A mean field control problem of PDMP and its application for smart charging

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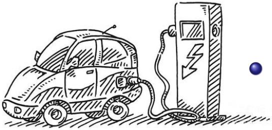
Séminaire du Fime

Outline

Motivations

A central planner wants to charge optimally a huge fleet of EVs over a finite time horizon. Different constraints must be taken into account:

- Satisfy EV owner requirements.
- Exploit EVs flexibility, in particular Vehicle-to-Grid (V2G).



- State variable : $X_t = (I_t, S_t)$
- I_t mode of charging (fast charging, idle, V2G...)
- S_t level of battery

Piecewise Deterministic Markov Process (PDMP)

Description of the process

Let $X_t = (I_t, S_t) \in \mathcal{I} \times [0,1]$ be a PDMP(b, α) :

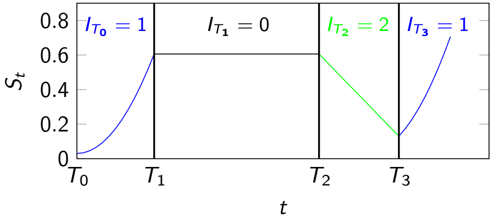


Figure 1: Evolution of the hybrid state variable $X_t = (I_t, S_t)$ over the time

- I_t is a jump process with values in $\mathcal{I} = \{0, 1, \ldots, d\}$, switching spontaneously, at jump times $\{T_k\}_{k \in \mathbb{N}}$ given by a Poisson process with intensity α.
- S_t follows a deterministic dynamics between two consecutive jumps:

$$\frac{d}{dt} S_t = b(I_t, S_t) \quad \forall t \in [T_k, T_{k+1})$$

Construction

Knowing T_k and $X_{T_k} = (I_{T_k}, S_{T_k})$, one obtains $(T_{k+1}, X_{T_{k+1}})$ as follows:

$$\begin{cases} \text{For any } j \in \mathcal{I} \\ T_{k+1,j} := \inf\left\{ t \geq T_k : \, E_{k+1,j} < \int_{T_k}^{t} \alpha_j(r, X_r) dr \right\} \quad \text{where } E_{k+1,j} \sim \text{Exp}(1) \\ T_{k+1} := \min_{j \in \mathcal{I}} T_{k+1,j} \\ I_{T_{k+1}} = \min\left\{ j \in \mathcal{I} \, : \, T_{k+1,j} = T_{k+1} \right\} \\ S_{T_{k+1}} := \int_{T_k}^{T_{k+1}} b(I_t, S_t) dt \\ X_{T_{k+1}} = (I_{T_{k+1}}, S_{T_{k+1}}) \end{cases}$$

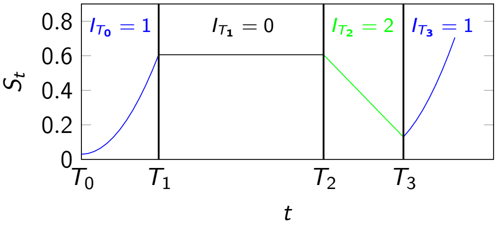


Figure 2: Evolution of the hybrid state variable $X_t = (I_t, S_t)$ over the time

- These processes are introduced rigorously in [Davis, 1984].
- Multiple applications in system **reliability and maintenance** [De Saporta and Zhang, 2013], **oil production** [Zhang et al., 2014], **biology** [Lin and Buchler, 2018], **insurance** [Marciniak and Palmowski, 2016], **communication networks**[Hespanha, 2005] etc...
- Existence of a large literature on the optimal control of PDMP using **dynamic programming** [Costa et al., 2016, De Saporta et al., 2017, Huang and Guo, 2019, Verms, 1985] or **BSDE representation** [Bandini, 2018].
- Existence of a growing literature on the analysis of the **mean field limit** of population of PDMPs [Diez, 2020, Monmarché, 2018].

The mean field control problem

The N agents problem

Let N PDMP $X^{1,\alpha^1}, \dots, X^{N,\alpha^N}$, with empirical initial distribution $m^0 \in \mathcal{P}^N(\mathcal{I} \times [0,1])$, controlled by $\alpha^1, \dots, \alpha^N \in$ $\mathbb{A}^N := \{\alpha \in C^0([0,T] \times (\mathcal{I} \times [0,1])^N, \mathbb{R}^d_+) : \forall i \in \mathcal{I}, \alpha_i(\cdot, i, \cdot) = 0\}$.
Objective function:

$$J^N(\alpha^1, \dots, \alpha^N) :=$$

$$\mathbb{E}\Big[\int_0^T \underbrace{f\left(t, \frac{1}{N}\sum_{n=1}^N p(t, X_t^{n,\alpha^n})\right)}_{\text{coupling cost}} dt\Big]$$

$$+\frac{1}{N}\sum_{n=1}^N \mathbb{E}\Big[\underbrace{\int_0^T c(t, X_t^{n,\alpha^n}) + \sum_{j\in I} L(\alpha_j^n(t, \{X_t^{k,\alpha^k}\}_k))dt + g(X_T^{n,\alpha^n})}_{\text{individual cost}}\Big]$$

The Mean Field Limit Control problem

Let X^α be a PDMP(b,α), with initial distribution $m^0 \in \mathcal{P}^N(\mathcal{I}\times[0,1])$, controlled by $\alpha \in \mathbb{A} := \{\alpha \in C^0([0,T]\times\mathcal{I}\times[0,1],\mathbb{R}^d_+) : \forall i \in \mathcal{I}, \alpha_i(\cdot,i,\cdot)=0\}$.
Objective function:

$$J(\alpha) := \int_0^T \underbrace{f\left(t, \mathbb{E}\left[p(t,X_t^\alpha)\right]\right)}_{\text{mean field interraction}} dt$$

$$+\mathbb{E}\Big[\int_0^T c(t,X_t^\alpha) + \sum_{j\in I} L(\alpha_j(t,X_t^\alpha))dt + g(X_T^\alpha)\Big]$$

Optimization problem:

$$\boxed{\min_{\alpha\in\mathbb{A}} J(\alpha)} \qquad (P)$$

1. Out of the scope of optimal control of PDMP.
2. Problem (P) is a priori **not convex**.
3. Numerical Approximation?

Mean field control literature

On the Mean Field Control literature:

- Itô processses : [Lacker, 2017, Carmona and Delarue, 2015, Carrillo et al., 2020, Pham and Wei, 2018]
- Common noise : [Djete et al., 2022]
- Discrete Markov processes : [Cecchin, 2021]
- Regime switching processes : [Bayraktar et al., 2021]
- Optimal stopping : [Talbi et al., 2021]

Assumptions

- Assumptions on the dynamics: $b \in C^1(I \times [0,1])$ and $b(i,0) = b(i,1) = 0$ for any $i \in I$.
- Assumptions on the coupling cost: $p \in C^1([0,T] \times I \times [0,1])$ and $f \in C^1([0,T] \times \mathbb{R})$ is strictly convex, with Lipschitz continuous gradient w.r.t. the second variable, and there exists $C > 0$ such that, for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\frac{x^2}{2C_f} - C_f \leq f(t,x) \leq C_f \frac{x^2}{2} + C_f.$$

- Assumptions on the local cost: $c \in C^1([0,T] \times I \times [0,1])$ and $g \in C^1(I \times [0,1])$. The function $L \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is increasing, strongly convex and there exists $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

Lagrangian decomposition

Let $\bar{J}$ be defined for any $v \in L^2(0, T)$ and $\alpha \in \mathbb{A}$ by:

$$\bar{J}(\alpha, v) := \int_0^T f(t, v(t))dt + \mathbb{E}\Big[\int_0^T c(t, X_t^\alpha) + \sum_{j \in I} L(\alpha_j(t, X_t^\alpha))dt + g(X_T^\alpha)\Big]$$

Problem (P) is equivalent to

$$\min_{\alpha \in \mathbb{A}, v \in L^2(0,T)} \bar{J}(\alpha, v), \quad \text{s.t } \mathbb{E}[p(t, X_t^\alpha)] - v(t) = 0 \text{ a.e on } [0, T] \qquad (\bar{P})$$

Lagrangian decomposition

Lagrangian $\mathcal{L} : \mathbb{A} \times L^2(0,T) \times L^2(0,T) \to \mathbb{R}$:

$$\mathcal{L}(\alpha, v, \lambda) := \bar{J}(\alpha, v) + \langle \mathbb{E}[p(t, X^\alpha)] - v, \lambda \rangle_{L^2(0,T)} = \mathcal{L}_1(\alpha, \lambda) + \mathcal{L}_2(v, \lambda),$$

where

$$\mathcal{L}_1(\alpha, \lambda) := \mathbb{E}\Big[\int_0^T c(t, X_t^\alpha) + \sum_{j \in I} L(\alpha_j(t, X_t^\alpha)) + p(t, X_t^\alpha)\lambda(t) dt + g(X_T^\alpha) \Big],$$

$$\mathcal{L}_2(v, \lambda) := \int_0^T f(t, v(t)) - v(t)\lambda(t) dt,$$

Dual function $\mathcal{W} : L^2(0,T) \to \mathbb{R}$:

$$\mathcal{W}(\lambda) := \underbrace{\inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)}_{\text{optimal control of PDMP}} + \underbrace{\inf_{v \in L^2(0,T)} \mathcal{L}_2(v, \lambda)}_{\text{convex problem}}. \tag{1}$$

Dual Problem

Dual Problem :

$$\boxed{\max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda).} \qquad (D)$$

Lemma

There exists a unique $\bar{\lambda} \in L^2(0, T)$ *such that* $\bar{\lambda} = \arg\max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda)$.

☞ Existence of a saddle point?

Dual Problem

Dual Problem :

$$\boxed{\max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda).} \qquad (D)$$

Theorem (Le Corre, Oudjane, S. (2022))

There is no duality gap associated with Problem (D), i.e.,

$$\max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda) = \min_{\alpha \in \mathbb{A}, v \in L^2(0,T)} \bar{J}(\alpha, v).$$

Besides,

- $\exists \bar{\alpha} \in \arg\min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \bar{\lambda})$, $\exists \bar{v} \in \arg\min_{v \in L^2(0,T)} \mathcal{L}_2(v, \bar{\lambda})$.
- $((\bar{\alpha}, \bar{v}), \bar{\lambda})$ *is a saddle point of the Lagrangian* $\mathcal{L}$.
- $\bar{\alpha}$ *is a solution of Problem* (P).

Sketch of the proof

Key result:

Lemma

The map $\lambda \mapsto \mathcal{W}(\lambda)$ is Gâteaux differentiable in $L^2(0,T)$.

Having: $\bar{\lambda} = \underset{\lambda \in L^2(0,T)}{\arg\max}\, \mathcal{W}(\lambda), \bar{\alpha} \in \underset{\alpha \in \mathbb{A}}{\arg\min}\, \mathcal{L}_1(\alpha, \bar{\lambda}), \bar{v} \in \underset{v \in L^2(0,T)}{\arg\min}\, \mathcal{L}_2(v, \bar{\lambda})$

Lemma implies:

- $\partial(-\mathcal{W})(\bar{\lambda})$ is a singleton.
- $\mathbb{E}[p(t, X_t^{\bar{\alpha}})] - \bar{v}(t) = 0$
- $(\bar{\alpha}, \bar{v})$ admissible for Problem $(\bar{P})$.
- No duality gap.
- $((\bar{\alpha}, \bar{v}), \bar{\lambda})$ is a saddle point of the Lagrangian $\mathcal{L}$.

Sketch of the proof

Key result:

Lemma

The map $\lambda \mapsto \mathcal{W}(\lambda)$ is Gâteaux differentiable in $L^2(0,T)$.

- $\lambda \mapsto \inf_{v \in L^2(0,T)} \mathcal{L}_2(v,\lambda)$ is Gâteaux differentiable in $L^2(0,T)$
  ☞ strict convexity of f.
- $\lambda \mapsto \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha,\lambda)$ is Gâteaux differentiable in $L^2(0,T)$.
  - There exists a selection $\lambda \mapsto \alpha[\lambda] \in \arg\min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha,\lambda)$, such that the map $\lambda \mapsto \alpha[\lambda]$ is locally Lipschitz continuous;
  - the map $\lambda \mapsto \mathbb{E}[p(X^{\alpha[\lambda]})]$ is continuous;
  - differentiability obtained by adapting the proof of Danskin's Theorem.

Overview

1. Initial problem (P) is a MFC of PDMP;
2. Introduction of an equivalent problem $(\bar{P})$;
3. Introduction of the associated Lagrangian $\mathcal{L}$ and dual function $\mathcal{W}$;
4. Existence of a saddle point for $\mathcal{L}$;
5. Distributed implementation : $\bar{\lambda}$ is sent to each EV which locally computes $\bar{\alpha} \in \arg\min_{\alpha} \mathcal{L}_1(\alpha, \bar{\lambda})$).

Application to smart charging

Settings (1/2)

We consider a large fleet of EVs controlled by a central planner during their charging period $[0, T]$ (with $T = 10$h). The central planner aims at:

- satisfying EV's owner requirement;
- making the consumption profile of the fleet to be close to a given profile $r = (r_t)_{0 \leq t \leq T}$.

The state of an Electric Vehicle (EV) $X^\alpha := (I^\alpha, S^\alpha)$ is a controlled PDMP(b, α) where

- $I_t^\alpha \in \mathcal{I} := \{-1, 0, 1\}$ is the mode of charging, 0 stands for idle mode, 1 for charging and -1 for injection.
- $S_t^\alpha \in [0, 1]$ is the State of Charge (SoC).

Settings (2/2)

The charging rate $b(i, \cdot)$ is proportional to the power consumption of the EV and is such that

- $i = -1$, V2G mode, with $b(-1, \cdot) \leq 0$.
- $i = 0$, non-charging mode, with $b(0, \cdot) = 0$,
- $i = 1$, charging mode, with $b(1, \cdot) \geq 0$.

Cost settings

- $c(t, i, s) = 0$, $L(a) = \frac{a^2}{2}$, $g(i, s) := \kappa_1 \times (1 - e^{\kappa_2 (s - 0.75)})^+$
- $p(t, i, s) := b(i, s)$, $f(v, t) := \kappa_3 (v - r(t))^2$

$$J(\alpha) := \int_0^T \kappa_3 \Big(\underbrace{\mathbb{E}\big[b(I_t^\alpha, S_t^\alpha)\big]}_{\text{mean consumption}} - r(t) \Big)^2 dt + \mathbb{E}\Big[\int_0^T \sum_{j \in \mathcal{I}} \frac{\big(\alpha_j(t, X_t^\alpha)\big)^2}{2} dt + g(X_T^\alpha) \Big]$$

Results (1/2)

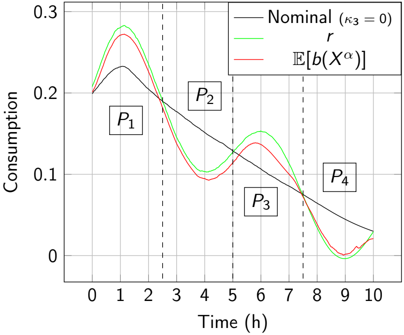


Figure 3: Controlled consumption compared to the profile and nominal

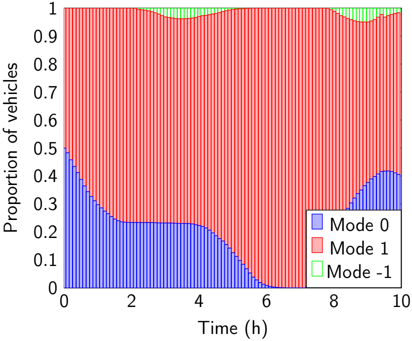


Figure 4: Evolution of the proportion of vehicles per mode

Results (2/2)

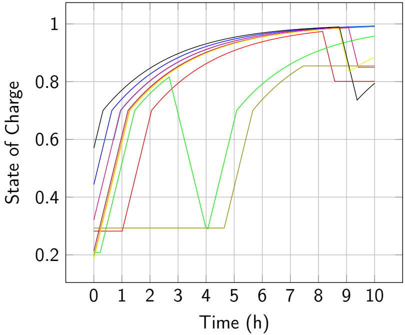


Figure 5: Representation of the SoC of 10 PDMP

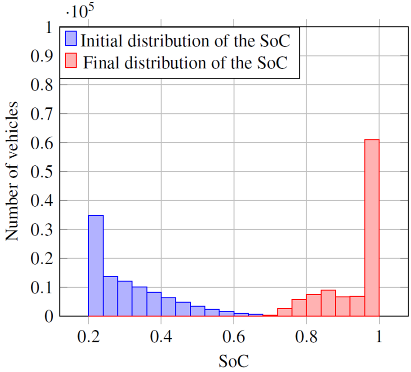


Figure 6: Initial and Final distribution of the SoC

PDE formulation

A constrained optimal control problem

Let $X^\alpha = (I^\alpha, S^\alpha)$ be a PDMP(b, α) controlled by $\alpha \in \mathbb{A}$.
Objective function:

$$J(\alpha) := \mathbb{E}\Big[\int_0^T c(t, X_t^\alpha) + \sum_{j \in \mathcal{I}} L(\alpha_j(t, X_t^\alpha))dt + g(X_T^\alpha)\Big].$$

Constraint, let $D \in C^0([0, T], \mathbb{R}_+^*)$,

$$\mathbb{P}(I_t^\alpha = i) \leq D_i(t) \quad \forall (t, i) \in [0, T] \times \mathcal{I} \tag{2}$$

Optimization problem:

$$\begin{array}{l} \min_{\alpha \in \mathbb{A}} J(\alpha) \\ \text{s.t. (2) is satisfied.} \end{array} \tag{P}$$

Constrained Mean field control literature

- Constraints of the type : $\Psi\big(\mathcal{L}(X_t)\big) \leq 0$ [Daudin, 2021, Germain et al., 2021]
- Constraints in Wasserstein spaces [Bonnet, 2019]
- Stochastic target problems [Soner and Touzi, 2002]
- Stochastic control problems with expectation constraints [Pfeiffer et al., 2021]
- Local constraints [Cardaliaguet et al., 2016]

Reformulation of the problem

Let $m(t) \in \mathcal{P}(\mathcal{I} \times [0,1])$ be the distribution of the mean field population of PDMP(α, b), with initial distribution $m^0 \in \mathcal{P}(\mathcal{I} \times [0,1])$.
The objective function

$$J(\alpha) := \mathbb{E}\Big[\int_0^T c(t, X_t^\alpha) + \sum_{j \in \mathcal{I}} L(\alpha_j(t, X_t^\alpha))dt + g(X_T^\alpha)\Big],$$

is equivalent to

$$J(m, \alpha) := \int_0^T \int_0^1 \sum_{i \in I} \left(c_i(t,s) m_i(t, ds) + \sum_{j \in I} L(\alpha_{i,j}(t,s)) \right) m_i(t, ds) dt$$
$$+ \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds).$$

Reformulation of the problem

The constraint

$$\mathbb{P}(I_t^\alpha = i) \leq D_i(t) \quad \forall (t,i) \in [0,T] \times \mathcal{I},$$

is equivalent to

$$\int_0^1 m_i(t, ds) \leq D_i(t) \quad \forall (t,i) \in [0,T] \times \mathcal{I} \tag{3}$$

(m, α) is a weak solution on $[0,T] \times \mathcal{I} \times [0,1]$ of the continuity equation:

$$\begin{aligned} &\partial_t m_i + \partial_s(m_i b_i) = - \sum_{j \in \mathcal{I}, j \neq i} (\alpha_j(i) m_i - \alpha_i(j) m_j), \\ &m_i(0) = m_i^0, \end{aligned} \tag{CE}$$

Problem (P) is equivalent to

$$\boxed{\begin{array}{l}\inf\limits_{(m,\alpha)} J(m,\alpha) \\ \text{s.t. } (m,\alpha) \text{ is a weak sol. (CE) and satisfies (3)}\end{array}} \qquad (\tilde{P})$$

Lemma

Problem $(\tilde{P})$ admits a solution.

- Characterization of the solutions of Problem $(\tilde{P})$?
- Regularity of the Lagrange multiplier?
- Numerical approximation?

Optimality conditions

Theorem (S. 2021)

Assume there exists $\varepsilon^0 > 0$ *such that:*

$$\varepsilon^0 < D_i(t) - m_i^0([0,1]) \quad \forall (t,i) \in [0,T] \times I,$$

then (m,α) *is a solution to* $(\tilde{P})$*, if and only if there exists a pair* $(\varphi, \lambda) \in \left(\mathrm{Lip}([0,T] \times I \times [0,1]) + BV([0,T] \times I) \right) \times \mathcal{M}^+([0,T] \times I)$ *such that* $\alpha_j(i) = H'(\varphi_i - \varphi_j)$ *and* (φ, λ, m) *is a weak solution of the following system on* $[0,T] \times I \times [0,1]$*:*

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) = 0 \\ \partial_t m_i + \partial_s (m_i b_i) + \sum_{j \neq i} (H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j) = 0 \\ m_i(0,s) = m_i^0(s),\ \varphi_i(T,s) = g_i(s) \\ \int_0^1 m_i(t,ds) - D_i(t) \leq 0,\ \lambda_i \geq 0 \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t,ds) - D_i(t) \right) \lambda_i(dt) = 0 \end{cases} \tag{S}$$

where H is the Fenchel conjugate of L and H' *its derivative.*

Regularity Results

Theorem (S. 2022)

If the congestion parameter D is time independent, and there exists $\varepsilon^0 > 0$ such that:

$$\varepsilon^0 < D_i - m_i^0([0,1]) \quad \forall i \in I,$$

then for any solution (m, α) of Problem $(\tilde{P})$, there exists $(\varphi, \lambda) \in \mathrm{Lip}([0,T] \times [0,1] \times I) \times \mathcal{M}^+([0,T] \times I)$ such that (φ, λ, m) is a weak solution of (S) and for any $i \in I$

$$\lambda_i = \lambda_i^{ac}\mathcal{L} + \beta_i \delta_T,$$

with $\lambda_i^{ac} \in L^\infty((0,T), \mathbb{R}_+)$ and $\beta_i \geq 0$. This yields $\alpha \in \mathrm{Lip}([0,T] \times [0,1] \times I)$.

Remark

- *If there exists $g \in C^1([0,1])$ such that $g = g_i$ for any $i \in I$, then $\beta = 0$.*
- *$L^\infty(0,T)$ is the best regularity that one can a priori expect.*

Numerical approximation

Find $(\varphi, \lambda, \beta)$

$$\begin{aligned} &\partial_t \varphi_i + b_i \partial_s \varphi_i + c_i + \lambda_i - \textstyle\sum_{j \in \mathcal{I}, j \neq i} H(\varphi_j - \varphi_i) \leq 0, \\ &\varphi_i(T) \leq g_i + \beta_i. \end{aligned} \tag{HJ}$$

$$\tilde{A}(\varphi, \lambda, \beta) := \sum_{i \in \mathcal{I}} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D(t)\lambda(t)dt + D_i(T)\beta.$$

$$\boxed{\begin{aligned} &\inf_{(\varphi,\lambda,\beta)} \tilde{A}(\varphi, \lambda, \beta) \\ &(\varphi, \lambda, \beta) \text{ weak sol (HJ)} \end{aligned}} \tag{D}$$

- Time and space discretization of Problem (D).
- Explicit finite difference scheme for the discretization of (HJ).

- 5h period, $I = \{0, 1\}$, where 0: idle; and 1: charging, $D_0 = 1$ and $D_1 = 1/5$, $g(s) := Ce^{c((0.7-s)^+)^2}$

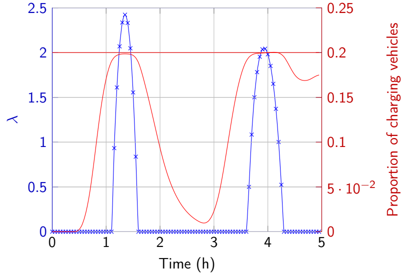


Figure 7: Optimal Lagrangian multiplier λ and proportion of EVs in mode 1 over the time

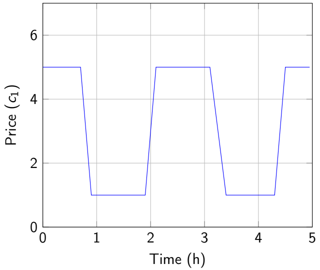


Figure 8: Price of electricity over the time

Thank you for your attention!

Appendix

Stochastic Uzawa algorithm

Objective: numerically approximate $\bar{\lambda} := \max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda)$.

Algorithm 1 Uzawa

Initialization $\lambda^0 \in L^\infty(0, T)$, set $\{\rho_k\}$ and $M \in \mathbb{N}^*$
$k \leftarrow 0$.
for $k = 0, 1, \ldots$ **do**
$\quad v^k \leftarrow \underset{v \in L^2(0,T)}{\arg\min}\ \mathcal{L}_2(v, \lambda^k)$.
$\quad \alpha^k \leftarrow \underset{\alpha \in \mathbb{A}}{\arg\min}\ \mathcal{L}_1(\alpha, \lambda^k)$.
$\quad U^{k+1} \leftarrow v^k - \mathbb{E}[p(\cdot, X_\cdot^{\alpha^k})]$.
$\quad \lambda^{k+1} \leftarrow \lambda^k + \rho_k\, U^{k+1}$.

Stochastic Uzawa algorithm

Objective: numerically approximate $\bar{\lambda} := \max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda)$.

Algorithm 4 Stochastic Uzawa

1: Initialization $\lambda^0 \in L^\infty(0,T)$, set $\{\rho_k\}$ and $M \in \mathbb{N}^*$
2: $k \leftarrow 0$.
3: **for** $k = 0, 1, \dots$ **do**
4: $\quad v^k \leftarrow \underset{v \in L^2(0,T)}{\arg\min}\ \mathcal{L}_2(v, \lambda^k)$.
5: $\quad \alpha^k \leftarrow \underset{\alpha \in \mathbb{A}}{\arg\min}\ \mathcal{L}_1(\alpha, \lambda^k)$.
6: $\quad$ Generate M independent states realizations $(X^{1,\alpha^k}, \dots, X^{M,\alpha^k})$.
7: $\quad U^{k+1} \leftarrow v^k - \frac{1}{M} \sum_{j=1}^{M} p(\cdot, X^{j,\alpha^k}_\cdot)$.
8: $\quad \lambda^{k+1} \leftarrow \lambda^k + \rho_k\, U^{k+1}$.

Convergence results

Theorem

Let $\{(\lambda^k, \alpha^k)\}_{k\in\mathbb{N}}$ be a sequence generated by Stochastic Uzawa Algorithm, then the following assertions hold

1. *The sequence $\{\lambda^k\}_k$ converges to $\bar{\lambda}$ a.s. in $L^2(0,T)$.*
2. *The sequence $\{\alpha^k\}_{k\in\mathbb{N}}$ converges a.s. to a solution of Problem (P) w.r.t. the norm $\|\cdot\|_\infty$.*
3. *The sequence $\{J(\alpha^k)\}_{k\in\mathbb{N}}$ converges a.s. to $\min_{\alpha\in\mathbb{A}} J(\alpha)$.*

Sketch of the proof:

1. Direct adaptation of Stochastic Gradient Algorithm in Hilbert space.
2. Continuity of the map: $\lambda \mapsto \alpha[\lambda] \in \arg\min_{\alpha\in\mathbb{A}} \mathcal{L}_1(\alpha,\lambda)$.
3. Continuity of the map $\alpha \mapsto J(\alpha)$.

Sketch of the proof

For $\delta, \varepsilon > 0$, we define the penalized problem

$$\boxed{\begin{array}{l}\inf_{(m,\alpha)} J(m,\alpha) + \sum_{i \in I} \int_0^T \frac{1}{\varepsilon} \Psi_i^+(m_i(t))dt + \sum_{i \in I} \frac{1}{\delta} \Psi_i^+(m_i(T)), \\ (m,\alpha) \text{ weak sol. (CE)}\end{array}} \qquad (D^{\varepsilon,\delta})$$

where $\Psi_i(\mu) := \mu_i([0,1]) - D_i$

- Optimality conditions of Problem $(D^{\varepsilon,\delta})$?
- Link between the solutions of Problem $(\tilde{P})$ and Problem $(D^{\varepsilon,\delta})$?

Proposition

Problem $(D^{\varepsilon,\delta})$ *has at least a solution and for any solution* (m,α) *there exists* $(\varphi,\lambda,\beta) \in \mathrm{Lip}([0,T]\times I\times[0,1]) \times L^\infty([0,T]\times I,\mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ *such that* $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ *on* $\{m_i > 0\}$ *and* $(\varphi,\lambda,\beta,m)$ *is a weak solution of the following system on* $[0,T]\times[0,1]\times I$*:*

$$\begin{cases} -\partial_t\varphi_i - b_i\partial_s\varphi_i - c_i - \dfrac{\lambda_i}{\varepsilon} + \displaystyle\sum_{j\in I, j\neq i} H(\varphi_i - \varphi_j) = 0, \\ \partial_t m_i + \partial_s(m_i b_i) + \displaystyle\sum_{j\in I} H'(\varphi_i-\varphi_j)m_i - H'(\varphi_j - \varphi_i)m_j = 0, \\ m_i(0) = m_i^0,\ \varphi_i(T) = g_i + \dfrac{\beta_i}{\delta}, \end{cases} \qquad (S^{\varepsilon,\delta})$$

and (λ,β) *satisfies*

$$\lambda_i(t) = \begin{cases} 0 & \text{if } \Psi_i(m(t)) < 0, \\ \in [0,1] & \text{if } \Psi_i(m(t)) = 0, \\ 1 & \text{if } \Psi_i(m(t)) > 0, \end{cases} \qquad \beta_i := \begin{cases} 0 & \text{if } \Psi_i(m(T)) < 0, \\ \in [0,1] & \text{if } \Psi_i(m(T)) = 0, \\ 1 & \text{if } \Psi_i(m(T)) > 0. \end{cases}$$

Proposition

There exists $\varepsilon^*, \delta^* > 0$, such that for any $(\varepsilon, \delta) \in (0, \varepsilon^*) \times (0, \delta^*)$, Problems $(\tilde{P})$ and $(D^{\varepsilon,\delta})$ have the same solutions.

Proof by contradiction:

- Uniform bound on $\|\alpha\|_\infty + \|\partial_s \alpha\|_\infty$, independently of ε and δ.
- For any $\delta < \delta^*$, $\Psi_i(m(T)) \leq 0$.
- Assume for any $\varepsilon > 0$, there exists $t^\varepsilon > 0$ such that $\Psi_i(m(t^\varepsilon)) > 0$
- For any $\varepsilon < \varepsilon^*$ and a.e. $t \in [0, T]$ satisfying $\Psi_i(m(t)) > 0$:

$$\frac{d^2}{dt^2}\Psi_i(m(t)) \geq C \sum_{j \in I} \int_0^1 \left(\frac{1}{C\sqrt{\varepsilon}} - C\right)(\alpha_{i,j} m_i(t) + \alpha_{j,i} m_j(t)) \geq 0$$

- Since $\Psi_i(m^0) < 0$, there exists $\tau \in (0, t^\varepsilon)$ such that $\Psi_i(m(\tau)) > 0$ and $\frac{d}{dt}\Psi_i(m(\tau)) > 0$.
- Then the map $t \mapsto \Psi_i(m(t))$ is strictly increasing on $[\tau, T]$. Then, $\Psi_i(m(T)) > 0$ (contradiction)

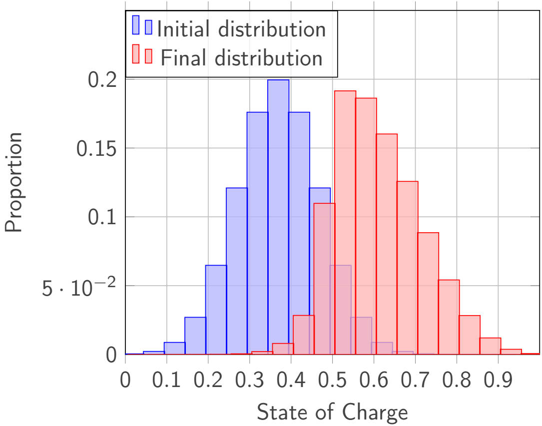


Figure 9: Marginal distribution of the State of Charge (s) at initial and final time

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