Decarbonization of financial markets: a mean-field game approach

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From: a decarbonization of large financial markets via the mean field game theory

To: a mean field game theory with quadratic BSDEs

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Decarbonization of financial markets: a mean-field game approach

Joint work Lavigne and Tankov (2023), available on Arxiv. Based on De Angelis et al. (2022).

Firms:

- ∞-number of heterogeneous firms;
- Control their **economical value** via their production level and maximize their **financial value**.
- **compete** for capital allocation;
- GHG emissions associated with production.

Investors:

- 2-investors: A brown investor and a green investor (averse to climate change);
- Optimize their wealth under budget constraint.

Equilibrium:

- Between firms and investors occurs through a **market clearing** condition.
- Nash equilibrium between the firms competing for capital allocation on the market.

Structure of interactions

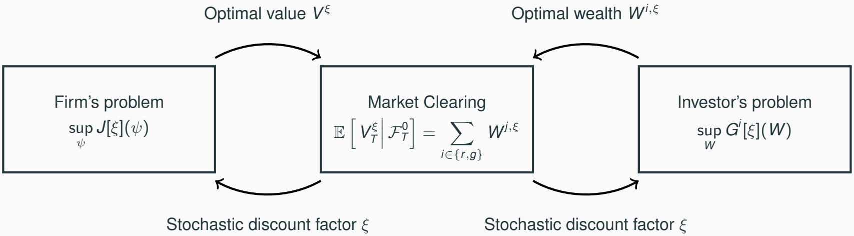

Figure 1: Structure of interactions.

Representative firm problem

Its criterion $J[\xi]$ is given by

$$J[\xi](\psi) = \mathbb{E}\left[S_0^{\xi,\psi} - \int_0^T \frac{\alpha_t|\psi_t|^2}{2}\mathrm{d}t\right], \quad S_t^{\xi,\psi} = \frac{1}{\xi_t}\mathbb{E}\left[\xi V_T^{\psi}\middle|\mathcal{F}_t\right],$$

where $S^{\xi,\psi}$ is the share price for a given discount factor of interaction ξ and a strategy of emissions ψ.

The value of the representative firm has the following dynamics:

$$\mathrm{d}V_t = V_t(\mu_t\mathrm{d}t + \sigma_t\mathrm{d}B_t + \sigma_t^0\mathrm{d}B_t^0) + \beta_t\psi_t\mathrm{d}t, \quad V_0 = V.$$

where $\mu, \sigma, \sigma^0, c, \psi$ are $\mathbb{F}$-adapted.

Lemma

Assume that $\mathbb{E}[\xi\mathcal{E}_T] < +\infty$, *and* $\psi^\xi \in L^2_\alpha(\mathbb{F})$, *where* $\psi_t^\xi := \frac{\beta_t}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]$. *Then the unique optimal solution of the representative firm problem is given by* ψ^ξ.

Investors' problem

The **regular** investor solves the optimization problem

$$\min_{W^r} \mathbb{E}_{\mathbb{P}}[e^{-\gamma^r W^r}] \quad \text{subject to} \quad \mathbb{E}[\xi W^r] \leq w^r.$$

The **green** investor solves

$$\min_{W^g} \mathbb{E}_{\mathbb{P}^g}[e^{-\gamma^g W^g}] \quad \text{subject to} \quad \mathbb{E}[\xi W^g] \leq w^g,$$

where the probability measure of the green investor $\mathbb{P}^g$, is defined by

$$\frac{\mathrm{d}\mathbb{P}^g}{\mathrm{d}\mathbb{P}} = Z, \quad Z = e^{-\int_0^T \lambda_s \mathrm{d}B_s^0 - \frac{1}{2}\int_0^T |\lambda_s|^2 \mathrm{d}s}.$$

The solutions to the optimization problems are of the form

$$W^g = \frac{1}{\gamma^g} \ln \frac{\xi}{Z} + c^g, \qquad W^r = \frac{1}{\gamma^r} \ln \xi + c^r,$$

for some constants c^g and c^r to be determined.

Market equilibrium

In the mean-field market, assuming unlimited supply of the risk-free asset, the market clearing condition takes the form:

$$W^g + W^r = \mathbb{E}\left[V_T | \mathcal{F}_T^0\right] + c$$

for an arbitrary constant c. Substituting the explicit formulas, this is equivalent to

$$\xi = \frac{\exp\left(\rho \ln(Z) - \gamma^* \mathbb{E}\left[V_T | \mathcal{F}_T^0\right]\right)}{\mathbb{E}\left[\exp\left(\rho \ln(Z) - \gamma^* \mathbb{E}\left[V_T | \mathcal{F}_T^0\right]\right)\right]},$$

with $\frac{1}{\gamma^*} = \frac{1}{\gamma^r} + \frac{1}{\gamma^g}$ and where $\rho := \frac{\gamma^r}{\gamma^g + \gamma^r} \in (0, 1)$ can be interpreted as the proportion of green investors in the market.

Mean-field game equilibrium

Mean-field game equilibrium problem: find a tuple $(\bar{\psi}, \bar{\xi}, \bar{m}) \in L^2_\alpha(\mathbb{F}) \times \Xi \times \mathcal{P}(\mathcal{P}(\Omega))$:

$$\bar{\psi} = \underset{\psi \in L^2_\alpha(\mathbb{F})}{\arg\min}\, J[\bar{\xi}](\psi), \quad \bar{\xi} = I(\bar{m}), \quad \bar{m} = \mathcal{L}\left(V_T^{\bar{\psi}} | \mathcal{F}_T^0\right). \tag{NE}$$

In our setting, it is enough to look for a fixed point in the space Ξ.

Then, Nash equilibrium problem reduces to the fixed point problem: find $\xi \in \Xi$, s. t.

$$\xi = \frac{\exp\left(\rho \ln(Z) - \gamma^* \mathbb{E}[V_T^\xi | \mathcal{F}_T^0]\right)}{\mathbb{E}\left[\exp\left(\rho \ln(Z) - \gamma^* \mathbb{E}[V_T^\xi | \mathcal{F}_T^0]\right)\right]}. \tag{FP}$$

Existence, uniqueness and algorithm

Under appropriate assumptions we have the following results.

Theorem

Then there a unique solution $\bar{\xi} \in \Xi$ to the fixed-point problem (FP)*.*

Algorithm

Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence in $[0, 1]$. Let $\xi_0 \in \mathcal{C}$. Consider the sequence $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ defined as follows

$$\eta_k = \frac{\exp\left(-\gamma^* \mathbb{E}[V_T^{\xi_k} | \mathcal{F}_T^0] + \rho \ln Z\right)}{\mathbb{E}\left[\exp\left(-\gamma^* \mathbb{E}[V_T^{\xi_k} | \mathcal{F}_T^0] + \rho \ln Z\right)\right]},$$

$$\xi_{k+1} = \alpha_k \eta_k + (1 - \alpha_k) \xi_k.$$

Theorem

The sequence $(\xi_k)_{k \in \mathbb{N}}$ weakly converges in $\mathcal{C}$ for $\alpha_k = 2/(k+2)$.

Extension

- In the problem studied, investors do not behave dynamically,
- The investors are impacted by the emission only through the discount factor ξ at equilibrium.

Refinements

Suppose one can apply the martingale representation theorem, and there exists $(\theta_t)_{t\in[0,T]}$ such that the tuple of price and volatility (S,σ) is solution to the following BSDE

$$\frac{\mathrm{d}S_t}{S_t} = \sigma_t(\theta_t \mathrm{d}t + \mathrm{d}W_t^0), \quad S_T = S.$$

Mean variance criterion

The investor now maximizes a penalized mean variance criterion,

$$\mathbb{E}\left[X_T^\delta - \int_0^T |\delta_t\sigma_t|^2 \mathrm{d}t - \int_0^T |\delta_t\varphi_t|^2 \mathrm{d}t\right],$$

where the dynamics of the portfolio $(X_t)_{t\in[0,T]}$ is given in quantities by

$$\mathrm{d}X_t = \delta_t \mathrm{d}S_t, \quad X_0 = X.$$

and $\varphi_t = \mathbb{E}[\psi_t|\mathcal{F}_t^0]$ is the total market emissions.

To a quadratic pricing BSDE

The first order condition and market clearing condition are given by

$$\theta_t \sigma_t S_t = \delta_t(\varphi_t^2 + \sigma_t^2), \quad \delta_t = 1, \quad S_T = \mathbb{E}\left[V_T | \mathcal{F}_T^0\right].$$

Plugging into the pricing equation, we are now looking for a solution (S, σ) to the quadratic BSDE parametrized by the flow of emission $(\varphi_t)_{t \in [0,T]}$

$$\mathrm{d}S_t = (\varphi_t^2 + \sigma_t^2)\mathrm{d}t + \sigma_t \mathrm{d}W_t^0, \quad S_T = \mathbb{E}\left[V_T | \mathcal{F}_T^0\right].$$

The mean field game problem of the firms is now to find (ψ, m, φ) such that

$$\psi \in \arg\max_{\psi'} \mathbb{E}\left[S_0^{m,\varphi,\psi'} - \int_0^T \frac{\alpha_t |\psi_t'|^2}{2} \mathrm{d}t\right],$$
$$m_t = \mathcal{L}(X_t^{\psi} | \mathcal{F}_t^0), \quad \varphi_t = \mathbb{E}[\psi_t | \mathcal{F}_t^0].$$

Such formulation is non-standard in mean field games.

Can we develop a ”general” theory with non-linear evaluation of the terminal interaction terms ?

To: a mean field game theory with quadratic BSDEs

Framework

For any couple of $\mathbb{F}^0$ adapted random law (ν, m), consider the individual problem

$$\inf_{\psi} \mathcal{J}[\nu, m](\psi) = \mathbb{E}\left[h(Y_0^{\psi,\nu,m}) + \int_0^T L(\psi_t)\mathrm{d}t\right].$$

where $(Y^{\psi,\nu,m}, Z^{\psi,\nu,m}, X^{\psi})$ is solution to the FBSDE

$$\begin{cases} -\mathrm{d}Y_t &= f(Y_t, Z_t, \nu_t)\mathrm{d}t - Z_t\mathrm{d}W_t^0, & Y_T = g(X_T, m_T), \\ \mathrm{d}X_t &= b(X_t, \psi_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \sigma^0(X_t)\mathrm{d}W_t^0, & X_0 = \eta. \end{cases}$$

Where we assume that

- L is convex quadratic, h, b, σ and σ^0 are Lipschitz continuous.
- g is bounded, i.e $g(X_T, m_T) \in L^\infty(\mathbb{F}^0)$.
- f is a quadratic driver, that is to say

$$|f(y, z, \nu)| \leq C\left(1 + |y| + |z|^2\right), \quad \text{almost surely.}$$

Mean field game problem

The mean field game problem is to find a tuple (ψ, Y, Z, X, ν, m) such that

$$\begin{cases} (i) & \psi \in \arg\min_{\psi'} \mathcal{J}[\nu, m](\psi'), \\ (ii) & (Y, Z, X) = (Y^{\psi,\nu,m}, Z^{\psi,\nu,m}, X^{\psi}), \\ (iii) & (\nu_t, m_t) = (\mathcal{L}(\psi_t|\mathcal{F}_t^0), \mathcal{L}(X_t|\mathcal{F}_t^0)), \end{cases} \tag{MFG}$$

Difficulty

1. Common noise setting.
2. The quadratic BSDE framework is not investigated in the MFG literature...
3. Even the Pontryagin maximum principle for this kind of optimization problem seems unavailable.

Approach

- Start from classical works on quadratic BSDE for existence, uniqueness of solutions, a priori estimates and stability results Kobylanski (2000), Ankirchner et al. (2007), Zhang (2017).
- Establish a Pontryagin maximum principle (for a fixed couple (m, ν)),
- Show existence and uniqueness of a MFG equilibrium.

Example: the entropic risk measure case

Example Consider the following simple case

$$\inf_{\psi} \mathcal{J}[\nu, m](\psi) = \mathbb{E}\left[Y_0^{\psi,\nu,m} + \int_0^T \frac{1}{2}|\psi_t|^2 \mathrm{d}t\right].$$

where $(Y^{\psi,\nu,m}, Z^{\psi,\nu,m}, X^{\psi})$ is solution to the FBSDE

$$\begin{cases} -\mathrm{d}Y_t = \frac{1}{2\gamma}|Z_t|^2 \mathrm{d}t - Z_t \mathrm{d}W_t^0, & Y_T = g(X_T, m_T), \\ \mathrm{d}X_t = \psi_t \mathrm{d}t + \sigma \mathrm{d}W_t + \sigma^0 \mathrm{d}W_t^0, & X_0 = \eta. \end{cases}$$

Explicit solution to the BSDE Consider a solution (P, Q) to the following linear BSDE

$$-\mathrm{d}P_t = Q_t \mathrm{d}W_t^0, \quad P_T = e^{-\frac{1}{\gamma} g(X_T, m_T)}.$$

(So $P_t = \mathbb{E}\left[e^{-\frac{1}{\gamma} g(X_T, m_T)} | \mathcal{F}_t^0\right]$). Then by the Hopf-Cole transformation, we have that (Y, Z) defined as

$$Y_t = -\gamma \ln(P_t), \quad Z_t = -\gamma Q_t / P_t,$$

is solution to the BSDE part.

Example: the entropic risk measure case

Then the control problem is now of the following form

$$\inf_{\psi} \mathcal{J}[\nu, m](\psi) = \rho_\gamma(g(X_T, m_T)) + \mathbb{E}\left[\int_0^T \frac{1}{2}|\psi_t|^2 \mathrm{d}t\right].$$

where ρ_γ is the entropic risk measure

$$\rho_\gamma(g(X_T, m_T)) = -\gamma \ln \mathbb{E}\left[e^{-\frac{1}{\gamma} g(X_T, m_T)}\right].$$

(Remark: the entropic risk measure has dual representation, see Barrieu and Karoui (2007)).

Pontryagin maximum principle

Fix (even forget about) the interaction terms (m, ν). Road to Pontryagin maximum principle:

- Consider an optimal control ψ and a perturbation $\varepsilon\varphi$.

$$\varepsilon^{-1}\left(\mathcal{J}(\psi+\varepsilon\varphi)-\mathcal{J}(\psi)\right) \geq 0.$$

- Establish derivatives (y, z, x) for the states variables (Y, Z, X) with respect to the control variable ψ that is to say show that

$$\lim_{\varepsilon \to 0} \|\Delta Y^{\varepsilon}\|_{S^{2p}(\mathbb{G})} + \|\Delta Z^{\varepsilon}\|_{M^{2p}(\mathbb{G})} + \|\Delta X^{\varepsilon}\|_{L^2(\mathbb{G})} = 0.$$

where

$$\Delta Y^{\varepsilon} = \varepsilon^{-1}(Y^{\psi+\varepsilon\varphi} - Y^{\psi}) - y, \quad \Delta Z^{\varepsilon} = \varepsilon^{-1}(Z^{\psi+\varepsilon\varphi} - Z^{\psi}) - z,$$
$$\Delta X^{\varepsilon} = \varepsilon^{-1}(X^{\psi+\varepsilon\varphi} - X^{\psi}) - x.$$

and

$$\begin{cases} -\mathrm{d}y_t &= (y_t\partial_y f_t + z_t\partial_z f_t)\,\mathrm{d}t - z_t\mathrm{d}W_t^0, \\ y_T &= \mathbb{E}\left[x_T\partial_x g_T | \mathcal{F}_T^0\right], \\ \mathrm{d}x_t &= \varphi_t\partial_\psi b_t\mathrm{d}t + x_t\left(\partial_x b_t\mathrm{d}t + \partial_x\sigma_t\mathrm{d}W_t + \partial_x\sigma_t^0\mathrm{d}W_t^0\right), \\ x_0 &= 0. \end{cases}$$

Pontryagin maximum principle

- Define adjoint variables (p, k, k^0, q) to the derivatives (y, z, x), which are themselves solutions to the FBSDE

$$\begin{cases} -\mathrm{d}p_t &= p_t \partial_x b_t \mathrm{d}t - k_t \mathrm{d}W_t - k_t^0 \mathrm{d}W_t^0, \quad p_T &= q_T \partial_x g_T, \\ \mathrm{d}q_t &= q_t \partial_y f_t \mathrm{d}t + q_t \partial_z f_t \mathrm{d}W_t^0, \quad q_0 &= \partial_y h(Y_0). \end{cases}$$

- Define the Hamiltonian

$$H(X_t, p_t, \psi_t) = L(\psi_t) + p_t b(X_t, \psi_t).$$

and show that any optimal control ψ satisfies

$$\langle \partial_\psi H(X_t, p_t, \psi_t), \varphi_t - \psi_t \rangle \geq 0,$$

a.s, for any direction φ.

Ongoing work

Existence and uniqueness of MFG equilibrium:

- Existence of MFG equilibrium : Picard fixed point ?
- Uniqueness holds under the following conditions:
  1. Lasry-Lions monotonicity condition
  $$\int_{\mathbb{R}} (g(x, m_1) - g(x, m_2)) \mathrm{d}(m_1 - m_2)(x) \geq 0.$$
  2. h is convex and strictly increasing, that is to say, there exists $C > 0$ such that
  $$h(y_1) \geq h(y_2) + \frac{1}{C}(y_1 - y_2).$$

Futur research

- Application: solve the initial green finance problem (and variants)
- Application to MFG with risk management (g-expectation criterion).
- Malliavin calculus for PMP principle and MFG.
- Numerical resolution via Euler schemes and regressions for the FBSDE system.
- Is there a potential formulation ?

Thank you for your attention.

Illustrations

Model specification

Let B^0 be a 2-dimensional Brownian motions with components $(B^{0,1}, B^{0,2})$.

- The emission efficacy (β_t) is constant in time for each firm, $\beta_t = C$, and the emission penalty (α_t) is the same for all firms (and $\mathbb{F}^0$-measurable).
- We model the differences in initial firm values and emission efficacies by introducing $\mathcal{F}_0$-measurable random variables V and C.
- The emission penalty (α_t) is a stochastic process defined by

$$\alpha_t = e^{\gamma B_t^{0,2} - \frac{\gamma^2}{2} t}, \quad \gamma \in \mathbb{R}_+.$$

- The density of the change of measure of the green investor is defined by

$$Z = e^{\lambda B_T^{0,2} - \lambda^2 T/2}, \quad \lambda \in \mathbb{R}_+.$$

Model specification

The parameters of the firm value dynamics σ, σ^0 and μ are constant, and we let

$$\overline{\mathcal{E}}_{t,T} = e^{\sigma^0 B_t^{0,1} + (\mu - (\sigma^0)^2/2)t}.$$

The fixed point equation then writes

$$\xi = \frac{\exp\left(-\gamma^* V_T^{\xi} + \rho(\lambda B_T^{0,2} - \lambda^2 t/2)\right)}{\mathbb{E}\left[\exp\left(-\gamma^* V_T^{\xi} + \rho(\lambda B_T^{0,2} - \lambda^2 T/2)\right)\right]},$$

$$V_T^{\xi} = \overline{V}\,\overline{\mathcal{E}}_T + \int_0^T \overline{C_0^2} e^{-\gamma B_t^{0,2} + |\gamma|^2 t/2} \overline{\mathcal{E}}_{t,T}\, \mathbb{E}[\xi \overline{\mathcal{E}}_{t,T} | \mathcal{F}_t^0] \mathrm{d}t,$$

where we denote $\overline{C_0^2} = \mathbb{E}[C_0^2]$ and $\overline{V} = \mathbb{E}[V]$.

Model parameters

The parameters ρ (proportion of green investors), γ (volatility of emissions penalty, a proxy for climate risk) and λ (environmental concern of green investors) are changing throughout the tests and the other parameters are given below:

Variable	**Value**	**Description**
T	5	Time horizon, years
γ^*	0.5	Risk aversion parameter
σ_0	10%	Volatility of the common noise part of firm value dynamics
μ	5%	Drift of the firm value dynamics
$\overline{V}$	1	Average initial firm value
$\overline{C^2}$	1	Average squared emission efficacy of production
$\overline{C}$	0.7	Average emission efficacy of production
n	20	Number of discretization steps
N	50 000	Number of sample trajectories
p	2	Weight in the fixed-point algorithm

Outputs of the algorithm

- The total average emissions, given by

$$\overline{\Psi}_T = \int_0^T \mathbb{E}[\psi_t|\mathcal{F}_T^0]\mathrm{d}t = \int_0^T \overline{C}_0 e^{-\gamma B_t^{0,2}+\gamma^2 t/2}\mathbb{E}[\xi\overline{\mathcal{E}}_{t,T}|\mathcal{F}_t^0]\mathrm{d}t.$$

- The expected emissions of the representative company at date t, given by

$$\mathbb{E}[\psi_t] = C_0\mathbb{E}\left[\xi e^{-\gamma B_t^{0,2}+\frac{\gamma^2 t}{2}}\overline{\mathcal{E}}_{t,T}\right].$$

- The initial stock price of the representative company, given by

$$S_0^\xi = V\mathbb{E}[\xi\mathcal{E}_T|\mathcal{F}_0] + \mathbb{E}\left[\int_0^T \frac{c_s^2}{\alpha_s}\mathbb{E}^2[\xi\mathcal{E}_{s,T}|\mathcal{F}_s]ds\middle|\mathcal{F}_0\right]$$
$$= V\mathbb{E}[\xi\overline{\mathcal{E}}_T] + C_0^2\mathbb{E}\left[\int_0^T e^{-\gamma B_t^{0,2}+\gamma^2 t/2}\mathbb{E}^2[\xi\overline{\mathcal{E}}_{t,T}|\mathcal{F}_t]\mathrm{d}t\right]$$

Convergence

Here $\gamma = 0.3$ and $\lambda = 0$ (no green investors).

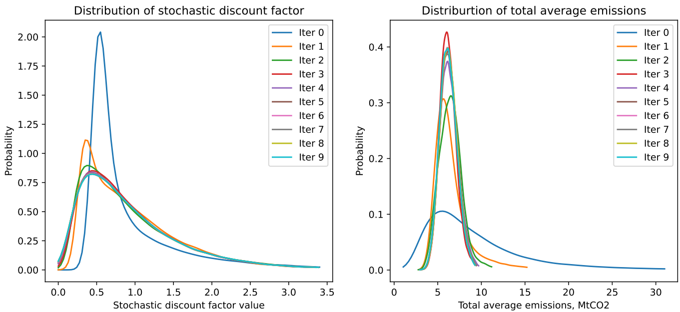

Figure 2: Left: convergence of the distribution of ξ. Right: convergence of the distribution of total average emissions $\overline{\Psi}_T = \int_0^T \mathbb{E}[\psi_t|\mathcal{F}_T^0]$; for the 10 first step of the algorithm.

Impact of uncertainty of climate policies γ on the decarbonization

Still in the absence of green investors ($\lambda = 0$).

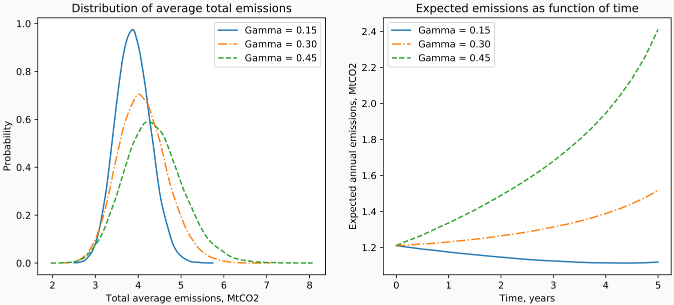

Figure 3: Left: distribution of total average emissions $\Psi_T = \int_0^T \mathbb{E}[\psi_t | \mathcal{F}_T^0] \mathrm{d}t$; right: expected emissions of the representative company per unit of time $\mathbb{E}[\psi_t]$; for different values uncertainty of climate policies γ.

Impact of the environmental concern λ of the green investors

The proportion of green investors is $\rho = 0.5$ and the risk $\gamma = 0.3$.

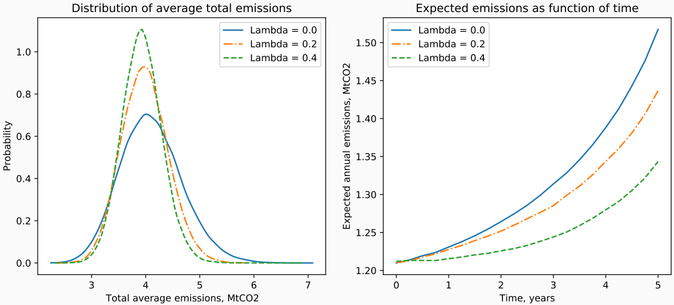

Figure 4: Left: distribution of total average emissions $\Psi_T = \int_0^T \mathbb{E}[\psi_t|\mathcal{F}_T^0]\mathrm{d}t$; right: expected emissions of the representative company per unit of time $\mathbb{E}[\psi_t]$; for different values of the environmental concern of green investors λ.

Impact of the proportion ρ of green investors

The environmental concern of the green investor is fixed at $\lambda = 0.4$.

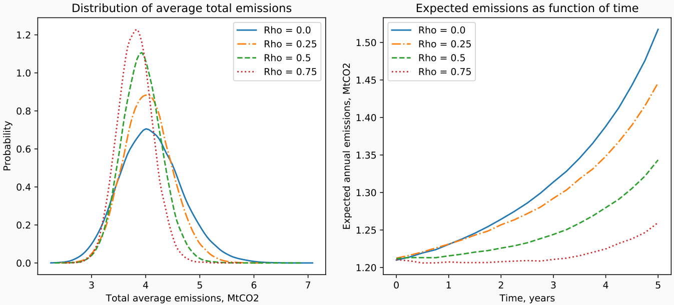

Figure 5: Left: distribution of total average emissions $\Psi_T = \int_0^T \mathbb{E}[\psi_t | \mathcal{F}_T^0] \mathrm{d}t$; right: expected emissions of the representative company per unit of time $\mathbb{E}[\psi_t]$; for different values of proportions of green investors ρ.

Impact of the green investors on share prices S_0^ξ

$$S_0^\xi = VP_1 + C_0^2 P_2, \quad P_1 = \mathbb{E}[\xi \overline{\mathcal{E}}_T], \quad P_2 = \mathbb{E}\left[\int_0^T e^{-\gamma B_t^{0,2} + \gamma^2 t/2} \mathbb{E}^2[\xi \overline{\mathcal{E}}_{t,T} | \mathcal{F}_t] \mathrm{d}t\right].$$

γ	λ	ρ	P_1	P_2
		Impact of γ		
0.15	0	0.5	1.2102 ± 0.0004	6.260 ± 0.011
0.3	0	0.5	1.2112 ± 0.0003	6.928 ± 0.014
0.45	0	0.5	1.2128 ± 0.0003	8.143 ± 0.010
		Impact of λ		
0.3	0	0.5	1.2112 ± 0.0003	6.928 ± 0.014
0.3	0.2	0.5	1.2103 ± 0.0003	6.834 ± 0.017
0.3	0.4	0.5	1.2115 ± 0.0003	6.730 ± 0.019
		Impact of ρ		
0.3	0.4	0.0	1.2112 ± 0.0003	6.928 ± 0.014
0.3	0.4	0.25	1.2108 ± 0.0005	6.825 ± 0.020
0.3	0.4	0.5	1.2115 ± 0.0003	6.730 ± 0.019
0.3	0.4	0.75	1.2119 ± 0.0004	6.710 ± 0.020
0.3	0.4	1.0	1.2113 ± 0.0003	6.595 ± 0.021

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