Does climate affect the intensity of defoliation in Québec ? Adjacent-Category AutoRegressive modelling approach.

Judicaël O. F. Osse (UQAT), Zinsou-Max Debaly, (CYU, Ecodep project), Philippe Marchand & M. Montoro Girona (UQAT).

Outline

Outline

Outline

1. Introduction
   - Power consumption
   - Example in forestry
2. Models and stability results
3. Estimation and asymptotic properties
4. Results

Electric grid stress

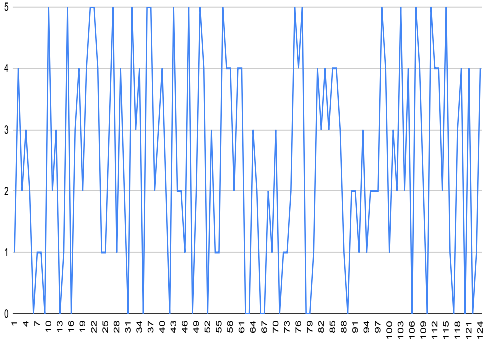


Figure: Daily balance between electricity demand and the amount of energy available within the grid

Electric grid stress

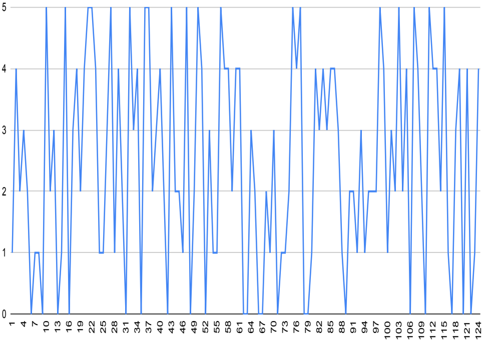


Figure: Daily balance between electricity demand and the amount of energy available within the grid

- Why not the power consumption ?

Electric grid stress

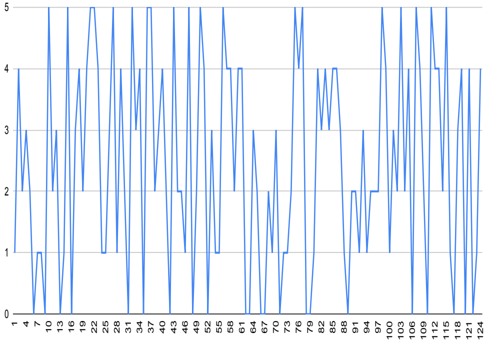


Figure: Daily balance between electricity demand and the amount of energy available within the grid

- Why not the power consumption ?
- Does Climate and Extreme Weather Impact Grid Stress?

Electricity prices

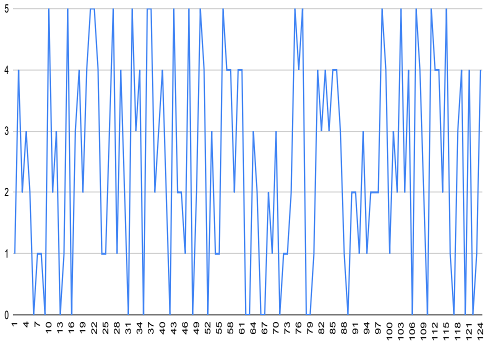


Figure: Daily variations of electricity price

- How economic indicators such that GDP impact the variations of electricity prices?

Outline

Diagram of Spruce budworm life cycle

Figure: Spruce budworm life cycle

Different states of invasion

June-july

4 successive cycles

Regeneration

- Early in the growing season, shoots and needles are encased in silk and turn a distinct reddish colour by late summer. The tops of some trees may appear needleless

Different states of invasion

June-july

4 successive cycles

Regeneration

- Early in the growing season, shoots and needles are encased in silk and turn a distinct reddish colour by late summer. The tops of some trees may appear needleless
- Most trees survive years of defoliation though grow slower than non-infested trees and often develop defects reducing their value as timber

Different states of invasion

June-july

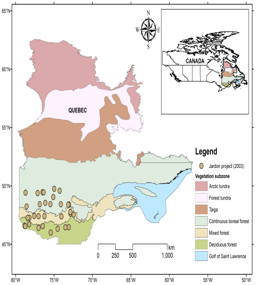
4 successive cycles

Regeneration

- Early in the growing season, shoots and needles are encased in silk and turn a distinct reddish colour by late summer. The tops of some trees may appear needleless
- Most trees survive years of defoliation though grow slower than non-infested trees and often develop defects reducing their value as timber
- After 4 to 5 years of persistent and severe defoliation trees usually die, especially understory trees

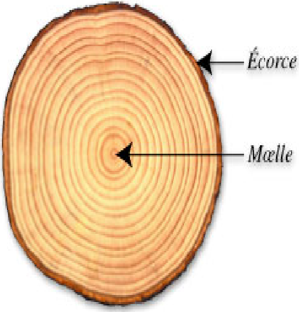
Material and data

Map of sites



Tree coring

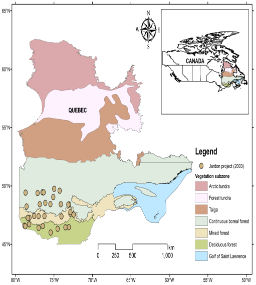
dendrochronology



- tree-ring width measurements for 631 white spruce (*Picea glauca*) trees distributed across 45 sites in southwestern Quebec, Canada, with 1 to 23 trees per site.

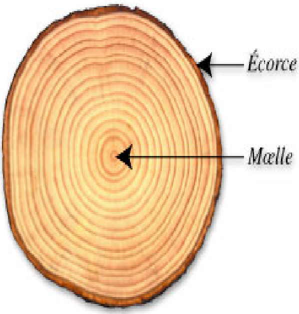
Material and data

Map of sites



Tree coring

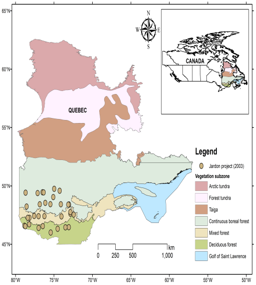
dendrochronology



- tree-ring width measurements for 631 white spruce (*Picea glauca*) trees distributed across 45 sites in southwestern Quebec, Canada, with 1 to 23 trees per site.
- The defoliation level for each area is classified on a scale of 1 to 3 corresponding to a low (approx. 1 – 35%), moderate (36 – 70%) or high (71 – 100%) fraction of the year's foliage defoliated by the SBW.

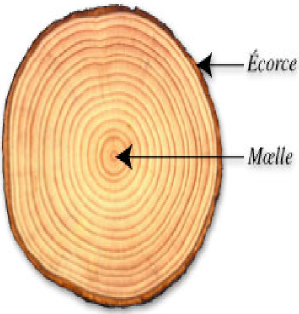
Material and data

Map of sites



Tree coring

dendrochronology



- tree-ring width measurements for 631 white spruce (*Picea glauca*) trees distributed across 45 sites in southwestern Quebec, Canada, with 1 to 23 trees per site.
- The defoliation level for each area is classified on a scale of 1 to 3 corresponding to a low (approx. 1 – 35%), moderate (36 – 70%) or high (71 – 100%) fraction of the year's foliage defoliated by the SBW.
- The mean daily maximum of temperature, the total precipitation and the mean CMI for the current and previous spring and summer.

Outline

Existing models for ordinal categorical time series (1/2)

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories :

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \leq j \leq K,\ Y_{j,t} = 1$ when $Y_t = j$.

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \leq j \leq K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{ \sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t} \right\}, \quad 0 \leq j \leq K, \tag{1}$$

For any sequence $(u_n)_{n\in\mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{2}$$

2. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \dots, Y_{K,t}) \in \{0,1\}^K$
   - Short term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{p} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{3}$$

   - Long term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{\infty} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{4}$$

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \le j \le K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \le U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \le j \le K, \tag{1}$$

For any sequence $(u_n)_{n\in\mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \leq j \leq K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K, \tag{1}$$

For any sequence $(u_n)_{n\in\mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \le j \le K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{ \sum_{k=0}^{j-1} \pi_{k,t} \le U_t < \sum_{k=0}^{j} \pi_{k,t} \right\}, \quad 0 \le j \le K, \tag{1}$$

For any sequence $(u_n)_{n \in \mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1}, \quad 1 \le j \le K, t \in \mathbb{Z} \tag{2}$$

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \leq j \leq K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K, \tag{1}$$

For any sequence $(u_n)_{n\in\mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{2}$$

2. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \dots, Y_{K,t}) \in \{0,1\}^K$
   - Short term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{p} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{3}$$

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \leq j \leq K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K, \tag{1}$$

For any sequence $(u_n)_{n\in\mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{2}$$

2. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \ldots, Y_{K,t}) \in \{0,1\}^K$
   - Short term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{p} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{3}$$

   - Long term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{\infty} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \leq j \leq K, t \in \mathbb{Z} \tag{4}$$

Existing models for ordinal categorical time series (1/2)

One-hot encoding for $K+1$ categories : for $0 \le j \le K$, $Y_{j,t} = 1$ when $Y_t = j$.

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top,$$

$$Y_{k,t} = \mathbb{1}\left\{ \sum_{k=0}^{j-1} \pi_{k,t} \le U_t < \sum_{k=0}^{j} \pi_{k,t} \right\}, \quad 0 \le j \le K, \tag{1}$$

For any sequence $(u_n)_{n \in \mathbb{N}}$, we adopt the convention $\sum_{i=0}^{-1} u_i = 0$.

1. Regression models

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1}, \quad 1 \le j \le K, t \in \mathbb{Z} \tag{2}$$

2. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \dots, Y_{K,t}) \in \{0,1\}^K$
   - Short term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{p} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \le j \le K, t \in \mathbb{Z} \tag{3}$$

   - Long term memory

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \sum_{i=1}^{\infty} \alpha_i^\top \overline{Y}_{t-i}, \quad 1 \le j \le K, t \in \mathbb{Z} \tag{4}$$

Models for ordinal categorical time series (2/2)

1. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \ldots, Y_{K,t}) \in \{0,1\}^K$
    - Long Short Term Memory (LSTM)

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \tag{5}$$

Models for ordinal categorical time series (2/2)

1. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \dots, Y_{K,t}) \in \{0,1\}^K$
   - Long Short Term Memory (LSTM)

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \tag{5}$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} = \omega_j + \gamma^\top X_{t-1} + \sum_{i \geq 1} \beta_j^i \left(\omega_j + \gamma^\top X_{t-1-i}\right) + \sum_{i \geq 1} \beta_j^{i-1} \alpha^\top \overline{Y}_{t-i} \tag{6}$$

$$1 \leq j \leq K, t \in \mathbb{Z}.$$

Models for ordinal categorical time series (2/2)

1. Autoregressions : $\overline{Y}_t = (Y_{1,t}, \dots, Y_{K,t}) \in \{0,1\}^K$
   - Long Short Term Memory (LSTM)

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \tag{5}$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} = \omega_j + \gamma^\top X_{t-1} + \sum_{i \geq 1} \beta_j^i \left(\omega_j + \gamma^\top X_{t-1-i}\right) + \sum_{i \geq 1} \beta_j^{i-1} \alpha^\top \overline{Y}_{t-i} \tag{6}$$

$$1 \leq j \leq K, t \in \mathbb{Z}.$$

where $\omega_j \in \mathbb{R}, \beta_j \in \mathbb{R}^*, j = 1, \dots, K, \gamma \in \mathbb{R}^P, \alpha \in \mathbb{R}^K$.

Stability conditions (1/2)

Stability conditions (1/2)

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top, Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K,$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z}.$$

$$\lambda_{j,t} := \log \frac{\pi_{j,t}}{\pi_{0,t}} = \sum_{k=0}^{j-1} \log \frac{\pi_{k+1,t}}{\pi_{k,t}} = \sum_{k=0}^{j-1} \eta_{j+1}. \tag{7}$$

Setting $\lambda_t = (\lambda_{1,t}, \ldots, \lambda_{K,t})^\top$, from equation (7),

$$\lambda_t = \tilde{\omega} + \tilde{A}\overline{Y}_{t-1} + \tilde{B}\lambda_{t-1}, t \in \mathbb{Z}$$

where $\tilde{\omega} = (\sum_{j=1}^{k} \omega_j)_{k=1,\ldots,K}, \tilde{A} = (\alpha|2\alpha|\cdots|K\alpha)^\top$ and

$$\tilde{B} = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \beta_2 & 0\cdots & \\ \vdots & & \ddots & \ddots \\ \beta_1 & \beta_2 & \cdots & \beta_K \end{pmatrix}.$$

Stability conditions (1/2)

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top, Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K,$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z}.$$

Stability conditions (1/2)

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top, Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \leq U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \leq j \leq K,$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \quad 1 \leq j \leq K, t \in \mathbb{Z}.$$

$$\lambda_{j,t} := \log \frac{\pi_{j,t}}{\pi_{0,t}} = \sum_{k=0}^{j-1} \log \frac{\pi_{k+1,t}}{\pi_{k,t}} = \sum_{k=0}^{j-1} \eta_{j+1}. \tag{7}$$

Stability conditions (1/2)

$$Y_t = (Y_{0,t}, \ldots, Y_{K,t})^\top, Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \le U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \le j \le K,$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \quad 1 \le j \le K, t \in \mathbb{Z}.$$

$$\lambda_{j,t} := \log \frac{\pi_{j,t}}{\pi_{0,t}} = \sum_{k=0}^{j-1} \log \frac{\pi_{k+1,t}}{\pi_{k,t}} = \sum_{k=0}^{j-1} \eta_{j+1}. \tag{7}$$

Setting $\lambda_t = (\lambda_{1,t}, \ldots, \lambda_{K,t})^\top$, from equation (7),

$$\lambda_t = \tilde{\omega} + \tilde{A}\overline{Y}_{t-1} + \tilde{B}\lambda_{t-1}, t \in \mathbb{Z}$$

Stability conditions (1/2)

$$Y_t = (Y_{0,t}, \dots, Y_{K,t})^\top, Y_{k,t} = \mathbb{1}\left\{\sum_{k=0}^{j-1} \pi_{k,t} \le U_t < \sum_{k=0}^{j} \pi_{k,t}\right\}, \quad 0 \le j \le K,$$

$$\log \frac{\pi_{j,t}}{\pi_{j-1,t}} =: \eta_{j,t} = \omega_j + \gamma^\top X_{t-1} + \alpha^\top \overline{Y}_{t-1} + \beta_j \eta_{j,t-1}, \quad 1 \le j \le K, t \in \mathbb{Z}.$$

$$\lambda_{j,t} := \log \frac{\pi_{j,t}}{\pi_{0,t}} = \sum_{k=0}^{j-1} \log \frac{\pi_{k+1,t}}{\pi_{k,t}} = \sum_{k=0}^{j-1} \eta_{j+1}. \tag{7}$$

Setting $\lambda_t = (\lambda_{1,t}, \dots, \lambda_{K,t})^\top$, from equation (7),

$$\lambda_t = \tilde{\omega} + \tilde{A}\overline{Y}_{t-1} + \tilde{B}\lambda_{t-1}, t \in \mathbb{Z}$$

where $\tilde{\omega} = (\sum_{j=1}^{k} \omega_j)_{k=1,\dots,K}, \tilde{A} = (\alpha|2\alpha|\cdots|K\alpha)^\top$ and

$$\tilde{B} = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \beta_2 & 0\cdots & \\ \vdots & & \ddots & \ddots \\ \beta_1 & \beta_2 & \cdots & \beta_K \end{pmatrix}.$$

Stability conditions (2/2)

Stability conditions (2/2)

Proposition

Consider the model (1)-(5) *and assume that* $(X_t, U_t)_{t\in\mathbb{Z}}$ *is stationary and ergodic with* $\mathbb{E}\log_+|X_0|_1 < \infty$. *If for* $k = 1, \ldots, K$, $|\beta_k| < 1$, *then there exists a unique non-anticipative, stationary and ergotic sequence* $(Y_t, X_t, \eta_t)_{t\in\mathbb{Z}}$ *solution of* (1)-(5). *In addition, for* $r \geq 1$, $\mathbb{E}|\eta_0|_1^r < \infty$ *provided that* $\mathbb{E}|X_0|_1^r < \infty$.

Stability conditions (2/2)

Proposition

Consider the model (1)-(5) *and assume that* $(X_t, U_t)_{t\in\mathbb{Z}}$ *is stationary and ergodic with* $\mathbb{E}\log_+|X_0|_1 < \infty$. *If for* $k = 1, \dots, K, |\beta_k| < 1$, *then there exists a unique non-anticipative, stationary and ergodic sequence* $(Y_t, X_t, \eta_t)_{t\in\mathbb{Z}}$ *solution of* (1)-(5). *In addition, for* $r \geq 1$, $\mathbb{E}|\eta_0|_1^r < \infty$ *provided that* $\mathbb{E}|X_0|_1^r < \infty$.

Hint :

Indeed, if for $k = 1, \dots, K, |\beta_k| < 1$, the roots $1/\beta_k, k = 1, \dots, K$ of polynomial

$$\mathcal{P}(z) = \det\left(I - \tilde{B}z\right) = \Pi_{j=1}^{K}(1 - \beta_j z) \tag{8}$$

are outside the unit disc. Moreover, the moments of latent process depend on that of $(X_t)_{t\in Z}$ since $Y_{k,t}$'s take only two values : 0 and 1.

Stability conditions (2/2)

Proposition

Consider the model (1)-(5) *and assume that* $(X_t, U_t)_{t\in\mathbb{Z}}$ *is stationary and ergodic with* $\mathbb{E}\log_+|X_0|_1 < \infty$*. If for* $k = 1, \ldots, K, |\beta_k| < 1$*, then there exists a unique non-anticipative, stationary and ergodic sequence* $(Y_t, X_t, \eta_t)_{t\in\mathbb{Z}}$ *solution of* (1)-(5)*. In addition, for* $r \geq 1$, $\mathbb{E}|\eta_0|_1^r < \infty$ *provided that* $\mathbb{E}|X_0|_1^r < \infty$.

Hint :
Indeed, if for $k = 1, \ldots, K, |\beta_k| < 1$, the roots $1/\beta_k, k = 1, \ldots, K$ of polynomial

$$\mathcal{P}(z) = \det\left(I - \tilde{B}z\right) = \Pi_{j=1}^{K}(1 - \beta_j z) \tag{8}$$

are outside the unit disc. Moreover, the moments of latent process depend on that of $(X_t)_{t\in Z}$ since $Y_{k,t}$'s take only two values : 0 and 1.
Notes:
Tightest condition available right now : $\rho(\tilde{B}) < 1$. Weak than $\rho(|\tilde{B}|_{\text{vec}} + |C|_{\text{vec}}|\tilde{A}|_{\text{vec}}) < 1$ (DT 2022) or $||\tilde{B}|| + ||C\tilde{A}|| < 1$ (MF 2017).

Outline

Quasi maximum likelihood estimator

From (5),

$$\eta_t(\theta) = \omega + \Gamma X_{t-1} + A\overline{Y}_{t-1} + B\eta_{t-1}(\theta)$$

where $\omega = (\omega_j)_{1\leq j\leq K}, \Gamma^\top = (\gamma|\cdots|\gamma), A^\top = (\alpha|\cdots|\alpha)$ and $B = \text{diag}(\beta_j, 1 \leq j \leq K)$.

Quasi maximum likelihood estimator

From (5),

$$\eta_t(\theta) = \omega + \Gamma X_{t-1} + A\overline{Y}_{t-1} + B\eta_{t-1}(\theta)$$

where $\omega = (\omega_j)_{1\leq j\leq K}, \Gamma^\top = (\gamma|\cdots|\gamma), A^\top = (\alpha|\cdots|\alpha)$ and $B = \mathrm{diag}(\beta_j, 1 \leq j \leq K)$.
The vector of the parameters of the model will be denoted by $\theta = (\omega^\top, \mathrm{vec}(\Gamma)^\top, \mathrm{vec}(A)^\top, \mathrm{vec}(B)^\top)^\top$ and assume that $\Theta(\ni \theta)$ is a compact set.
The true value of of the parameter will be θ_0.

Quasi maximum likelihood estimator

From (5),

$$\eta_t(\theta) = \omega + \Gamma X_{t-1} + A\overline{Y}_{t-1} + B\eta_{t-1}(\theta)$$

where $\omega = (\omega_j)_{1\leq j\leq K}, \Gamma^\top = (\gamma|\cdots|\gamma), A^\top = (\alpha|\cdots|\alpha)$ and $B = \mathrm{diag}(\beta_j, 1 \leq j \leq K)$.
The vector of the parameters of the model will be denoted by $\theta = (\omega^\top, \mathrm{vec}(\Gamma)^\top, \mathrm{vec}(A)^\top, \mathrm{vec}(B)^\top)^\top$ and assume that $\Theta(\ni \theta)$ is a compact set.
The true value of of the parameter will be θ_0.
For $k = 1, \ldots, K$,

$$\pi_{k,t}(\theta) = \frac{\mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}}{1 + \sum_{i=1}^{K} \mathrm{e}^{\sum_{j=1}^{i} \eta_{j,t}(\theta)}}, \quad \theta \in \Theta$$

From (5),

$$\eta_t(\theta) = \omega + \Gamma X_{t-1} + A\overline{Y}_{t-1} + B\eta_{t-1}(\theta)$$

where $\omega = (\omega_j)_{1\le j\le K}, \Gamma^\top = (\gamma|\cdots|\gamma), A^\top = (\alpha|\cdots|\alpha)$ and $B = \mathrm{diag}(\beta_j, 1 \le j \le K)$.
The vector of the parameters of the model will be denoted by $\theta = (\omega^\top, \mathrm{vec}(\Gamma)^\top, \mathrm{vec}(A)^\top, \mathrm{vec}(B)^\top)^\top$ and assume that $\Theta(\ni \theta)$ is a compact set.
The true value of of the parameter will be θ_0.
For $k = 1, \dots, K$,

$$\pi_{k,t}(\theta) = \frac{\mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}}{1 + \sum_{i=1}^{K} \mathrm{e}^{\sum_{j=1}^{i} \eta_{j,t}(\theta)}}, \quad \theta \in \Theta$$

and the parameter θ can by estimated by

$$\hat{\theta}_n = \operatorname*{argmin}_{\theta\in\theta} - \sum_{t=1}^{n} \left(\sum_{k=1}^{K} Y_{k,t} \left(\sum_{j=1}^{k} \eta_{j,t}(\theta) \right) - \log \left(1 + \sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)} \right) \right) \quad (9)$$

Asymptotic properties (1/2)

Asymptotic properties (1/2)

Let us set

$$s_t(\theta) = -\sum_{k=1}^{K}\left(\sum_{j\geq k} Y_{j,t} - \frac{\sum_{j\geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}}{1+\sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}}\right) \nabla_\theta \eta_{k,t}(\theta)$$

and

Asymptotic properties (1/2)

Let us set

$$s_t(\theta) = -\sum_{k=1}^{K}\left(\sum_{j\geq k} Y_{j,t} - \frac{\sum_{j\geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}}{1+\sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}}\right)\nabla_\theta \eta_{k,t}(\theta)$$

and

$$\begin{aligned} h_t(\theta) &= \sum_{k=1}^{K}\left\{\frac{1}{\left(1+\sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}\right)^2}\left[\left(\sum_{j\geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}\right)\sum_{\ell=1}^{k-1}\left(1+\sum_{j=1}^{\ell-1} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}\right)\nabla_\theta \eta_{\ell,t}(\theta) + \right.\right. \\ & \left.\left.\left(1+\sum_{j=1}^{k-1} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}\right)\sum_{\ell=k}^{K}\left(\sum_{j\geq \ell} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}\right)\nabla_\theta \eta_{\ell,t}(\theta)\right]\right\}\nabla_\theta^\top \eta_{k,t}(\theta) \end{aligned}$$

Asymptotic properties (2/2)

Proposition

1. *Suppose that*
   - *the conditions of proposition 1 hold for any* $\theta \in \Theta$ *with* $\mathbb{E}|X_0|_1 < \infty$ *and*
   - $\eta_0(\theta) = \eta_0(\theta_0)\ \mathbb{P}_{\theta_0} - a.s \Rightarrow \theta = \theta_0$.

   Then, the estimator (9) *is consistent, i.e a.s*

$$\lim_{n\to\infty} \hat{\theta}_n = \theta_0.$$

Asymptotic properties (2/2)

Proposition

1. *Suppose that*
   - *the conditions of proposition 1 hold for any* $\theta \in \Theta$ *with* $\mathbb{E}|X_0|_1 < \infty$ *and*
   - $\eta_0(\theta) = \eta_0(\theta_0)\ \mathbb{P}_{\theta_0} - a.s \Rightarrow \theta = \theta_0$.

   Then, the estimator (9) is consistent, i.e a.s

$$\lim_{n \to \infty} \hat{\theta}_n = \theta_0.$$

2. *If in addition,* θ_0 *is located in the interior of* Θ, $\mathbb{E}|X_0|_1^2 < \infty$ *and the matrix* $\mathbb{E}h_0(\theta_0)$ *is invertible,*

$$\lim_{n \to \infty} \sqrt{n}\left(\hat{\theta}_n - \theta_0\right) = \mathcal{N}\left(0, J^{-1}LJ^{-1\top}\right)$$

   where $L = \mathbb{E}s_0(\theta_0)s_0^\top(\theta_0)$ *and* $J = \mathbb{E}h_0(\theta_0)$.

Diagnostics checking

Diagnostics checking

Null hypothesis

The null hypothesis is (1)-(5) hold true with $\theta = \theta_0$.

Diagnostics checking

Null hypothesis

The null hypothesis is (1)-(5) hold true with $\theta = \theta_0$.

One will consider the residuals

$$e_t(\theta) = \left(\sum_{j \geq k} Y_{j,t} - \frac{\sum_{j \geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}}{1 + \sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}} \right)_{1 \leq k \leq K}, t \in \mathbb{Z}, \theta \in \Theta, \tag{10}$$

Diagnostics checking

Null hypothesis

The null hypothesis is (1)-(5) hold true with $\theta = \theta_0$.

One will consider the residuals

$$e_t(\theta) = \left(\sum_{j \geq k} Y_{j,t} - \frac{\sum_{j \geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}}{1 + \sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}} \right)_{1 \leq k \leq K}, t \in \mathbb{Z}, \theta \in \Theta, \quad (10)$$

the autocorrelations

$$\rho_h(\theta) = \frac{1}{n} \sum_{t=1}^{n} e_t(\theta) \odot e_{t-h}(\theta), h = 1, \ldots, n, \theta \in \Theta. \quad (11)$$

Null hypothesis

The null hypothesis is (1)-(5) hold true with $\theta = \theta_0$.

One will consider the residuals

$$e_t(\theta) = \left(\sum_{j \geq k} Y_{j,t} - \frac{\sum_{j \geq k} \mathrm{e}^{\sum_{i=1}^{j} \eta_{i,t}(\theta)}}{1 + \sum_{k=1}^{K} \mathrm{e}^{\sum_{j=1}^{k} \eta_{j,t}(\theta)}} \right)_{1 \leq k \leq K}, t \in \mathbb{Z}, \theta \in \Theta, \qquad (10)$$

the autocorrelations

$$\rho_h(\theta) = \frac{1}{n} \sum_{t=1}^{n} e_t(\theta) \odot e_{t-h}(\theta), h = 1, \ldots, n, \theta \in \Theta. \qquad (11)$$

The vector of q successive autocorrelations is $\rho_{1:q}(\theta) = \mathrm{vec}((\rho_1(\theta)| \cdots |\rho_q(\theta))^\top)$

Diagnostics checking (2/2)

Proposition

Under the assumptions of proposition 2 with additionnal moment conditions on X_0 *and* W *invertible, as* n *tends to* ∞,

$$n\hat{\rho}_{1:q}^{\top}\hat{W}^{-1}\hat{\rho}_{1:q} \Rightarrow \chi^2_{Kq}$$

and then the adequacy of the ACAR model (1)-(5) *is then rejected at the asymptotic level* α *if*

$$n\hat{\rho}_{1:q}^{\top}\hat{W}^{-1}\hat{\rho}_{1:q} > \chi^2_{Kq}(1-\alpha).$$

Simulations

Table 1. Estimation results for the quasi maximum likelihood estimation

n	θ_0	1.2	0.7	0.5	-0.8	1.5	-1.5	2.0	2.0	0.3	-0.3	0.5	0.8	-0.2	0.3
100	CMLE	1.642	0.854	0.625	-1.097	1.908	-1.966	2.581	2.577	0.180	-0.709	0.649	0.809	-0.171	0.273
	TSE	1.011	1.317	1.130	0.400	0.588	0.622	0.784	0.774	1.203	1.346	1.314	0.064	0.337	0.292
	MAE	0.756	0.641	0.646	1.200	0.925	2.122	1.258	1.258	0.903	1.646	0.819	0.736	0.537	0.097
	MSE	1.666	1.809	1.835	0.219	0.313	0.345	0.470	0.451	1.828	1.819	1.863	0.034	0.129	0.130
300	CMLE	1.237	0.636	0.580	-0.805	1.492	-1.531	1.997	2.012	0.250	-0.394	0.460	0.807	-0.207	0.308
	TSE	0.359	0.520	0.404	0.179	0.277	0.284	0.356	0.358	0.432	0.510	0.494	0.035	0.192	0.158
	MAE	0.843	0.191	0.123	0.979	1.223	1.784	1.644	1.642	0.134	0.810	0.103	0.765	0.392	0.143
	MSE	0.122	0.100	0.114	0.051	0.089	0.097	0.123	0.117	0.130	0.122	0.153	0.011	0.049	0.042
500	CMLE	1.182	0.578	0.518	-0.801	1.486	-1.492	1.973	1.966	0.277	-0.292	0.539	0.799	-0.207	0.291
	TSE	0.265	0.392	0.307	0.137	0.216	0.216	0.276	0.275	0.329	0.376	0.373	0.028	0.152	0.127
	MAE	0.935	0.308	0.193	0.937	1.284	1.716	1.724	1.725	0.060	0.676	0.137	0.772	0.352	0.173
	MSE	0.064	0.061	0.067	0.032	0.057	0.053	0.072	0.071	0.075	0.077	0.089	0.007	0.032	0.028
	θ_0	1.2	0.7	0.5	0.8	-1.5	1.5	-2.0	-2.0	-0.3	0.3	-0.5	-0.8	0.2	-0.3
100	CMLE	2.286	1.077	0.401	1.237	-2.345	2.356	-3.117	-3.186	-0.305	1.209	-0.686	-0.803	0.157	-0.298
	TSE	1.956	0.918	0.986	0.451	0.718	0.738	0.925	0.941	0.897	1.403	0.926	0.050	0.319	0.303
	MAE	1.808	0.889	0.956	0.547	0.932	0.946	1.205	1.274	0.842	1.492	1.111	0.043	0.151	0.127
	MSE	3.910	1.730	1.603	1.373	2.356	2.377	3.276	3.578	1.256	5.799	2.350	0.058	0.198	0.166
300	CMLE	1.171	0.638	0.401	0.849	-1.611	1.600	-2.123	-2.112	-0.198	0.490	-0.352	-0.808	0.189	-0.319
	TSE	0.803	0.378	0.480	0.171	0.273	0.271	0.348	0.347	0.365	0.438	0.360	0.028	0.184	0.167
	MAE	0.651	0.349	0.393	0.124	0.192	0.175	0.234	0.245	0.378	0.455	0.373	0.020	0.072	0.073
	MSE	0.845	0.461	0.513	0.162	0.260	0.252	0.318	0.324	0.472	0.578	0.459	0.026	0.092	0.094
500	CMLE	1.143	0.654	0.461	0.821	-1.538	1.556	-2.053	-2.065	-0.196	0.455	-0.389	-0.806	0.184	-0.310
	TSE	0.607	0.282	0.360	0.127	0.203	0.205	0.262	0.263	0.274	0.320	0.265	0.021	0.144	0.131
	MAE	0.601	0.266	0.280	0.093	0.136	0.155	0.176	0.190	0.268	0.346	0.264	0.018	0.056	0.053
	MSE	0.738	0.328	0.365	0.119	0.180	0.194	0.224	0.247	0.340	0.462	0.326	0.021	0.072	0.065
	θ_0	1.2	0.7	1.5	0.8	-1.5	-1.5	2.0	-2.0	0.3	-0.3	-0.5	-0.8	0.2	-0.3
100	CMLE	1.870	0.861	1.385	1.249	-2.278	-2.265	2.965	-3.053	0.852	0.041	-0.284	-0.805	0.138	-0.319
	TSE	1.834	0.993	1.142	0.457	0.720	0.728	0.919	0.950	1.017	1.517	0.866	0.050	0.359	0.304
	MAE	1.542	0.886	0.922	0.524	0.837	0.828	1.025	1.107	1.101	1.413	0.950	0.043	0.162	0.134
	MSE	2.017	1.142	1.275	0.848	1.421	1.467	1.697	1.971	1.608	2.059	1.329	0.057	0.221	0.171
300	CMLE	1.324	0.795	1.448	0.874	-1.653	-1.650	2.196	-2.218	0.378	-0.247	-0.468	-0.802	0.183	-0.308
	TSE	0.853	0.438	0.545	0.181	0.293	0.292	0.379	0.381	0.378	0.499	0.371	0.028	0.192	0.181
	MAE	0.717	0.336	0.393	0.138	0.239	0.229	0.294	0.313	0.388	0.508	0.336	0.021	0.077	0.065
	MSE	0.861	0.451	0.504	0.188	0.332	0.311	0.406	0.418	0.490	0.647	0.422	0.027	0.096	0.083
500	CMLE	1.199	0.718	1.458	0.842	-1.580	-1.587	2.109	-2.116	0.380	-0.207	-0.431	-0.803	0.183	-0.307
	TSE	0.647	0.325	0.418	0.132	0.214	0.216	0.279	0.280	0.286	0.373	0.274	0.022	0.151	0.140
	MAE	0.537	0.298	0.353	0.103	0.161	0.168	0.207	0.201	0.313	0.364	0.264	0.017	0.060	0.051
	MSE	0.671	0.391	0.442	0.135	0.213	0.218	0.273	0.267	0.405	0.468	0.345	0.021	0.074	0.064

Outline

Results (1/2)

Table: Best Model for Témiscamingue

	number of parameters = 14	AIC = 61.712	n = 75	
	portemanteau-test :	statistics = 1.167	p-value = 0.737	
	parameters	std	t-statistics	sig. code
ω_1	-1.741	0.929	-1.875	*
ω_2	-8.866	3.53	-2.511	**
ω_3	-80.08	192.275	-0.416	
Δ*temp. max. spring*	0.058	0.082	0.703	
Δ*temp. min. spring*	0.02	0.108	0.187	
Δ*temp. max. summer*	-0.134	0.115	-1.166	
Δ*temp. min. summer*	-0.057	0.217	-0.264	
Δ*log. tot. precipitation*	-4.063	0.991	-4.101	****
α_1	4.17	1.614	2.584	***
α_2	9.528	2.895	3.291	****
α_3	84.678	192.613	0.44	
β_1	0.517	0.193	2.677	***
β_2	0.027	0.578	0.046	
β_3	-0.999	0.005	-206.937	****

sig. code : 0.85 · 0.90 * 0.95 ** 0.99 *** 0.999 **** 1

Results (2/2)

1. Best model for the region of Témiscamingue is the one with the inter-annual variation of maximum and minimum seasonal temperature and inter-annual log-variation of the amount of precipitations as covariates.
2. The magnitude and statistical significance of autoregressive parameter α_1 and α_2 show that the levels of low and moderate intensities of defoliation affect the defoliation mechanism.
3. Statistical significance of the feedback parameters except β_2 indicating a link between Y_t and its all history.

References I

[1] Bunyeth Chan, Peng Bun Ngor, Zeb S. Hogan, Nam So, Sébastien Brosse, and Sovan Lek. Temporal dynamics of fish assemblages as a reflection of policy shift from fishing concession to co-management in one of the world's largest tropical flood pulse fisheries. *Water*, 12(11), 2020.

[2] Julie-Pascale Labrecque-Foy, Hubert Morin, and Miguel Montoro Girona. Dynamics of territorial occupation by north american beavers in canadian boreal forests: A novel dendroecological approach. *Forests*, 11(2):221, 2020.

[3] A Colin Cameron and Pravin K Trivedi. *Regression analysis of count data*, volume 53. Cambridge university press, 2013.

[4] Christian H Weiß, Fukang Zhu, and Aisouda Hoshiyar. Softplus ingarch models. *Statistica Sinica*, 32(3), 2022.

[5] George L. Vourlitis, Osvaldo Borges Pinto, Higo J. Dalmagro, Paulo Enrique Zanella de Arruda, Francisco de Almeida Lobo, and José de Souza Nogueira. Tree growth responses to climate variation in upland and seasonally flooded forests and woodlands of the cerrado-pantanal transition of brazil. *Forest Ecology and Management*, 505:119917, 2022.