Optimal liquidation

with transient impact and signals

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Based on joint work with

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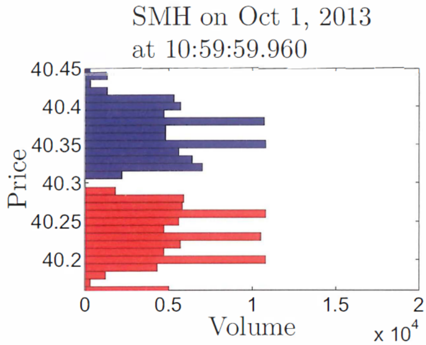
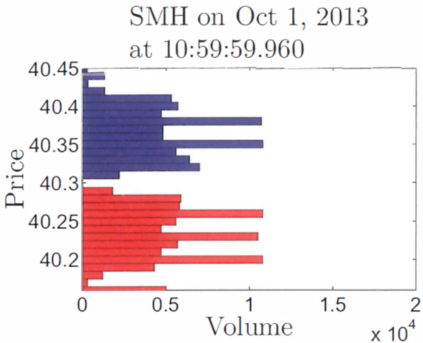
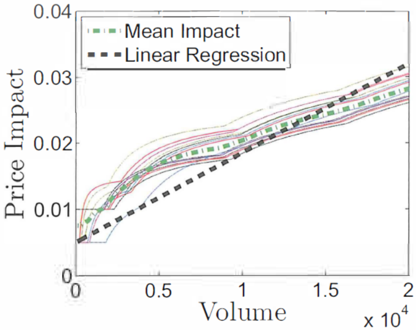
- Trader that wishes to liquidate on $[0, T]$ an initial position q_0 of shares
- will face market impact: price will drop
- will have to split the order to reduce his impact
- Trade-off between price impact (trade quicker) and price risk (take longer to complete all trades)
- Take into account some signals

Q What type of market impact?
A temporary/instantaneous and transient/permanent



Taken from the book of Cartea, Á., Jaimungal, S., & Penalva, J. (2015)





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Transient impact justified by:

- Linear equilibrium model by Kyle (85): the optimal strategy for an investor with insider information on the fundamental price of an asset is to trade incrementally through time: to minimize costs and revelation of information to the rest of the market.
- Empirical studies indicate a persistence of power-law behavior in time of Market Impact.

Initial position: $q > 0$ shares in a risky asset, at time $t \in [0, T]$:

$$Q_t^u = q - \int_0^t u_s ds,$$

where $(u_s)_{s\in[0,T]}$ denotes the trading speed.
The actual price at which the orders are executed is

$$S_t := \underbrace{P_t}_{\text{signal inclusion}} - \underbrace{\lambda u_t}_{\text{instanteneous}} - \underbrace{Z_t^u}_{\text{transient}}, \qquad 0 \leq t \leq T,$$

where P denotes some unaffected price process and

$$Z_t^u = h_0(t) + \int_0^t G(t,s) u_s ds, \qquad 0 \leq t \leq T,$$

for some square integrable deterministic function h_0 and transient impact kernel G.

Large class of transient impact kernels that offer great modeling flexibility:

- Permanent impact
$$G(t,s) = 1_{s<t}$$
- Exponential kernel
$$G(t,s) = e^{-\lambda(t-s)} 1_{s<t}, \quad \lambda \in \mathbb{R}$$
- Fractional kernel
$$G(t,s) = (t-s)^{H-1/2} 1_{s<t}, \quad H \in (0,1),$$
- Non-convolution for bond with maturity T trading
$$G(t,s) = f(t-T)H(t-s)$$
with $f(0) = 0$ due to the terminal condition on the bond price.

Maximize objective functional:

$$J(u) := \mathbb{E}\left[\int_0^T (P_t - Z_t^u)u_t dt - \lambda \int_0^T u_t^2 dt + Q_T^u P_T - \phi \int_0^T (Q_t^u)^2 dt - \varrho (Q_T^u)^2\right],$$

- The first three terms represent the trader's terminal wealth: final cash position including the accrued trading costs which are induced by temporary and transient price impact, as well as remaining final risky asset position's book value:

$$-\int_0^T S_t dQ_t^u \approx -\sum_i S_{t_i}(Q_{t_{i+1}}^u - Q_{t_i}^u)$$

- The fourth and fifth terms implement a penalty $\phi > 0$ and $\varrho > 0$ on running and terminal inventory.

Controlled process in $\mathbb{R}^2$

$$Q_t^u = q - \int_0^t u_s ds,$$
$$Z_t^u = h_0(t) + \int_0^t G(t,s) u_s ds,$$

where $u_t \in \mathbb{R}$ belongs to some admissible set $\mathcal{A}$, G square-integrable

Performance functional

$$J(u) := \mathbb{E}\left[\int_0^T (P_t - Z_t^u) u_t dt - \lambda \int_0^T u_t^2 dt + Q_T^u P_T - \phi \int_0^T (Q_t^u)^2 dt - \varrho (Q_T^u)^2\right],$$

Optimization problem

$$V_0 = \sup_{u \in \mathcal{A}} J(u)$$

(!) Linear-Quadratic Volterra control problem with stochastic coefficients and control only in the drift

Literature on transient impact and signals Gatheral & Schied (2012), Alfonsi & Schied (2013), Cartea & Jaimungal (2016), Bouchaud et al. (2018), Lehalle & Neuman (2019), Belak, Muhle-Karbe & Ou (2019), Brigo, Graceffa & Neuman (2022), Forde, Sánchez-Betancourt & Smith (2022), Neuman & Voss (2022).

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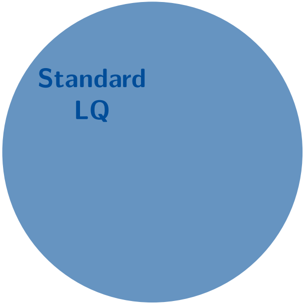
Challenges and Limitations

1. Difficulty in dealing with (non-convolution) singular kernels, non-Markovianity, non-semimartingality
2. Lack of numerical methods,
3. **In the LQ framework:** underlying LQ structure not well identified/understood.
4. Explicit solution?

Aim Treat all challenges in one go.

Conventional LQ problems



- Revisit conventional LQ problems: Basic linear-quadratic (LQ) problems are solved by different methods relying on Itô stochastic calculus including standard dynamic programming, maximum principle, HJB equation, ...

Basic linear-quadratic (LQ) regulator problem with BM noise W

$$X_t^{\alpha} = \int_0^t \alpha_s ds + W_t,$$

and a quadratic cost functional on finite horizon T to minimize

$$J(\alpha) = \mathbb{E}\Big[\int_0^T \big(|X_t^{\alpha}|^2 + \alpha_t^2\big)\,dt\Big].$$

Optimal control

$$\alpha_t^* = -\psi_{t,T} X_t^{\alpha^*}, \quad 0 \leq t \leq T,$$

where ψ is a deterministic nonnegative function:

$$\psi_{t,T} = \tanh(T - t),$$

that is solution to the Riccati equation:

$$\dot{\psi}_{t,T} = -1 + \psi_{t,T}^2, \quad \psi_{T,T} = 0,$$

and thus the associated optimal state process X^{α^*} is a mean-reverting Markov process.

Dimension $d = 1$:

$$dX_s^\alpha = \alpha_s ds + dW_s$$

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left((X_s^\alpha)^2 + \alpha_s^2\right) ds\right]$$

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Ansatz for value function:

$$V_t^\alpha = \psi_t X_t^2$$

for some deterministic functions $t \to \psi_t$ to be determined such that $\Psi_T = 0$.

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Strategy: Inspired by martingale verification argument: Find Γ such that

$$S_t^\alpha := V_t^\alpha + \int_0^t \left(X_s^2 + \alpha_s^2\right) ds$$

is a submartingale.

$$dX_s^\alpha = \alpha_s ds + dW_s$$
$$S_t^\alpha := V_t^\alpha + \int_0^t \left(X_s^2 + \alpha_s^2\right) ds$$
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By Itô:

$$\begin{aligned} dS_t^\alpha = X_t^2 \left(\dot{\psi}_t + 1\right) dt \\ + \left(\alpha_t^2 + 2\psi_t \alpha_t X_t\right) dt \\ + 2X_t \psi_t dW_t \end{aligned}$$

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Completion of squares: on red term

$$(\bullet) = \left(\alpha_t - \alpha_t^*\right)^2 - \psi_t^2 X_t^2$$

with

$$\alpha_t^* = -\psi_t X_t$$

$$
\begin{aligned}
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Vanishing first term if ψ_t solves the Backward Riccati equation:

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\dot{\psi}_t = -1 + \psi_t^2, \quad \psi_T = 0.
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$\Rightarrow M^\alpha = S^\alpha - \int_0^\cdot (\alpha_s - \alpha_s^*)^2 ds$ is a local martingale.

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True martingale under suitable admissible controls

Writing the martingale property $\mathbb{E}[M_T^\alpha|\mathcal{F}_t] = M_t^\alpha$ we obtain

$$J_t(\alpha) - V_t^\alpha = \mathbb{E}\left[\int_t^T (\alpha_s - \alpha_s^*)^2 ds \;\middle|\; \mathcal{F}_t\right] \geq 0,$$

where

$$J_t(\alpha) := \mathbb{E}\left[\int_t^T \left(X_s^2 + \alpha_s^2\right) ds \;\middle|\; \mathcal{F}_t\right].$$

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where

$$J_t(\alpha) := \mathbb{E}\left[\int_t^T \left(X_s^2 + \alpha_s^2\right) ds \,\middle|\, \mathcal{F}_t\right].$$

This shows that α^* is an optimal control and $V_t^{\alpha^*}$ is the value function of the problem:

$$V_t^{\alpha^*} = \inf_{\alpha \in \mathcal{A}_t(\alpha^*)} J_t(\alpha)$$

where

$$\mathcal{A}_t(\alpha') := \{\alpha \in \mathcal{A} : \alpha_s = \alpha'_s, \quad s \leq t\}.$$

$$\mathcal{A} = \left\{ \alpha : \Omega \times [0,T] \to \mathbb{R} \text{ progressive such that } \sup_{0 \le t \le T} \mathbb{E}\left[\int_0^T |\alpha_t|^2 \right] dt < \infty \right\}$$

Verification result

Assume that

1. There exists a solution ψ to the Riccati equation:

$$\dot{\psi}_t = -1 + \psi_t^2, \quad \psi_T = 0.$$

2. There exists an admissible control α^* satisfying

$$\alpha_t^* = -\psi_t X_t^{\alpha^*}$$

Then, α^* is an optimal control and $V_t^{\alpha^*} = \Gamma_t (X_t^{\alpha^*})^2$ is the value function of the problem:

$$V_t^{\alpha^*} = \inf_{\alpha} J_t(\alpha)$$

Crucial observation

The same Riccati equation

$$\dot{\psi}_t = -1 + \psi_t^2, \quad \psi_T = 0.$$

also appears in the computation of the Laplace transform of the integrated square of Brownian motion $X = \frac{1}{\sqrt{2}} W$:

$$\mathbb{E}\left[\exp\left(-\int_0^T X_s^2 ds\right)\right] = \exp\left(\phi_t + \psi_t X_t^2\right)$$

Laplace transform

$$X_t = \int_0^t K(t,s)dW_s$$

- Non-Markovian/ non-semimartingale

Compute Laplace transform of functionals of the square of X to understand LQ structure and how to recover Markovianity.

Fix $K : [0, T] \to \mathbb{R}$, W standard Brownian motion and

$$X_t = X_0 + \int_0^t K(t, s) dW_s.$$

where $\int_0^T \int_0^T |K(t, s)|^2 dtds < \infty$. Define

$$g_t(s) := \mathbb{E}_t[X_s]\mathbf{1}_{s \geq t} = \left(X_0 + \int_0^t K(s, u) dW_u \right) 1_{s \geq t},$$

$$\Sigma_t(s, u) := \int_t^{s \wedge u} K(s, r) K(u, r) dr, \quad t \leq s, u \leq T.$$

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Aim: compute Laplace transform of integrated squared process

$$L_{t,T} = \mathbb{E}_t\left[\exp\left(w\int_t^T X_s^2 ds\right)\right], \quad w \le 0.$$

Idea: Exploit Gaussianity.

Laplace transform

Formal derivation

$$X_t = X_0 + \int_0^t K(t,s)dW_s, \quad L_{t,T} = \mathbb{E}_t\left[\exp\left(w\int_t^T X_s^2 ds\right)\right].$$

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1. From dynamic to static:

$$w\int_t^T X_s^2 ds \approx w\frac{(T-t)}{n}\sum_{i=1}^n X_{t_i^n}^2 = w\frac{(T-t)}{n}\mathrm{Tr}(X_n X_n^\top),$$

where $X_n = (X_{t_1^n}, \ldots, X_{t_n^n})^\top$.

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3. $X_n X_n^\top$ follows a Wishart distribution with explicit Laplace transform (cf next slide):

$$L_{t,T}^n = \mathbb{E}_t\left[\exp\left(w\frac{(T-t)}{n}\mathrm{Tr}\left(X_n X_n^\top\right)\right)\right] = \exp\left(\phi_{t,T}^n + (\mathbf{g}_t^n)^\top \Psi_{t,T}^n \mathbf{g}_t^n\right),$$

Laplace transform
Formal derivation

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4. Send $n\to\infty$ and expect $L_{t,T}^n \to L_{t,T}$

$$L^n_{t,T} = \exp\left(\phi^n_{t,T} + (\mathbf{g}^n_t)^\top \Psi^n_{t,T} \mathbf{g}^n_t\right)$$

where

$$\Psi^n_{t,T} = w\frac{(T-t)}{n}\left(I_n - 2w\frac{(T-t)}{n}\mathbf{\Sigma}^n_t\right)^{-1}$$

$$\phi^n_{t,T} = -\frac{1}{2}\log\det(I_n - 2w\frac{(T-t)}{n}\mathbf{\Sigma}^n_t)$$

$$L^n_{t,T} = \exp\left(\phi^n_{t,T} + (\mathbf{g}^n_t)^\top \Psi^n_{t,T}\mathbf{g}^n_t\right) \to \exp\left(\phi_{t,T} + \langle g_t, \boldsymbol{\Psi}_{t,T} g_t\rangle_{L^2}\right),$$

where

$$\Psi^n_{t,T} = w\frac{(T-t)}{n}\left(I_n - 2w\frac{(T-t)}{n}\boldsymbol{\Sigma}^n_t\right)^{-1} \to w\left(\mathrm{id} - 2w\boldsymbol{\Sigma}_t\right)^{-1} := \boldsymbol{\Psi}_{t,T}$$

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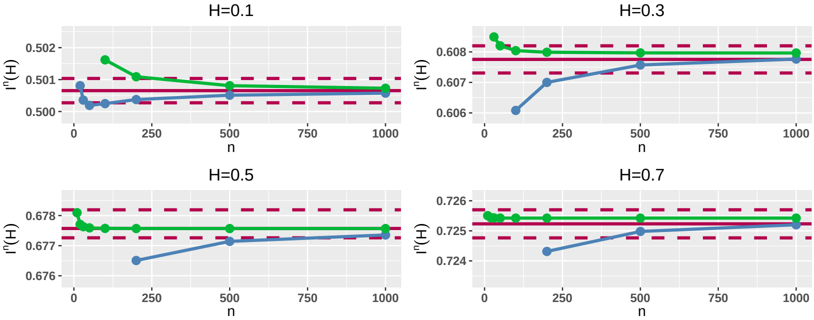
where

- $\langle f, g\rangle_{L^2} = \int_0^T f(s)g(s)ds$
- $\forall K \in L^2\left([0,T]^2, \mathbb{R}^{N\times N}\right)$, $\boldsymbol{K}$ is the integral operator on $L^2([0,T])$ induced by the kernel K $(\boldsymbol{K}g)(s) = \int_0^T K(s,u)g(u)du.$

Let W^H denote a fractional Brownian motion with Hurst index $H \in (0,1)$. Approximation via closed form Wishart marginals

$$\mathbb{E}\left[\exp\left(-\frac{1}{n}\sum_{i=1}^{n}\left(W_{t_i}^H\right)^2\right)\right] \to \mathbb{E}\left[\exp\left(-\int_0^1 \left(W_s^H\right)^2 ds\right)\right]$$



Riemann sum (blue), Gauss Legendre quadrature (green), and benchmark MC (red) with 90% confidence.

Back to 1903

Fredholm determinant

From Wikipedia, the free encyclopedia

In mathematics, the **Fredholm determinant** is a complex-valued function which generalizes the determinant of a finite dimensional linear operator. It is defined for bounded operators on a Hilbert space which differ from the identity operator by a trace-class operator. The function is named after the mathematician Erik Ivar Fredholm.

Definition [edit]

Let *H* be a Hilbert space and *G* the set of bounded invertible operators on *H* of the form *I* + *T*, where *T* is a trace-class operator. *G* is a group because

$$(I+T)^{-1} - I = -T(I+T)^{-1},$$

so $(I+T)^{-1}-I$ is trace class if *T* is. It has a natural metric given by $d(X, Y) = \|X - Y\|_1$, where $\| \cdot \|_1$ is the trace-class norm.

If *H* is a Hilbert space with inner product $(\cdot, \cdot)$, then so too is the *k*th exterior power $\Lambda^k H$ with inner product

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k) = \det(v_i, w_j).$$

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Déterminant de Fredholm

4 langues

Article Discussion Plus

$$\phi(x) + \int_0^1 K(x,t)\phi(t)\,dt = f(x) \quad (*).$$

On peut essayer de discrétiser cette équation :

- en l'évaluant sur une famille de points $(x_l)_{0\leq l\leq n-1}$ équirépartis dans l'intervalle $[0;1]$: $x_k = \frac{k}{n},\ 1 \leq l \leq n-1.$
- en approchant l'intégrale par une somme de Riemann : $\int_0^1 k(x_i,t)\phi(t)\,dt \approx \sum_{j=0}^{n-1} \frac{1}{n} k(x_i,x_j)\phi(x_j).$

On obtient alors, pour chaque $n \in \mathbb{N}$, un système linéaire (E_n) d'équations

$$(E_n) : \begin{cases} \phi(x_1) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_1,x_j)\phi(x_j) = f(x_1) \\ \dots \\ \phi(x_i) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_i,x_j)\phi(x_j) = f(x_i) \\ \dots \\ \phi(x_n) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_n,x_j)\phi(x_j) = f(x_n) \end{cases}$$

- **Ivar Fredholm** (1903). Sur une classe d'équations fonctionnelles. Acta mathematica, 27(1), 365-390.
- His work foreshadowed the theory of **David Hilbert** on Hilbert spaces.
- **Henry McKean** (2010): "Nowadays, Fredholm's determinant is explained mostly in an abstract and, if I may say so, a **superficial way**."
- **Peter Lax** (2002) book: entire chapter for Fredholm's theory: "it is time to **resurrect** Fredholm's determinant."

$$X_t = X_0 + \int_0^t K(t,s)dW_s \in \mathbb{R}^{d\times m}$$

Assumption on K: $\sup_{t\le T}\int_0^T |K(t,s)|^2 ds < \infty, \quad \lim_{h\to 0}\int_0^T |K(t+h,s)-K(t,s)|^2 ds = 0$

Laplace transform

Fix $w \in \mathbb{S}^d_-$. Then,

$$\mathbb{E}_t\left[\exp\left(\int_t^T \operatorname{Tr}\left(X_s^\top w X_s\right) ds\right)\right] = \frac{\exp\left(\langle g_t, \boldsymbol{\Psi}_{t,T} g_t\rangle_{L^2}\right)}{\det(\mathrm{id} - 2\sqrt{w}\boldsymbol{\Sigma}_t\sqrt{w})^{m/2}},$$

where $\quad \boldsymbol{\Psi}_{t,T} = \sqrt{w}\left(\mathrm{id} - 2\sqrt{w}\boldsymbol{\Sigma}_t\sqrt{w}\right)^{-1}\sqrt{w}.$

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Dynamic viewpoint $\dot{\boldsymbol{\Psi}}_t = 2\boldsymbol{\Psi}_t \dot{\boldsymbol{\Sigma}}_t \boldsymbol{\Psi}_t$ (Operator Riccati 'linearizable')

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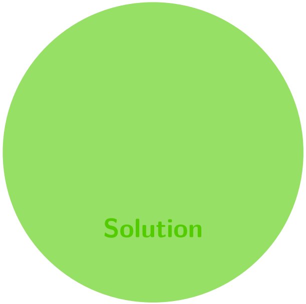
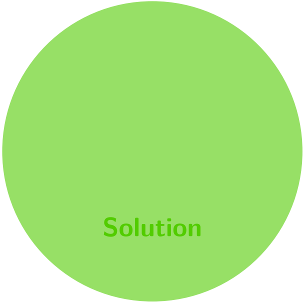
Quadratic ansatz $\exp(\phi_t + \langle g_t, \boldsymbol{\Psi}_{t,T} g_t\rangle_{L^2})$

Solution to the execution problem



Intuition for the *solution*:

Solution to the execution problem



Intuition for the *solution*:

1. Ansatz for the value function on suitable g_t process
2. Use same Martingale approach as in standard LQ,
3. Solve the Riccati equations explicitely

Controlled process in $\mathbb{R}^2$

$$Q_t^u = q - \int_0^t u_s ds,$$

$$Z_t^u = h_0(t) + \int_0^t G(t,s) u_s ds,$$

where $u_t \in \mathbb{R}$ belongs to some admissible set $\mathcal{A}$, G square-integrable

Performance functional

$$J(u) := \mathbb{E}\left[\int_0^T (P_t - Z_t^u) u_t dt - \lambda \int_0^T u_t^2 dt + Q_T^u P_T - \phi \int_0^T (Q_t^u)^2 dt - \varrho (Q_T^u)^2\right],$$

Optimization problem

$$V_0 = \sup_{u \in \mathcal{A}} J(u)$$

(!) Linear-Quadratic Volterra control problem with stochastic coefficients and control only in the drift

Define

$$\tilde{G}(t,s) = 2\varrho \mathbb{1}_{\{s<t\}} + G(t,s)$$

and

$$\boldsymbol{D}_t := 2\lambda \mathrm{id} + (\tilde{\boldsymbol{G}}_t + \tilde{\boldsymbol{G}}_t^*) + 2\phi \mathbf{1}_t^* \mathbf{1}_t,$$

where $\mathbf{1}_t$ is the integral operator induced by the kernel

$$\mathbb{1}_t(u,s) = \mathbb{1}_{\{s\le u\}} 1_{\{s\ge t\}}.$$

$$\boldsymbol{\Gamma}_t^{-1} = \begin{pmatrix} \boldsymbol{D}_t^{-1} & -2\phi \boldsymbol{D}_t^{-1}\mathbf{1}_t^* \\ -2\phi \mathbf{1}_t \boldsymbol{D}_t^{-1} & -2\phi \mathrm{id} + 4\phi^2 \mathbf{1}_t \boldsymbol{D}_t^{-1} \mathbf{1}_t^* \end{pmatrix}.$$

And define the process

$$\Theta_t(s) = -\left(\boldsymbol{\Gamma}_t^{-1} \mathbb{1}_t \mathbb{E}_t[P_\cdot - P_T] e_1\right)(s).$$

Optimal trading speed

$$u_t^* = a_t + \int_0^t B(t,s) u_s^* ds, \quad t \in [0,T],$$

where

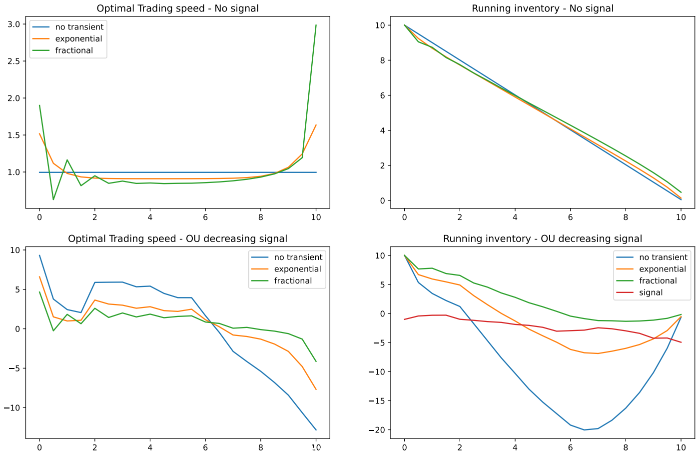
$$a_t = \frac{1}{2\lambda}\left(\mathbb{E}[(P_t - P_T) \mid \mathcal{F}_t] - \tilde{h}_0(t) + \langle \Theta_t, K_t \rangle_{L^2} + \langle \boldsymbol{\Gamma}_t^{-1} K_t, 1_t(\tilde{h}_0, q)^\top \rangle_{L^2}\right),$$

$$B(t,s) = 1_{\{s<t\}} \frac{1}{2\lambda}\left(\langle \boldsymbol{\Gamma}_t^{-1} K_t, 1_t(\tilde{G}(\cdot,s), -1)^\top \rangle_{L^2} - \tilde{G}(t,s)\right).$$

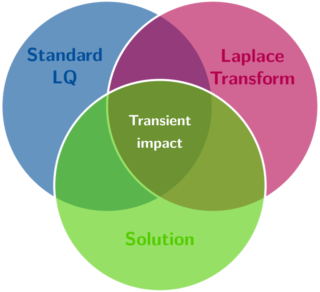
where

$$K_t(s) = (\tilde{G}(s,t), -\mathbb{1}_{t \le s})^\top \quad \text{and} \quad \tilde{h}_0(t) = h_0(t) - 2\varrho q.$$

Solution to the execution problem
Numerics



Wrap-up



- Martingale verification argument as in conventional case.
- Infinite dimensional control on suitable lifted process
- Analytic solutions for operator Riccati equations,
- Efficient and straightforward numerical methods.

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