Quelques nouvelles des MFG

Pierre-Louis LIONS

Collège de France, Paris

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SOMMAIRE

I. INTRODUCTION

- SELF CONSISTENT MODELS FOR LARGE POP.
- CONT. TIME/SPACE

1) MEASURES :

$M_+(\mathbb{R}^d) \to m \quad (P), x \in \mathbb{R}^d, t \in [0, T], u(x, m, t) \in \mathbb{R}$

$$\frac{\partial u}{\partial t} + D + T + N = 0(...)$$

D Decision. ex. Opt. control

T Transport

N Noise

<u>Rk.</u> : 2^{nd} order MFG where $D \leftrightarrow N$

2) HILBERT (or BANACH): $L^2(\Omega, \mathcal{F}, P) \ni X$

$U(X, t) \in H; A, B : H \times H \to H$

$\frac{\partial U}{\partial t} + (A(X, U).D)U = B(X, U) +$ noise

- FINITE STATE: $U : \mathbb{R}^k \times [0, T] \to \mathbb{R}^k$

$$\frac{\partial U}{\partial t} + (A(X, U) \cdot \nabla) \, U = B(X, U) + \text{ noise}$$

- Example (measures) $\nu, \alpha \geqslant 0$

$$\frac{\partial u}{\partial t} + A_0(x, \partial_x u, m) + \langle \partial_m u, -\partial_x \cdot (B_0(x, \partial_x u, m)m) \rangle$$

$$-(\nu + \alpha)\Delta_x u + \langle \partial_m u, -(\nu + \alpha)\Delta_x m \rangle$$

$$-\alpha \frac{\partial_u^2}{\partial_m^2}(\nabla_x m, \nabla_x m) + 2\alpha \langle \partial_m \nabla_x u, \nabla_x m \rangle = \dots$$

- $U(x,m,t) \to \mathbb{R}^d \quad (\partial_x u)$

$$\frac{\partial U}{\partial t} + (A(x,U,m)\cdot\partial_x)U + C(x,U,m) +$$

$$+\langle \partial_m U, -\partial_x \cdot (B(x,U,m)m)\rangle + \quad \text{“Noise”} \quad = 0$$

- Rks. : i) indt noise OK Hilbertian (extended proba.): $\varphi \in BUC(H)$ dep. only on the law, $\tilde{\varphi}(X) = \varphi(\mathcal{L}(X))$ if $X(\omega,\omega'), X \in L^2(\Omega\times\Omega, P\otimes P)$, let $G_0 = G_0(\omega')$ $(E[G_0] = \theta, E[G_0^2] = 1)$, and define

$$\tilde{D}^2\varphi(X) = D^2\tilde{\varphi}(X)(G_0, G_0)!$$

ii) full interaction: A, B, C depend on $\nabla U \# m$

- Claim: well-posed problems $(C\cdot - L\cdot)$ with $IC \quad U\,|_{t=0} = U_0$ in $C_t(C_b) \cap L_t^\infty$ (Lip.) (assuming all data A, B, C, V_0 are Lip.)
- Lip. $/m$: $MK - 1$ $W_1 \approx W^{-1,1}$ norm
- Rks : i) cont in H $\iff$ cont in P_1 but
  Lip. in $P_1 \implies$ Lip. in H

  ii) $W_1(U_1\#m_1, U_2\#m_2) \leqslant \sup|V_1 - V_2|(\int m_1)$

  $+\|U_2\|_{\mathrm{Lip}} W_1(m_1, m_2)$

  same result and proof.

II. UN PEU DE THÉORIE

ALL EQUATIONS, A, B, C Lip.

<u>THM</u> : Let U_0 Lip (and bounded). Then $\exists T_0 \in]0, +\infty]$ such that for all $T < T_0, \exists!$ sol. $U \in C([0, T] ; BUC) \cap L^\infty(0, T, \text{Lip})$.
If $T_0 < \infty$, then $\|U(t)|_{\text{Lip}} \to +\infty$ as $t \to T_0$.

<u>Rks</u> : i) below Lip ?

ii) $T_0 = +\infty$ if “monotonicity” assumptions

iii) in general, $T_0 < \infty$

iv) set of solutions beyond T_0 (potential case, examples) ?

v) usual Master Eq. on P : $\partial_x u \text{ Lip in } (x, m)$!

vi) meaning of solution later on!

Ex. : FINITE STATE SPACE

$$\frac{\partial U}{\partial t} + (A(X,U)\cdot\nabla)U = B(X,U)$$

U_0 Lip

Fixed point : $\frac{\partial U}{\partial t} + A(X,V)\cdot\nabla U = B(X,U)$

Estimates : $(U_1,V_1),(U_2,V_2)$

i) $\frac{d}{dt}\sup|U_1 - U_2| \leqslant C_0(\sup(V_1 - V_2))\|U_1\|_{\mathrm{Lip}} + C_0 \sup(V_1 - V_2)$

ii) $\frac{d}{dt}\|U\|_{\mathrm{Lip}} \leqslant C_0\|U\|_{\mathrm{Lip}}\ (1 + \|V\|_{\mathrm{Lip}}\ )$

For T small, $\exists$ ball in Lip inv. by $V \to U$ and then contraction in sup norm

IN ALL CASES SAME ESTIMATES THAT IN THE FINITE STATE SPACE.

Ex. : $\frac{\partial U}{\partial t} + (A(x, V, m).\partial_x)U + C(x, V, m) + < \partial_m U, -\partial_x \cdot (B(x, V, m)m) > +$ noise $(U) = 0$ and $U \equiv V$.

Hence the formulation requires only to define and estimate solutions of LINEAR equations with V Lip. through

$$dx = -A(x, V(x, m, t), m)dt + \dots$$

$$dm + \mathrm{div}(B(x, V(x, m, t), m))dt + \dots$$

standard estimates on $|x_1 - x_2|, W_1(m_1, m_2)$!

- MONOTONICITY CONDITIONS YIELD GLOBAL SOL. ("SMOOTH")
- IN GENERAL, SINGULARITIES APPEAR IN FINITE TIME
- FINITE STATE SPACE (deterministic case)

$$\frac{\partial U}{\partial t} + (A(X,U)\cdot\nabla)U = B(X,U) \quad U : \mathbb{R}^k \to \mathbb{R}^k$$

Ex : $k = 1, A(x,z) = z, B = 0 : \frac{\partial u}{\partial t} + uu_x = 0$!

singularity unless $u \uparrow$

THM : (B,A) MON. $\mathbb{R}^k \to \mathbb{R}^k \implies T_0 = +\infty$

- NON INCREASING SOLUTIONS : $U_i(x_1, \ldots, x_k) \searrow$ in $x_j \ \forall i, j$
  (cond. on A, B : ex. $\frac{\partial U}{\partial t} + (U \cdot \nabla)U = B(x) \ \ B \searrow$)
  joint work with B. SEEGER: notion of solutions
  (semi-continuous viscosity solutions)

  THM: i) $\exists$ max. sol $\bar{U}$, $\exists$ min. sol $\underline{U}$ (distinct for $T > T_0$)
  ii) $\exists$ regular. $U_\varepsilon \to \bar{U}, U_\varepsilon \to \underline{U}$, vanishing viscosity
  $\to U \neq \bar{U}, \underline{U}$
  iii) catalogue of sol.

  ex. $\frac{\partial u}{\partial t} + u \, \frac{\partial u}{\partial x} = 0 \ \ , u|_{t=0} = 1_{(x>0)}$
  then set of solutions $\quad u_0(x - c(t))$ with $0 \leqslant c' \leqslant 1$
  entropy sol. $c(t) \equiv t/2$, for each solution $\exists$ regularization

- CONJECTURE: THIS IS GENERAL (even in inf. dem.) ! ?
  Rk. : models require some extra "arbitrage-like" assumption . . . ?

III. APPLICATIONS

- Development of existing models : Edmond $\rightarrow$ global simulator of commodities (energy...)
- Regulation/major agents (very) incomplete markets
- MFG vs principal agent
- New topics
  - multi agent generator A.I.
  - agriculture (and land...) : water, competing uses of land...