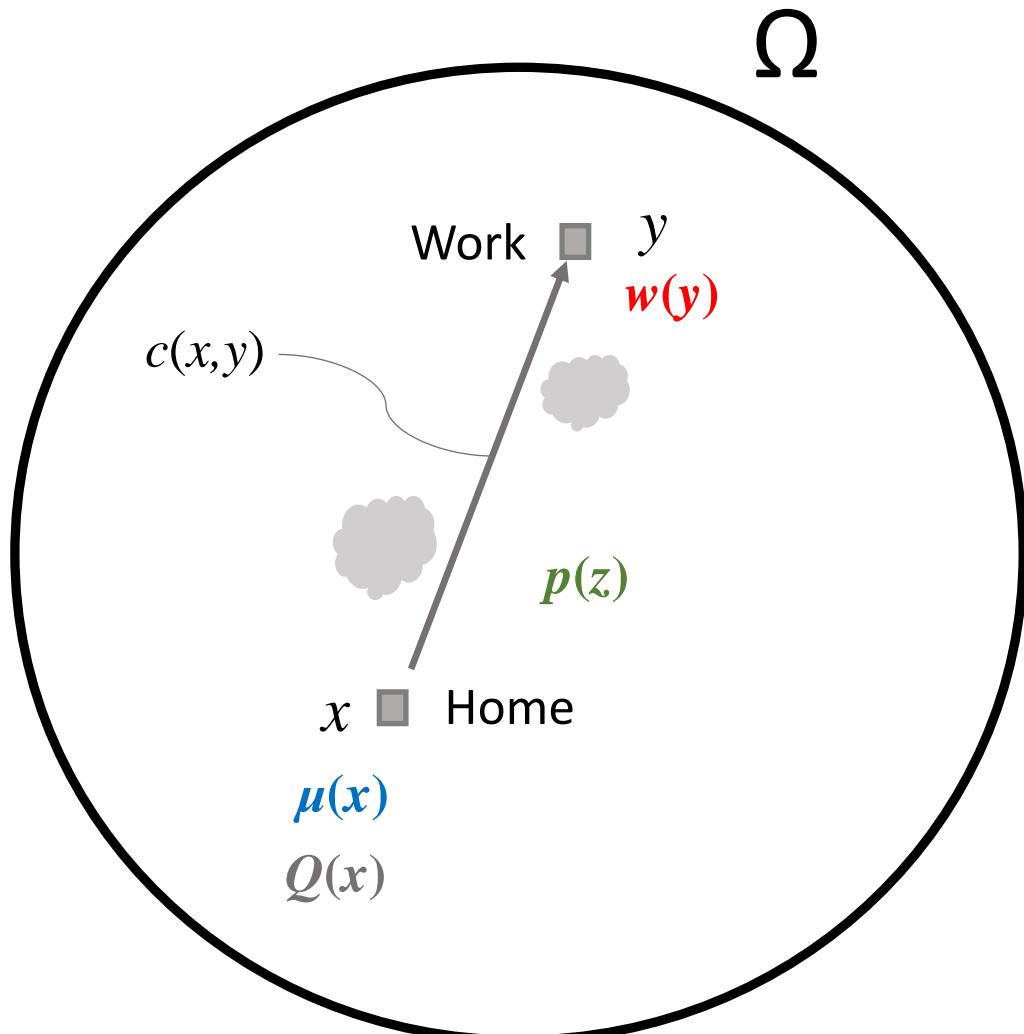
Model : Pollution



POLLUTION DISPERSION

$$\Delta \mathbf{E}(z) - \nabla \cdot (\mathbf{V}(z) \mathbf{E}(z)) + f_{\mu, w}(z) - \Lambda \mathbf{E} = 0, z \in \Omega$$

Model : Pollution

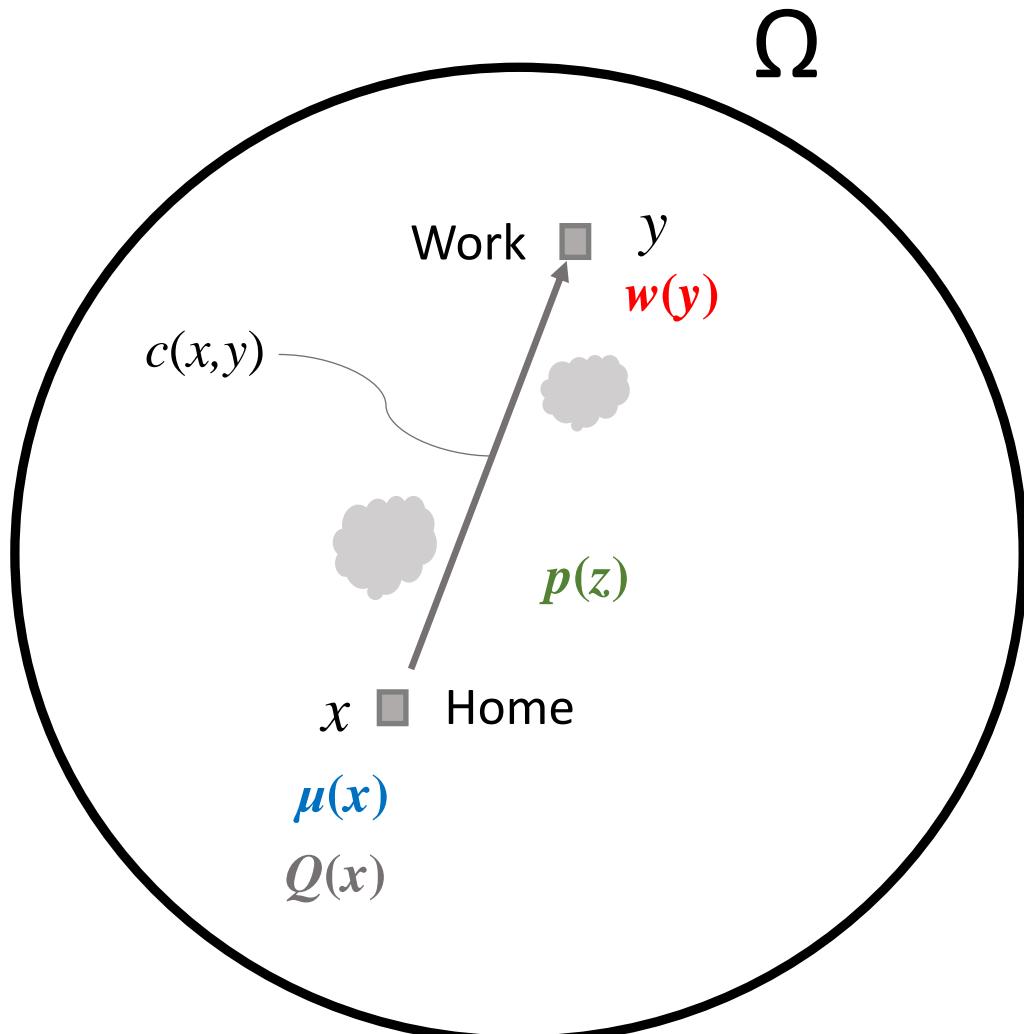


POLLUTION DISPERSION

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$$\mathbf{E}(s) = 0, \quad s \in \partial\Omega$$

Model : equilibrium



EQUILIBRIUM

$$\int_{x \in \Omega} G(x, y, \mathbf{w}) d\mu(x) = L(y, \mathbf{w}(y)), \quad y \in \bar{\Omega}$$

$$S_\theta(R(x, \mathbf{w}), \mathbf{Q}(x)) \mu(x) = 1, \quad x \in \bar{\Omega}$$

$$\begin{cases} -\Delta \mathbf{E}(z) + \nabla \cdot (\mathbf{V}(z) \mathbf{E}(z)) + \Lambda \mathbf{E} = f_{\mu, \mathbf{w}}(z), & z \in \Omega \\ \mathbf{E}(s) = 0, & s \in \partial\Omega \end{cases}$$

$$\text{supp } \mu \subset \text{argmax}_{x \in \bar{\Omega}} U_{\theta, \gamma}(R(x, \mathbf{w}), \mathbf{Q}(x), \mathbf{E}(x))$$

Model : Existence of equilibria

There exists at least one equilibrium

$$(\mathbf{w}, Q, \mathbf{E}, \boldsymbol{\mu}) \in C(\bar{\Omega}, \mathbb{R}_+^*)^2 \times H_0^1(\Omega) \times P_c(\bar{\Omega})$$

(Under some assumptions on L and c)

Proof: We apply Brouwer fixed-point theorem

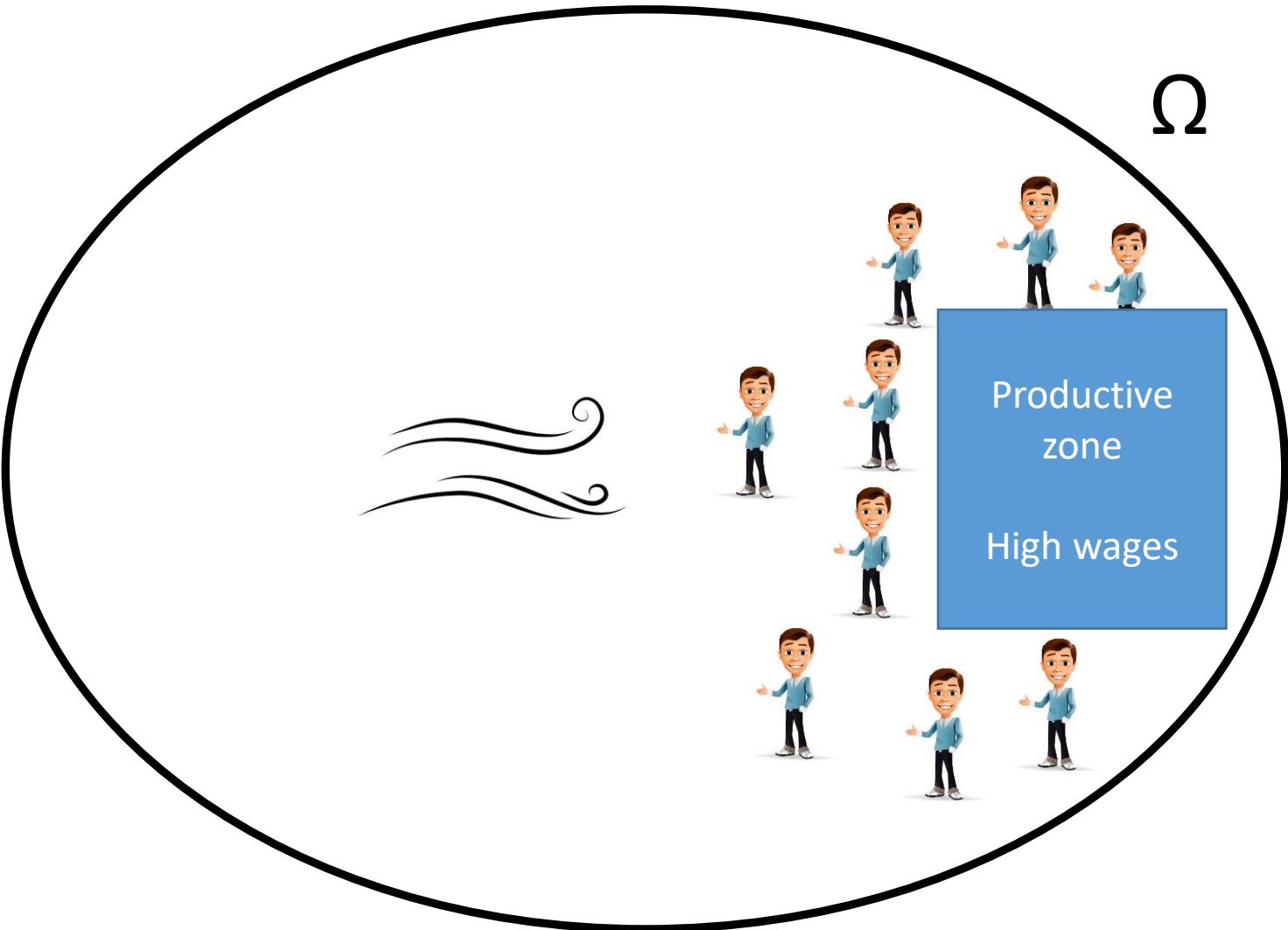
Analytical results

- $\boldsymbol{\mu}(x) = \frac{R_\sigma(x, \mathbf{w})^{\frac{\theta}{1-\theta}} \mathbf{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int R_\sigma(z, \mathbf{w})^{\frac{\theta}{1-\theta}} \mathbf{E}(z)^{-\frac{\gamma}{1-\theta}} dz}$
- $\int_{\Omega} \mathbf{E} = \lambda^{-1} \int_{\Omega^2} |x - y| G(x, y, \mathbf{w}) \boldsymbol{\mu}(x) dx dy$
- $\int_{\Omega} \mathbf{E} \boldsymbol{\mu} = \lambda^{-1} \int_{\Omega} f_{\boldsymbol{\mu}, \mathbf{w}} \boldsymbol{\mu} - \lambda^{-1} \int_{\Omega} \nabla \mathbf{E} \cdot \nabla \boldsymbol{\mu} + \lambda^{-1} \boxed{\frac{\theta}{1-\theta+\gamma} \int_{\Omega} [\mathbf{V} \cdot \nabla R_\sigma(\cdot, \mathbf{w})] \mathbf{E} \boldsymbol{\mu}}$

A combined effect of economic
and meteorological factors

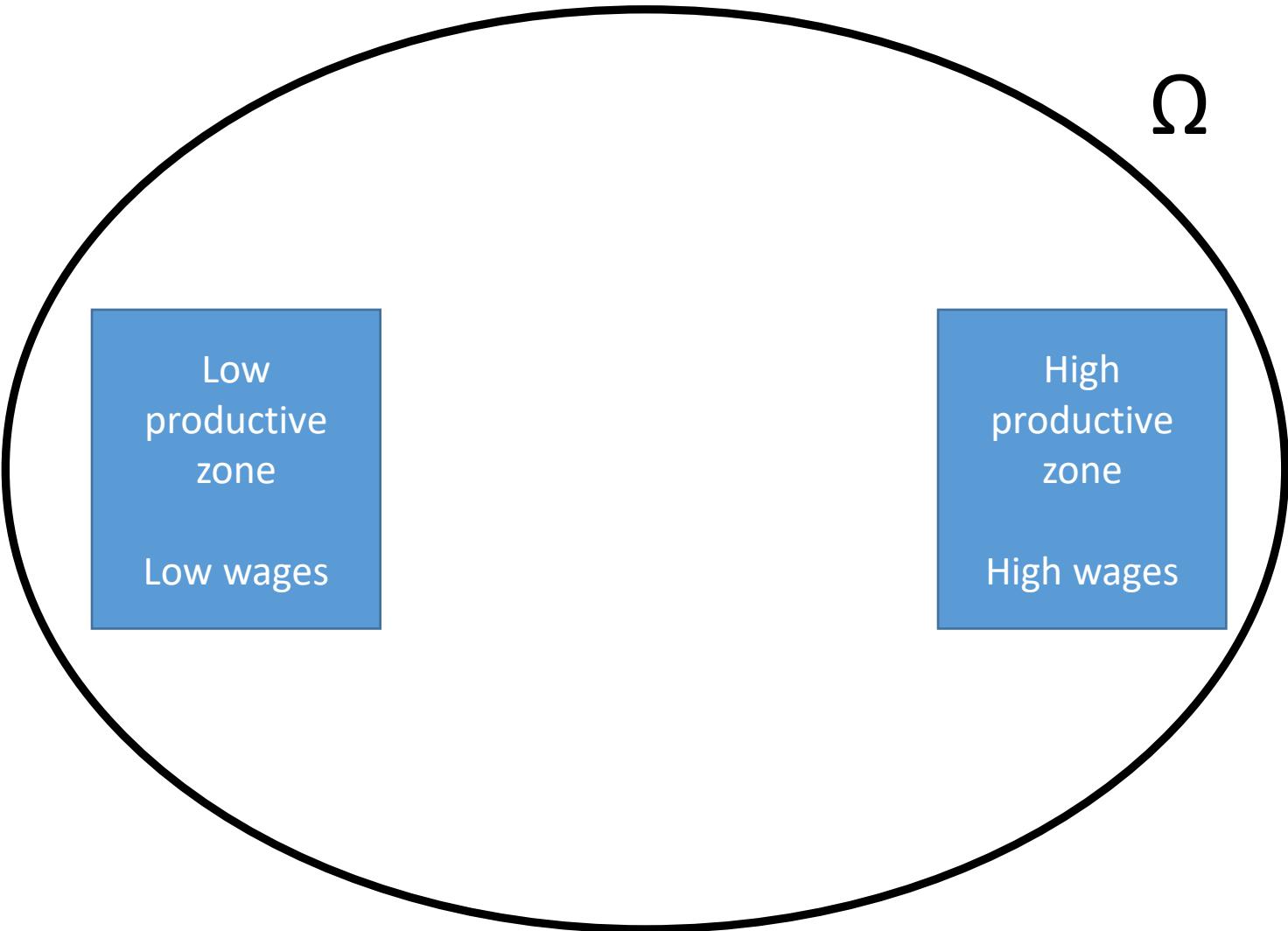
Analytical results

$$\begin{aligned}\nabla R_\sigma(\cdot, \mathbf{w}) & \quad 0 \longrightarrow E \\ V & \quad 0 \longrightarrow E\end{aligned}$$



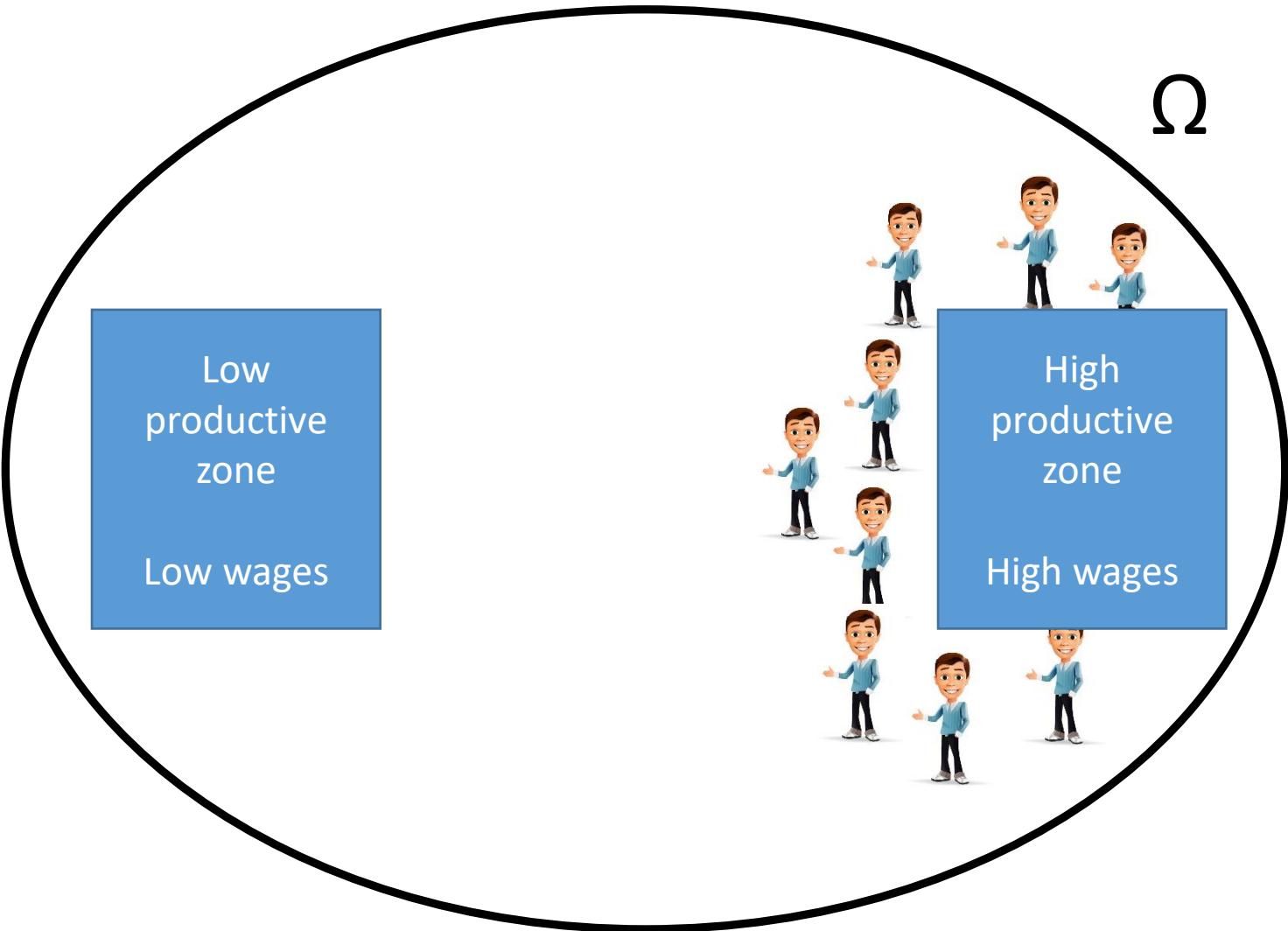
Analytical results

$$\begin{array}{l} \nabla R_\sigma(\cdot, \mathbf{w}) \\ \mathbf{V} \end{array} \quad \begin{array}{l} \mathbf{0} \longrightarrow \mathbf{E} \\ \mathbf{0} \longrightarrow \mathbf{E} \end{array}$$



Analytical results

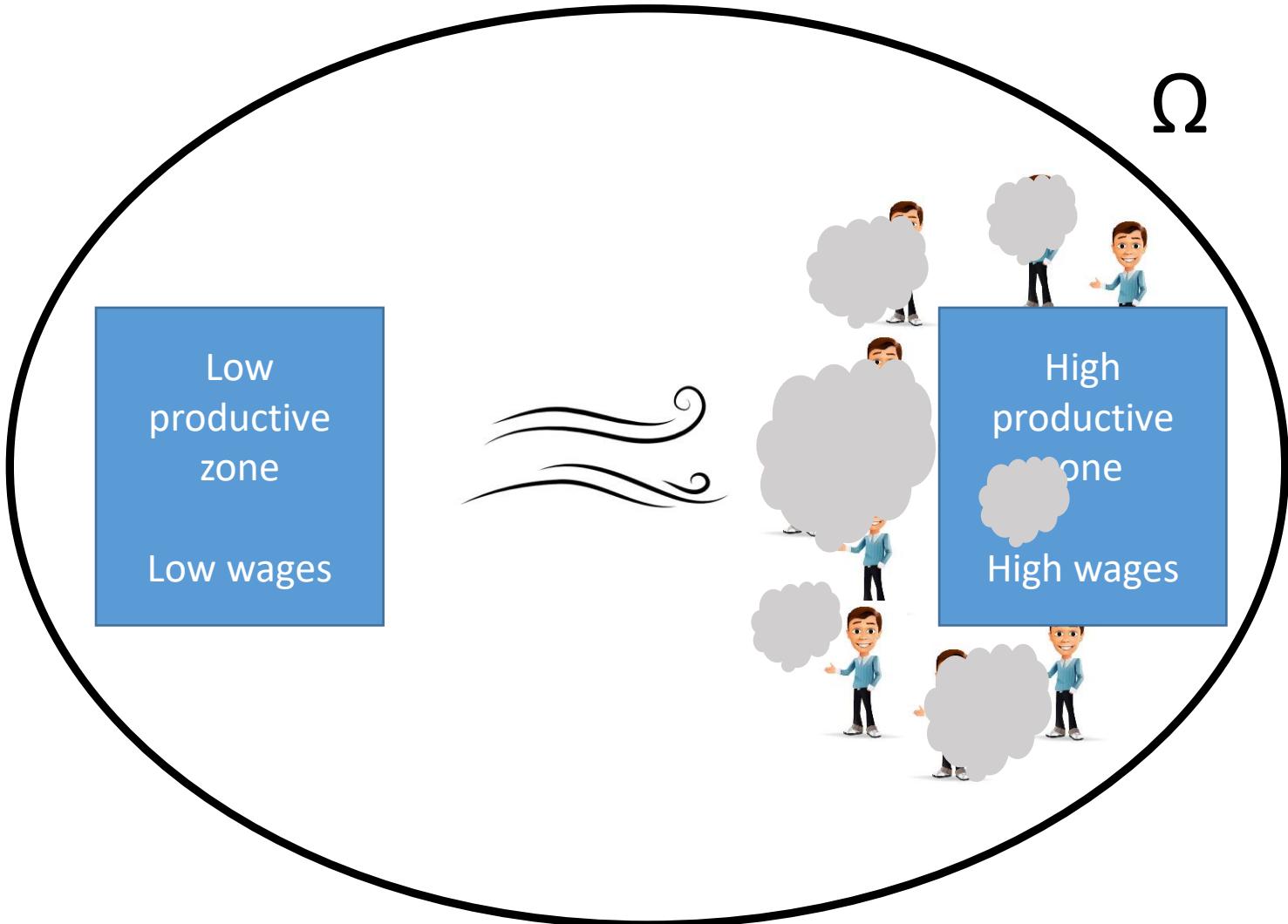
$$\begin{array}{l} \nabla R_\sigma(\cdot, \mathbf{w}) \\ \mathbf{V} \end{array} \quad \begin{array}{l} \mathbf{0} \longrightarrow \mathbf{E} \\ \mathbf{0} \longrightarrow \mathbf{E} \end{array}$$



Analytical results

$$\begin{array}{l} \nabla R_\sigma(\cdot, \mathbf{w}) \quad 0 \longrightarrow E \\ V \quad \quad \quad 0 \longrightarrow E \end{array}$$

In this case, wind tends to increase the level of pollution suffered by the residents

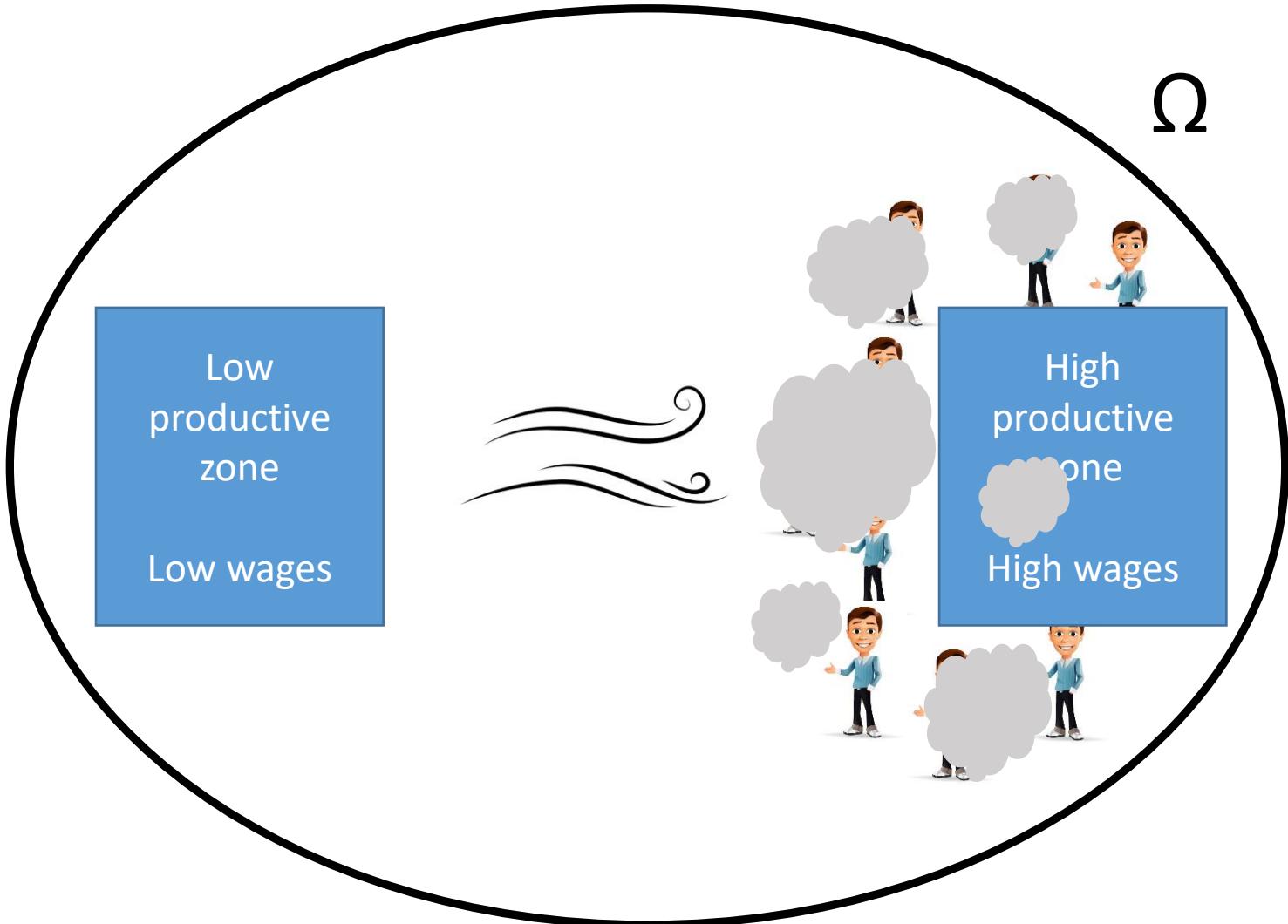


Analytical results

$$\begin{array}{l} \nabla R_\sigma(\cdot, \mathbf{w}) \quad 0 \longrightarrow E \\ V \quad \quad \quad 0 \longrightarrow E \end{array}$$

Now, γ becomes large

People are more sensitive
to air pollution

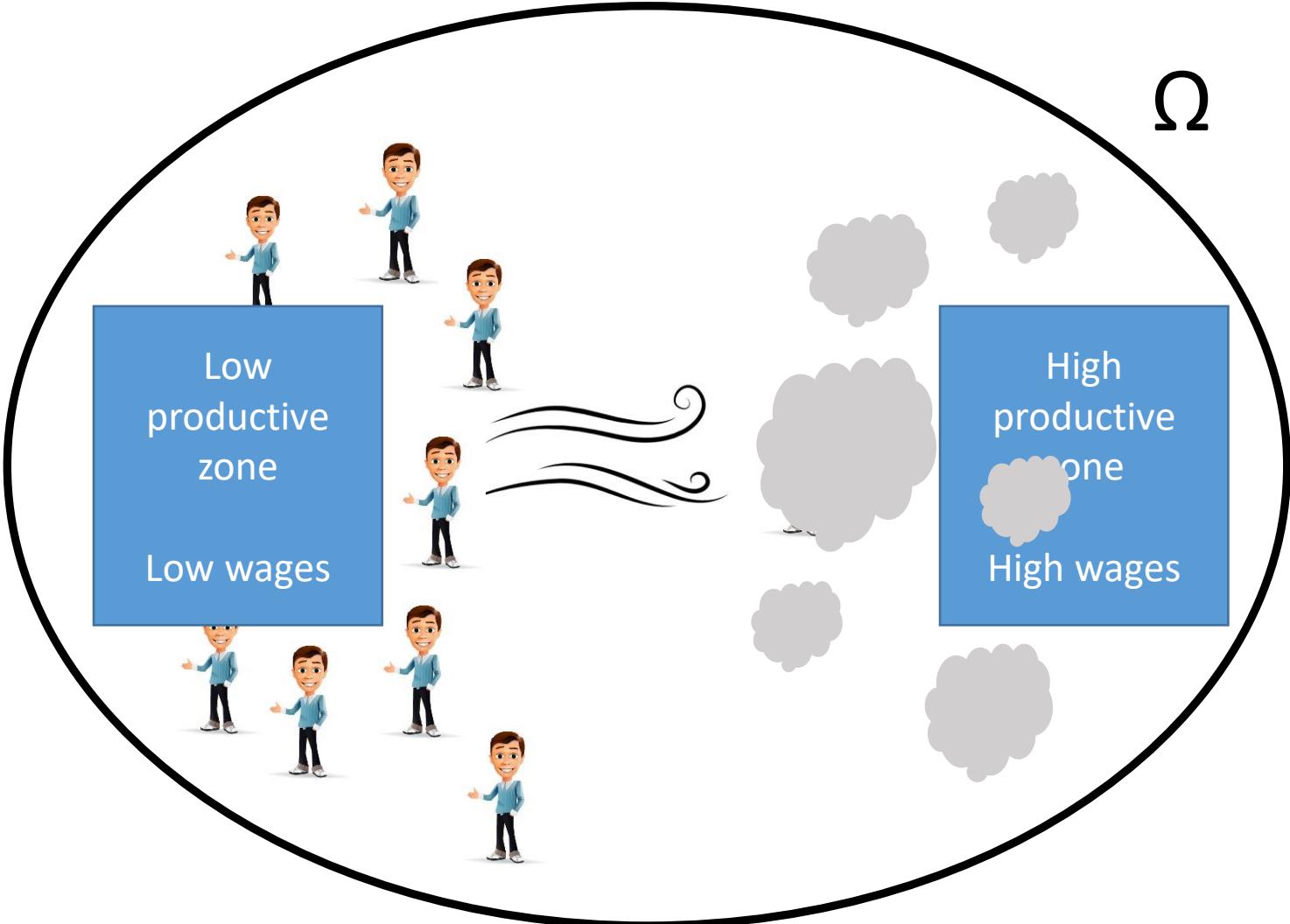


Analytical results

$$\begin{array}{l} \nabla R_\sigma(\cdot, \mathbf{w}) \quad 0 \longrightarrow E \\ V \quad 0 \longrightarrow E \end{array}$$

Now, γ becomes large

People are more sensitive
to air pollution



Next steps

- Uniqueness is not easy to get
- We are currently running numerical simulations in 2D
- The description of pollution dispersion can still be improved

Thank you