Mean field coarse correlated equilibria with applications to emissions' abatement

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– Based on joint works with F. Cannerozzi[1], F. Cartellier[2] and M. Fischer[3] –

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Outline

Mean Field Games (MFGs)

- Introduced by [Huang, Malhamé and Caines, 2006] and [Lasry and Lions, 2007]
- Arise as limits as $N \to \infty$ of certain N-player games exhibiting
  - symmetry (statistically indistinguishable components/exchangeable joint laws)
  - mean field interaction (the influence of each single player on the whole system diminishes as $N \to \infty$)
- The passage to the limit is analogous to McKean-Vlasov limit for weakly interacting particle systems but here the states are controlled.
- Notion of solution at prelimit level: (approximate) Nash equilibria.
- Goal of the talk: looking for more general equilibria with better properties.

Continuous time setting

Consider N players, whose individual $\mathbb{R}^d$ -valued states follow

$$dX_t^{N,i} = b(t, X_t^{N,i}, \bar{\mu}_t^N, \alpha_t^i)dt + dW_t^i, \quad X_0^{N,i} = \xi^i \sim \nu$$

for $i = 1, \ldots, N$, where

- $(\xi^i)_{i\geq 1}$ are i.i.d., $(W^i)_{i\geq 1}$ are independent BMs, all defined on the canonical probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$
- $\mathbb{F}^{1,N}$ is the (completed) filtration generated by $(W^1, \ldots, W^N)$ and $(\xi^i)_{i=1}^N$
- $\bar{\mu}_t^N = \frac{1}{N}\sum_i \delta_{X_t^{N,i}}$ is the empirical measure of players' states
- $\alpha^i \in \mathcal{A}_N = \mathcal{A}(\mathbb{F}^{1,N})$ is an admissible strategy for player i, i.e. $\mathbb{F}^{1,N}$-prog measurable process with values in some compact $A \subset \mathbb{R}^l$, $l \geq 1$

The expected cost of each player is

$$J_i(\alpha) = J(\alpha^i, \alpha^{-i}) = \mathbb{E}^{\mathbb{P}^1}\left[\int_0^T f(t, X_t^{N,i}, \bar{\mu}_t^N, \alpha_t^i)dt + g(X_T^{N,i}, \bar{\mu}_T^N)\right] \to \min_{\alpha^i}$$

Continuous time setting

An admissible strategy profile $\hat{\alpha} = (\hat{\alpha}^1, \dots, \hat{\alpha}^N) \in \mathcal{A}_N^N$ is a Nash eq (NE) if for all $i = 1, \dots, N$

$$J_i(\hat{\alpha}) \leq J_i(\alpha^i, \hat{\alpha}^{-i}), \quad \forall \alpha^i \in \mathcal{A}_N$$

When $N \to \infty$ Nash equilibria "converge" towards MFG equilibria, i.e. pairs $(\hat{\alpha}, \hat{\mu})$ such that:

- (Optimality) Given the flow of measure $\hat{\mu}_t$

$$J_{MFG}(\hat{\alpha}) = \inf_{\alpha} \mathbb{E}\left[\int_0^T f(t, X_t, \hat{\mu}_t, \alpha_t)dt + g(X_T, \hat{\mu}_T)\right]$$

  under $dX_t = b(t, X_t, \hat{\mu}_t, \alpha_t)dt + dW_t$, $X_0 = \xi \sim \nu$

- (Consistency) Let $\hat{X}$ satisfy

$$dX_t = b(t, X_t, \hat{\mu}_t, \hat{\alpha}_t)dt + dW_t, \quad X_0 = \xi$$

  then $\mathcal{L}(\hat{X}_t) = \hat{\mu}_t$ for all $t \in [0, T]$.

Main ref (probabilistic approach): the book [Carmona and Delarue, 2018]

Beyond Nash equilibria

Typical issues with NE:

- hard to compute (fixed point!)
- not very efficient (compared to Pareto optima)
- hard to justify if players have limited rationality

That's why in game theory we also have

- Correlated equilibria (CE) [Aumann, 1974] and
- Coarse correlated equilibria (CCE) [Hannan, 1957], [Moulin and Vial, 1978]

Observe: NE $\subset$ CE $\subset$ CCE

Main idea [Aumann, 1974]: before the game starts, a "mediator" chooses a profile of possibly correlated strategies according to some distribution and recommends them privately to the players.

E.g. traffic lights in routing games [Roughgarden, 2016]

(Coarse) correlated equilibria – Advantages

CE vs CCE

- CE: the players decide to follow or not the recommendations after receiving them (so ex-post)
- CCE: the players decide to follow or not the recommendations before receiving them, i.e. they commit to the mediator ex-ante

Advantages of CE and CCE:

- Easier to compute (linear programming under linear constraints)
- Lower expected costs (higher efficiency)
- Easier to justify via learning algorithms [Hart and Mas-Colell, 2001]

Applications:

- Economics: oligopoly, emissions' abatement [Moulin et al, 2014], [Dokka et al, 2022]
- Computer science: Babichenko, Papadimitriou, Roughgarden ...

CCE and mean field games – Assumptions

Main standing assumptions: (other than A compact)

- A1 Initial distribution $\nu \in \mathcal{P}^{\bar{p}}(\mathbb{R}^d)$ some $\bar{p} > 4$
- A2 b, f, g jointly measurable in all their variables
- A3 $(x, m, a) \mapsto b(t, x, m, a)$ Lipschitz uniformly in t
- A4 $t \mapsto (b, f)(t, 0, \delta_0, a_0)$ bounded for some $a_0 \in A$
- A5 f, g locally Lipschitz and quadratic growth in (x, m, a) (unif. in t)

Extra-assumption for the existence of CCE in the MFG

- B The set

$$K(t, x, m) := \{(b(t, x, m, a), z) : a \in A, f(t, x, m, a) \leq z\} \subset \mathbb{R}^d \times \mathbb{R}$$

is closed and convex $\forall (t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d)$

CCE in the N-player game – Setting

- We add a pre-game phase at time $t = 0-$
- Let $(\Omega^0, \mathcal{F}^{0-}, \mathbb{P}^0)$ be a Polish probability space supporting some extra-randomization, to be chosen as part of the equilibrium
- Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^0, \mathcal{F}^{0-}, \mathbb{P}^0) \otimes (\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ equipped with the filtration $\mathbb{F} = \mathcal{F}^{0-} \otimes \mathbb{F}^{1,N}$
- Mediator's recommandations: any $\mathbb{F}$-progressively measurable

$$\lambda_t : \Omega^0 \times \Omega^1 \to A^N, \quad \lambda_\cdot(\omega_0, \cdot) \in \mathcal{A}_N^N$$

inducing states' dynamics under $\mathbb{P}$

$$dX_t^{N,i} = b(t, X_t^{N,i}, \bar{\mu}_t^N, \lambda_t^i)dt + dW_t^i, \quad X_0^{N,i} = \xi^i \sim \nu$$

and individual costs (when all players follow the recommendation)

$$J_i(\lambda) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^{N,i}, \bar{\mu}_t^N, \lambda_t^i)dt + g(X_T^{N,i}, \bar{\mu}_T^N)\right]$$

CCE in the N-player game – Definition

CCE Let $\varepsilon \geq 0$. An ε-coarse correlated equilibrium is any admissible profile $\hat{\lambda}$ such that for all player $i = 1, \dots, N$

$$J_i(\hat{\lambda}) \leq J_i(\beta^i, \hat{\lambda}^{-i}) + \varepsilon$$

for all deviations $\beta^i \in \mathcal{A}_N$.

Remark (NE $\subset$ CCE)

- If $\mathbb{P}^0$ is a Dirac, then $\hat{\lambda}$ is a NE in pure strategies
- If $(\Omega^0, \mathcal{F}^{0-}, \mathbb{P}^0) = \otimes_{i=1}^N (\Omega_i^0, \mathcal{F}_i^{0-}, \mathbb{P}_i^0)$ and $\lambda^i(\omega^0, \cdot) = \tilde{\lambda}^i(\omega_i^0, \cdot)$ for each player i, then $\hat{\lambda}$ is a NE in mixed strategies.

MFG literature (For CE in discrete time, finite state and action spaces):

- [C. and Fischer, 2022], [Bonesini, C. and Fischer, 2022],
- Google Brain & Deep Mind [Muller et al, 2021] [Muller et al, 2022]

CCE and mean field games: $N \to \infty$

Let $(\Omega^*, \mathcal{F}^*, \mathbb{F}^*, \mathbb{P}^*)$ be the (completed) filtered canonical space for Brownian motion and initial condition

A coarse correlated MFG equilibrium is any admissible tuple

$$((\Omega^0, \mathcal{F}^{0-}, \mathbb{P}^0), \hat{\lambda}, \hat{\mu})$$

such that on the (completed) product space

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) = (\Omega^0, \mathcal{F}^{0-}, \mathcal{F}^{0-}, \mathbb{P}^0) \otimes (\Omega^*, \mathcal{F}^*, \mathbb{F}^*, \mathbb{P}^*)$$

we have

1. Optimality: $J(\hat{\lambda}, \hat{\mu}) \leq \inf_{\beta \in \mathcal{A}} J(\beta, \hat{\mu})$, where

$$J(\beta, \hat{\mu}) := \mathbb{E}^{\mathbb{P}} \left[\int_0^T f(t, X_t, \hat{\mu}_t, \beta_t) dt + g(X_T, \hat{\mu}_T) \right]$$

with $dX_t = b(t, X_t, \hat{\mu}_t, \beta_t)dt + dW_t$, $X_0 = \xi \sim \nu$.

2. Consistency: $\mathcal{L}(\hat{X}_t \mid \hat{\mu}) = \hat{\mu}_t$ for all $t \in [0, T]$

CCE and mean field games – Existence

Theorem

Assume A1 to A5 and B, then there exists a coarse correlated MFG equilibrium.

The proof is based on an argument inspired by [Hart and Schmeidler, 1989]:

Let $(\hat{\lambda}, \hat{\mu})$ be a coarse correlated MFG eq

$$J(\hat{\lambda}, \hat{\mu}) \leq \inf_{\beta} J(\beta, \hat{\mu}) \iff \inf_{\beta}(J(\beta, \hat{\mu}) - J(\hat{\lambda}, \hat{\mu})) \geq 0$$

Hence, by Fan's minimax theorem we show (up to some compactification)

$$\sup_{\lambda,\mu \text{ s.t. (2) holds}} \inf_{\beta}(J(\beta, \mu) - J(\lambda, \mu)) = \inf_{\beta} \sup_{\lambda,\mu \text{ s.t. (2) holds}} (J(\beta, \mu) - J(\lambda, \mu))$$

and prove that $\inf \sup(\ldots) \geq 0$ (easier), so getting existence (no fixed point theorems!)

Remark: Assumption B is used to "uncompactify" to go back to strict controls.

CCE and mean field games – Approximation

Theorem

Assume A1 to A5 and let $((\Omega^0, \mathcal{F}^{0-}, \mathbb{P}^0), \hat{\lambda}, \hat{\mu})$ *be a coarse corr MFG eq.*

Then for all $N \geq 2$ *there exists a tuple*

$$((\Omega^{0,N}, \mathcal{F}^{0-,N}, \mathbb{P}^{0,N}), \underline{\lambda}^N)$$

such that $\underline{\lambda}^N = (\lambda^1, \ldots, \lambda^N)$ *is a* ε_N*-CCE for the* N*-player game with* $\varepsilon_N \to 0$ *as* $N \to \infty$.

Proof's main steps:

- Construct infinitely many copies $(\hat{\lambda}^i)_{i \geq 1}$ of $\hat{\lambda}$, which are conditionally independent given $\hat{\mu}$
- Recommend those copies to the players in the N-player game
- Check, via propagation of chaos arguments, that the error player 1 makes when following $\hat{\lambda}^1$ (instead of deviating) vanishes as $N \to \infty$

Motivation: [Barrett, 1994] model

It's a very popular game for environmental abatement (≈ 2000 citations in Google Scholar):

- N agents (countries or firms) choose how much to abate
- each agent has a personal (abatement) cost ...
- ... but the total abatement generates a benefit to all agents
- x_i is the abatement level chose by agent i, $x = \sum_i x_i$ is the total abatement
- agent i's benefit is $B_i(x) = b(ax - \frac{x^2}{2})N$ while the cost is $C_i(x_i) = c\frac{x_i^2}{2}$
- Tension between two goals: high abatement level and high payoffs
- [Dokka et al, 2022] consider static CCE for Barrett's game

Abatement game: classical MFG setting

Each player maximises the following objective wrt α [Dokka et al, 2022]

$$J(\alpha, m) = \mathbb{E}\Big[\int_0^T \Big(\underbrace{a m_t - \frac{b}{2} {m_t}^2}_{\text{Abatement benefit}} \; \underbrace{-\frac{1}{2}{\alpha_t}^2}_{\text{Ab. private cost}} \; \underbrace{- \frac{\varepsilon}{2}(m_t - X_t)^2}_{\text{Reputational cost}} \Big) dt \Big],$$

where

$$dX_t = \alpha_t dt + dW_t, \qquad 0 \le t \le T, \qquad X_0 = \xi, \quad a, b \ge 0, \varepsilon > 0.$$

- $\xi \sim \nu$, W is a 1-dim BM, both defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\xi \perp\!\!\!\perp W$,
- $\mathbb{F}^1$ is the (completed) filtration generated by (ξ, W),
- $\alpha \in \mathcal{A}$ the set of $\mathbb{F}^1$-prog measurable square-integrable processes with values in $\mathbb{R}$,
- $(m_t)_{0 \le t \le T}$ the flow of average state in the population $(= (\mathbb{E}[X_t])_{0 \le t \le T}$ at equilibrium).

Standard MFG solution

Definition (Nash MFG equilibrium)

A couple $(\hat{\alpha}, \hat{m})$ is a Nash MFG equilibrium for the abatement game if:

(i) Optimality: for every deviation $\alpha \in \mathcal{A}$ (adapted to $\mathbb{F}^1$),

$$J(\hat{\alpha}, \hat{m}) \geq J(\alpha, \hat{m}).$$

(ii) Consistency: for every time $t \in [0, T]$,

$$\hat{m}_t = \mathbb{E}[\hat{X}_t] \quad \text{with} \quad d\hat{X}_t = \hat{\alpha}_t dt + dW_t.$$

Proposition

In the abatement game, there exists a unique Nash MFG equilibrium, given as follows:

$$\hat{\alpha}_t = \phi_t(\hat{m}_t - \hat{X}_t), \quad \hat{m}_t = \nu_1, \qquad \dot{\phi}_t = \phi_t^2 - \varepsilon, \ \phi_T = 0.$$

Remarks:

- $\hat{m}$ is deterministic
- optimal strategy is feedback in X_t (adapted to $\mathbb{F}^1$)

Social optimum: Mean field control

Definition (Social optimum)

Social optimum is defined by the central's planner optimal strategy, obtained through mean field control: maximize over $\alpha \in \mathcal{A}$ the payoff functional

$$J^{MFC}(\alpha) = \mathbb{E}\left[\int_0^T \left(a\mathbb{E}[X_t] - \frac{b}{2}\mathbb{E}[X_t]^2 - \frac{1}{2}\alpha_t^2 - \frac{\varepsilon}{2}(\mathbb{E}[X_t] - X_t)^2\right) dt\right],$$

$$dX_t = \alpha_t dt + dW_t, \quad X_0 = \xi.$$

Proposition

There exists a unique maximizer for the MFC problem, which is given by

$$\hat{\alpha}_t^{MFC} = \phi_t(\mathbb{E}[X_t^{MFC}] - X_t^{MFC}) - \bar{\eta}_t \mathbb{E}[X_t^{MFC}] - \bar{\chi}_t,$$

where $\bar{\eta}$ and $\bar{\chi}$ have the following deterministic dynamics

$$\dot{\bar{\eta}}_t = \bar{\eta}_t^2 - b, \quad \bar{\eta}_T = 0, \qquad \dot{\bar{\chi}}_t = \bar{\eta}_t \bar{\chi}_t + a, \quad \bar{\chi}_T = 0.$$

CCE: introducing correlation

Definition (Correlated flow)

A correlated flow is a pair (λ, μ) satisfying the following properties:

(i) $\lambda = (\lambda_t)_{t\in[0,T]}$ is an $\mathbb{F}$-progressively measurable square integrable process taking real values. We call λ the *suggested strategy*.

(ii) $\mu = (\mu_t)_{t\in[0,T]}$ is an $\mathcal{F}_0$-measurable $\mathcal{C}([0,T];\mathbb{R})$-random variable;

(iii) μ is independent of both ξ and W,

Abuse of notation: μ_t is not a distribution! (Compare to the general case)

Objective of a player following the *suggested strategy*:

$$J(\lambda,\mu) = \mathbb{E}\Big[\int_0^T \Big(a\mu_t - \frac{b}{2}{\mu_t}^2 - \frac{1}{2}{\lambda_t}^2 - \frac{\varepsilon}{2}(\mu_t - X_t)^2\Big)\,dt\Big],$$

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$$dX_t = \lambda_t dt + dW_t, \quad X_0 = \xi.$$

The deviating player

Objective of the *deviating player*

$$J(\beta,\mu) = \mathbb{E}\Big[\int_0^T \Big(a\mu_t - \frac{b}{2}{\mu_t}^2 - \frac{1}{2}{\beta_t}^2 - \frac{\varepsilon}{2}(\mu_t - X_t)^2\Big)\,dt\Big],$$

$$dX_t = \beta_t dt + dW_t, \quad X_0 = \xi.$$

with $\beta \in \mathcal{A}$ (prog. measurable to $\mathbb{F}^1$, the filtration of (ξ, W)).

$\Rightarrow$ The deviating player only knows the law of (λ,μ) (the "mechanism device"), and not its actual realisation !

Optimal deviation: $\hat{\beta} \in \mathcal{A}$ which maximises $J(\beta,\mu)$.

CCE definition

Definition (Coarse correlated solution of the MFG)

A *correlated flow* (λ, μ) is a coarse correlated solution of the MFG if the following holds:

(i) Optimality: for every deviation $\beta \in \mathcal{A}$,

$$J(\lambda, \mu) \geq J(\beta, \mu).$$

(ii) Consistency: for every time $t \in [0, T]$,

$$\mu_t = \mathbb{E}[X_t | \mu] \quad \mathbb{P}\text{-a.s.}$$

Remarks:

- μ is stochastic
- The Nash MFG equilibrium is a CCE
- This def is taylor-made for the LQ case (compare the general one)

Procedure to identify CCEs analytically

1. Compute the best deviation $\hat{\beta}$ for a given μ
2. Define a suggested strategy λ for this given μ:
   a) with a shape similar to the best deviation $\hat{\beta}$ ...
   b) ... and which verifies the consistency condition
3. Check the optimality condition

$\Rightarrow$ Rationale: To focus on a set of correlated flows (λ, μ) whose objective is comparable analytically with the best deviation payoff.

Optimal deviation *(Step 1)*

Proposition (Optimal strategy for the deviating player)

The optimal strategy for the deviating player in the abatement game is given by

$$\hat{\beta}_t = \phi_t(\mathbb{E}[\mu_t] - \hat{X}_t) - \chi_t,$$

where χ *is the solution of the ODE*

$$\dot{\chi}_t = \phi_t\left(\chi_t + \mathbb{E}\left[\frac{d\mu}{dt}\right]\right), \quad \chi_T = 0.$$

In particular, the optimally controlled state of the deviating player is given by

$$d\hat{X}_t = (\phi_t(\mathbb{E}[\mu_t] - \hat{X}_t) - \chi_t)dt + dW_t, \quad X_0 = \xi.$$

Proof. First note that the objective can be written

$$J(\beta, \mu) = \mathbb{E}\Big[\int_0^T \mathbb{E}\left[a\mu_t - \frac{b}{2}\mu_t^2 - \frac{1}{2}\beta_t^2 - \frac{\varepsilon}{2}(\mu_t - X_t)^2 \mid \mathcal{F}_t^1\right] dt\Big]$$

Simplify this expression and apply Pontryagin maximum principle. □

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Simplify this expression and apply Pontryagin maximum principle. □

Characterizing a set of CCEs

(Step 2) Let's focus on a subset $\mathcal{G}$ of correlated flows where

$$\lambda_t = \phi_t(\mu_t - X_t) + \frac{d\mu}{dt}, \qquad dX_t = \lambda_t dt + dW_t.$$

Proposition (Optimality condition - *Step 3*)

The correlated flow $(\lambda, \mu) \in \mathcal{G}$ is a CCE for the abatement game if, and only if,

$$\int_0^T \mathbb{E}\left[\left(\frac{d\mu}{dt}\right)^2\right] dt \leq \int_0^T \Big(\mathbb{E}\left[(\phi_t f_t(\mu) + \chi_t)^2\right] + \phi_t^2 \mathbb{E}\left[(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))^2\right]$$
$$+ (\varepsilon - \phi_t^2)\mathbb{E}\left[f_t^2(\mu)\right] \Big) dt,$$

$$\dot{f}_t(\mu) = -\left(\phi_t(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu)) + \frac{d\mu}{dt} + \chi_t\right), \quad f_0(\mu) = 0$$

NB: Consistency condition (ii) is verified by all correlated flows in $\mathcal{G}$.

Remains to characterize optimality condition (i) $J(\hat{\beta}, \mu) \leq J(\lambda, \mu)$.

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$$+(\varepsilon - \phi_t^2)\mathbb{E}\left[f_t^2(\mu)\right]\Big) dt,$$

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Comparison with Nash and MFC eq

Proposition (When no global "abatement benefit")

If $a = b = 0$,

a) *The Nash MFG equilibrium* $(\hat{\alpha}, m)$ *equals the MFC optimum* $(\alpha^{MFC}, \bar{x})$.

b) *For all correlated flows* (λ, μ), $J(\lambda, \mu) \leq J(\hat{\alpha}, \hat{m})$.

Proposition (With global "abatement benefit")

If $a > 0$ *or* $b > 0$,

c) $J(\hat{\alpha}, \hat{m}) < J(\alpha^{MFC}, \bar{x})$.

d) (i) *There exists a CCE* (λ, μ) *with* $J(\lambda, \mu) > J(\hat{\alpha}, \hat{m})$.

(ii) *For all CCEs* (λ, μ), $J(\lambda, \mu) < J(\alpha^{MFC}, \bar{x})$. *(in progress ...)*

An example: linear in time flow of moments

Class of *linear in time* correlated flows (CFs) $\mathcal{G}_L \subset \mathcal{G}$:

$$\mathcal{G}_L := \{(\lambda, \mu) \in \mathcal{G} : \mu_t = \mathbb{E}[\xi] + tZ, \; Z \in \mathcal{Z}\},$$

$$\mathcal{Z} := \{Z \in L^2(\mathcal{F}_0) : Z \perp\!\!\!\perp (\xi, W)\}.$$

Proposition (Optimality condition)

$(\lambda, \mu) \in \mathcal{G}_L$ built out of $Z \in \mathcal{Z}$ is a CCE iff

$$c_M \mathbb{E}[Z]^2 + c_V \mathbb{V}[Z] \geq 0$$

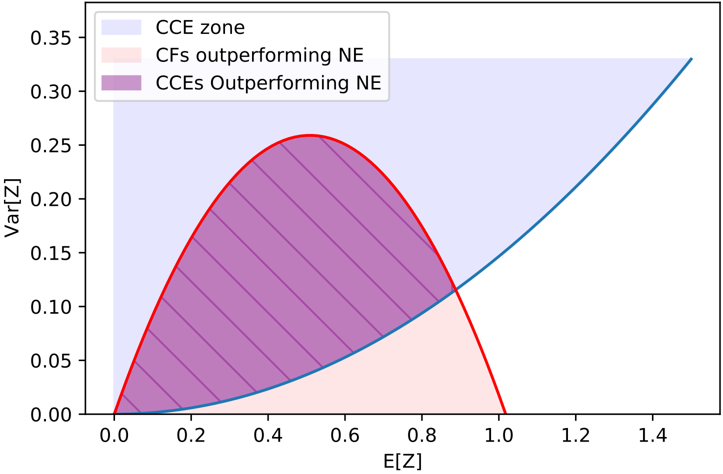
with c_M, c_V coefficients depending on T and ε.

Proposition (Comparison with NE payoff)

For $(\lambda, \mu) \in \mathcal{G}_L$ built out of $Z \in \mathcal{Z}$,

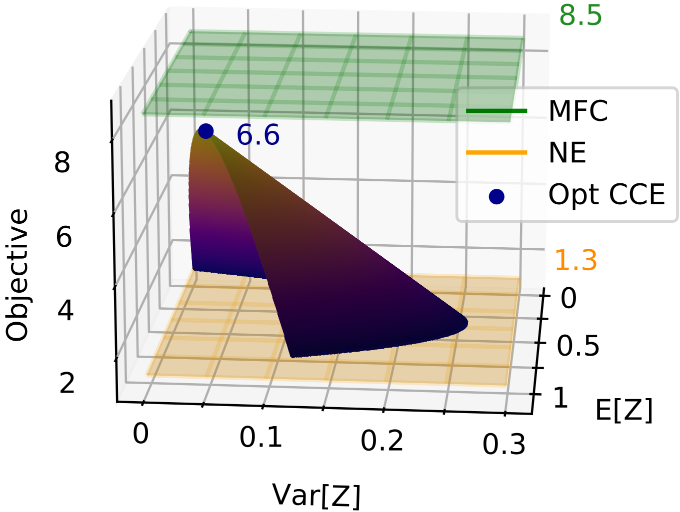
$$J(\lambda, \mu) = J(\hat{\alpha}, \hat{m}) + \frac{T^2}{2}(a - b\nu_1)\,\mathbb{E}[Z] - \left(b\frac{T^3}{3} + T\right)\frac{\mathbb{E}[Z^2]}{2}.$$

Zone of CCEs outperforming NE in the plane $(\mathbb{E}[Z], \mathbb{V}[Z])$



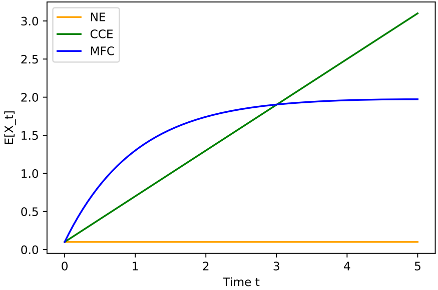
$$T = 5, a = 2, \ b = 1, \ \varepsilon = 1$$

The CCE payoff where it outperforms the NE

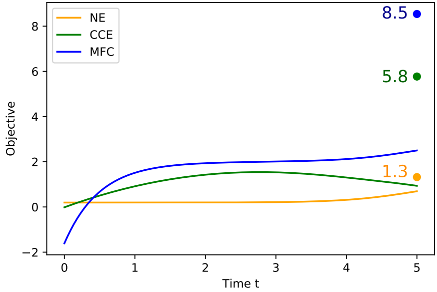


CCEs bridging the gap between NE and MFC

Average abatement effort ($\mathbb{E}[X_t]$)



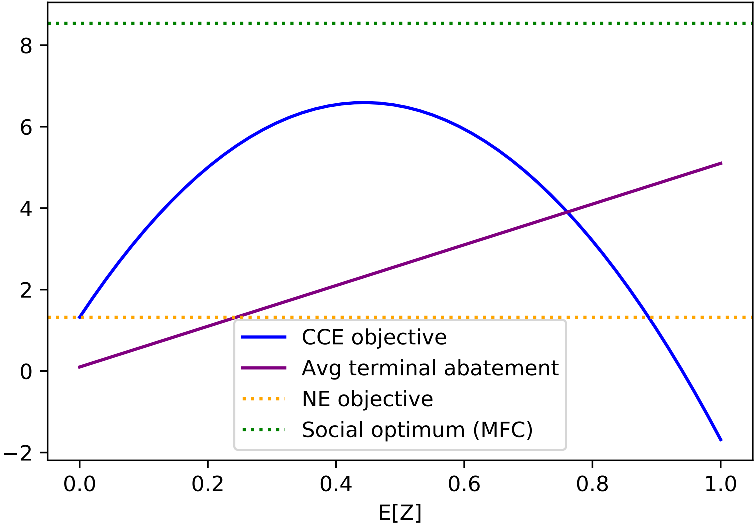
Running and total objective



$$\mathbb{E}[Z] = 0.6, \ \mathbb{V}[Z] = 0.06$$

NE: free-riding equilibrium

Trade-off between social utility and abatement goal



Conclusion

⇒ CCEs are non-cooperative equilibria which can help bridge the gap between NE and social optimum

⇒ Particularly relevant in context of common goods which encourage free-riding such as climate change

More general results in multi-dimensional linear quadratic MFGs:

- We have characterized the CCEs for a given class of correlated flows (optimality condition)
- We have obtained a condition for outperformance on NE

To-do list:

- Connection to N-player game
- Consider other types of CFs
- Select an optimal (λ, μ) in some sense

Bibliography I

Carmona, R. and Delarue, F.
Probabilistic Theory of Mean Field Games with Applications
Probability Theory and Stochastic Modelling, 83, Springer, 2018.

Roughgarden, T.
Twenty Lectures on Algorithmic Game Theory
Cambridge University Press, 2016.

Aumann, R.J.
Subjectivity and correlation in randomized strategies
Journal of Mathematical Economics, 1, pp. 67–96, 1974.

Bardi, M., and Fischer, M.
On non-uniqueness and uniqueness of solutions in finite-horizon mean field games
ESAIM: Control, Optimisation and Calculus of Variations 25 (2019): 44.

Barrett, S.
Self-enforcing international environmental agreements
Oxford Economic Papers, 46 (1994): 878-894.

Bonesini, O., Campi, L. and Fischer, M.
Correlated equilibria in mean field games with progressive strategies
arXiv:2212.01656, 2022.

Campi, L. and Fischer, M.
Correlated equilibria and mean field games: a simple model
Mathematics of Operations Research, 47(3), 2022.

Bibliography II

Campi, L., Cannerozzi, F. and Fischer, M.
Coarse correlated equilibria for continuous time mean field games in open loop strategies
arXiv:2303.16728, 2023

Dokka, T., Moulin, H., Ray, I., SenGupta, S.
Equilibrium design in an n-player quadratic game
Rev Econ Design (2022). https://doi.org/10.1007/s10058-022-00299-2

Hannan, J.
Approximation to Bayes risk in repeated play
In Contributions to the theory of games, vol. 3, Annals of Mathematics Studies, no. 39, 97-139. Princeton University Press, Princeton, 1957.

Hart, S. and Schmeidler, D.
Existence of correlated equilibria
Mathematics of Operations Research, 14(1), pp. 18-25, 1989.

Hart, S. and Mas-Colell, A.
A General Class of Adaptive Strategies
Journal of Economic Theory 98, 26-54, 2001.

Lacker, D.
A general characterization of the mean field limit for stochastic differential games
Probability Theory and Related Fields 165 (2016): 581-648.

Lasry, J.-M. and Lions, P.-L.
Mean field games,
Jpn. J. Math., 2(1), 229-260, 2007.

Bibliography III

Huang, M., Malhamé, R.P., and Caines P.E.
Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle
Commun. Inf. Syst., 6(3), 221-251, 2006.

Moulin, H., Ray, I. and Sen Gupta, S.
Improving Nash by coarse correlation
Journal of Economic Theory 150, 852-865, 2014.

Moulin, H. and Vial, J.-P.
Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon
Internat. J. Game Theory, 7(3-4), 201-221, 1978.

Muller, P., Rowland, M., Elie, R., Piliouras, G., Perolat, J., Lauriere, M., Marinier, R., Pietquin, O., Tuyls, K.
Learning Equilibria in Mean-Field Games: Introducing Mean-Field PSRO
arXiv:2111.08350v1, 2021.

Muller, P., Elie, R., Rowland, Lauriere, M., M., Perolat, J., Perrin, S., Geist, M., Piliouras, G., Pietquin, O., Tuyls, K.
Learning Correlated Equilibria in Mean-Field Games
arXiv:2208.10138, 2022.