Disentangling endogenous and exogenous correlation effects via high frequency information

Marc Hoffmann

Université Paris-Dauphine, PSL

IHP, séminaire FIME, January 2024

- In progress !
- Based on a joint project with E. Bacry, T. Deschatre, J.F. Muzy and R. Ruan.

Setting

- We observe two times series :

$$\xi_1^n, \ldots, \xi_n^n, \quad \text{and} \quad \zeta_1^n, \ldots, \zeta_n^n, \quad n \text{ large}$$

- Continuous time model embedding :

$$\xi_i^n = X_{iD}^1, \ldots, \zeta_i^n = X_{iD}^2, \quad i = 1, \ldots, n, \; T = nD,$$

$X = (X_t^1, X_t^2)_{T \in [0,T]}$ continuous Itô semimartingale :

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s,$$

- $b_t \in \mathbb{R}^2, \sigma_t = (\sigma_t^{kl})_{1 \leq k,l \leq 2} \in \mathbb{R}^{\otimes 2}$, càdlàg adapted, $B_t \in \mathbb{R}^2$ Brownian motion.

Setting

- This is a macroscopic time setting $[0,T] \rightsquigarrow [0,1]$, $n = T/D \to \infty$.
- Correlation estimator based on quadratic variation

$$\rho_n = \frac{\langle X^1, X^2\rangle_n}{\langle X^1, X^1\rangle_n^{1/2}\langle X^2, X^2\rangle_n^{1/2}},$$

where $\langle X^k, X^l\rangle_n = \sum_{i=1}^n (X^k_{iD} - X^k_{(i-1)D})(X^l_{iD} - X^l_{(i-1)D})$.
- Classical semimartingale theory

$$\rho_n \xrightarrow{\mathbb{P}} \frac{\int_0^1 (\sigma_s^{11} + \sigma_s^{22})\sigma_s^{12} ds}{\big(\int_0^1 \big((\sigma_s^{11})^2 + (\sigma_s^{12})^2\big) ds\big)^{1/2}\big(\int_0^1 \big((\sigma_s^{22})^2 + (\sigma_s^{12})^2\big) ds\big)^{1/2}}$$

with rate $n^{-1/2}$ and an associated CLT.

(Ignoring the drift.)

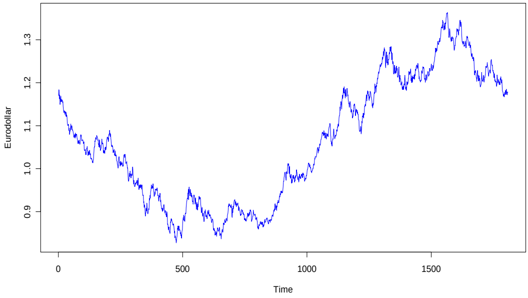

Figure – EuroUSD FX, $\Delta = 1$ day (traded price), from 01 Jan. 1999 to 06 Dec. 2005.

(Ignoring the drift.)

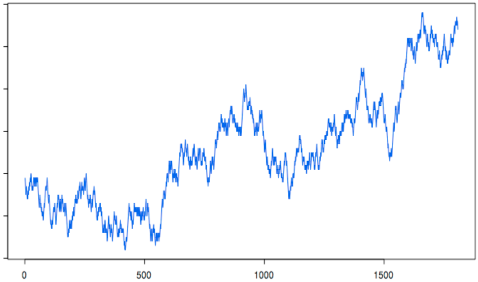

Figure – Sample path of a Bernoulli random walk “EuroUSD", $\Delta = 1$ day, from 01 Jan. 1999 to 06 Dec. 2005.

(Ignoring the drift.)

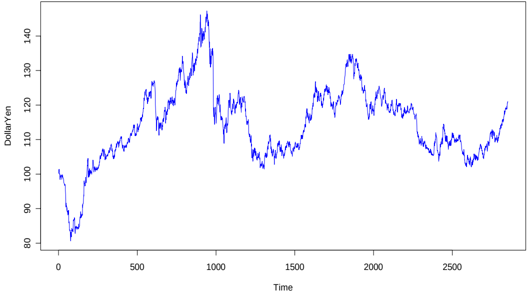

Figure – FX USD–Yen, $\Delta = 1$ day (traded price), from 02 Jan. 1995 to 06 Dec. 2005.

(Ignoring the drift.)

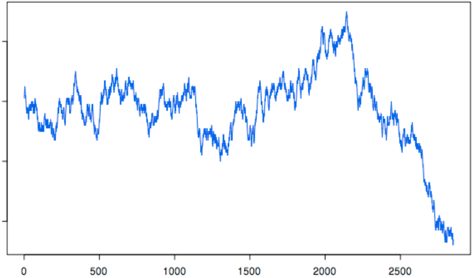

Figure – Sample path of a Bernoulli random walk "USD–Yen", $\Delta = 1$ day, 02 Jan. 1995 to 06 Dec. 2005.

(Ignoring the drift.)

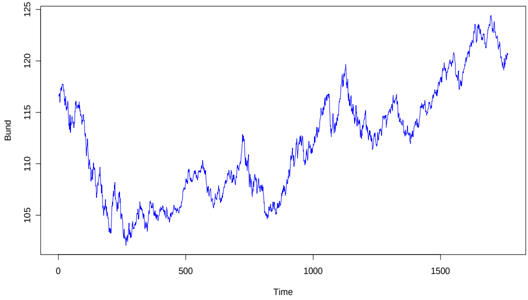

Figure – 10Y German Bund (FGBL) with $\Delta = 1$ day (traded price), from 04 Apr. 1999 to 06 Dec. 2005.

(Ignoring the drift.)

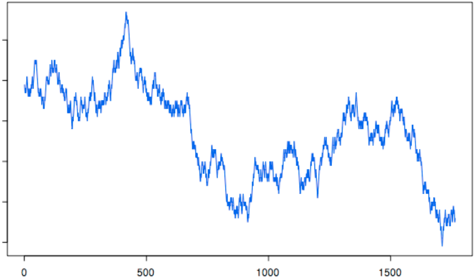

Figure – Sample path of a Bernoulli random walk "FGBL", $\Delta = 1$ day, 04 Avr. 1999 to 06 Dec. 2005.

Objective

- The simplest model : constant diffusion matrix.
- Reparametrisation (diffusion matrix in $\mathbb{R}^{1\times 1}$ versus $\mathbb{R}^{\otimes 2}$.)

$$\begin{cases} X_t^1 & = \sigma_1 B_t^1 \\ X_t^2 & = \sigma_2(\rho B_t^1 + \sqrt{1-\rho^2} B_t^2) \end{cases}$$

$\sigma_i > 0, \rho \in [-1, 1]$ so that

$$\boxed{\rho_n \xrightarrow{\mathbb{P}} \rho, \quad n \to \infty.}$$

- The quantity ρ accounts for both endogenous and exogenous effects.
- How to disentangle them ? Does it even make any sense ?

Objective

- Without extraneous information, hopeless purpose !
- We look for information in higher frequencies of the signal $\rightsquigarrow$ modification of the modelling.
- Limitation : microstructure noise (variance effect) and Epps effect (covariance effect).

Observation on a macroscopic scale

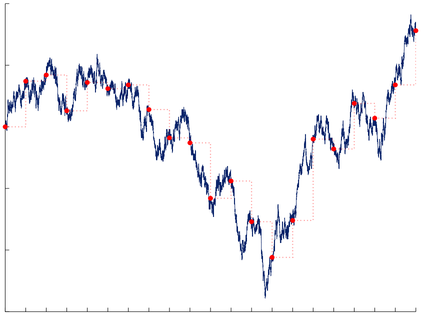

Figure – $\Delta_T \to \infty$ and $\Delta_T / T \to 0$ as $T \to \infty$: the diffusion aproximation becomes valid.

Coarse-to-fine modelling

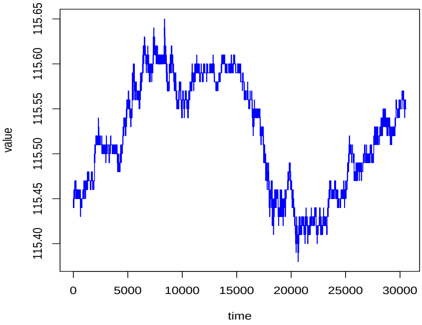

Figure – FGBL, 06 Feb 2007, 08 :30-17 :00 (UTC) sampled with $D = 1$ second. The candidate for the underlying process X is rather a marked point process that we observe at times iD.

Coarse-to-fine modelling (cont.)

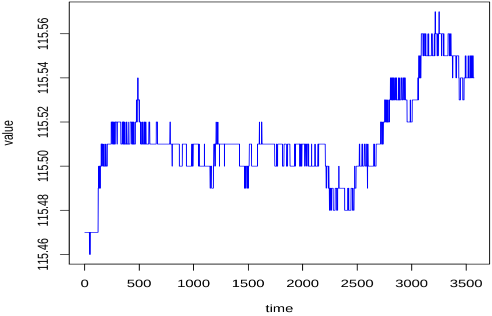

Figure – FGBL, 06 Feb 2007, 09 :00–10 :00 (UTC) 1 data every second.
The underlying process looks more complex than a simple CTRW.

Observation on a intermediate scale

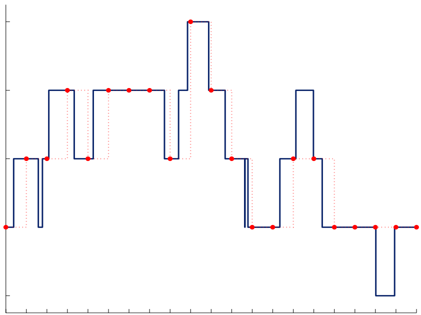

Figure – Statistically the hardest case $\Delta_T \approx 1$.

Observation on a microscopic scale

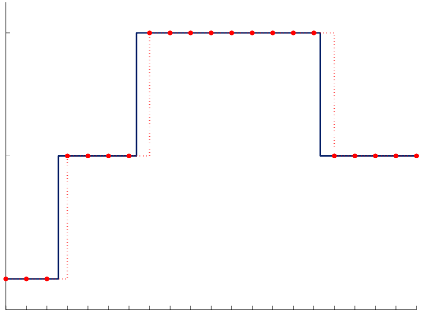

Figure – $\Delta_T \to 0$ as $T \to \infty$: one can “locate" the jump times.

Basic setting : price model in dimension 1

- Ideal HF model : marked point processes with jumps of size 1 on a lattice (tick-by-tick prices).
-
$$X_t = N_t^+ - N_t^-, t \in [0, T].$$
- $(N_t^\pm)$: dependent counting processes that reproduce the variance effect of microstructure noise.
- Simplest construction : 2-dimensional Hawkes process.
- Incorporates microstructure noise effects.

Signature plot

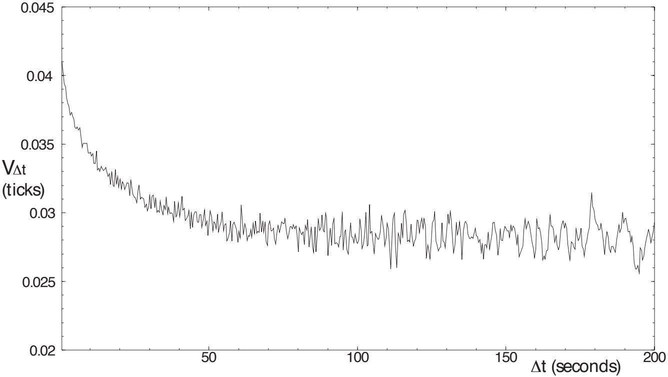

Figure – $D \mapsto \langle X, X \rangle_{n,D}$ for FGBL (43 days, 9-11 AM) on Last Traded Ask.

Short recap on Hawkes processes

- $(\pi^k(ds, dz))_{1 \le k \le d}$ IID Poisson random measures with intensity $dsdz$ on $[0, \infty)^2$.
- $N = (N_t^k)_{1 \le k \le d}$ Hawkes process with baseline $\mu \ge 0$ and kernel $\varphi = (\varphi^{lk})_{1 \le l,k \le d}$ if

$$\boxed{N_t^k = \int_0^t \int_0^\infty \mathbf{1}_{\left\{z \le \mu + \int_0^{t-} \sum_{l=1}^d \varphi^{lk}(s-u) dN_u^l\right\}} \pi^k(ds, dz), \quad 1 \le k \le d.}$$

- φ^{lk} locally integrable yields existence + uniqueness of N such that $\mathbb{E}[N_t^k] < \infty$ for every $t \ge 0$. (Picard + Grönwall.)

Simplest price model in dimension 1

- $N^k_t - \int_0^t \lambda_s ds$ is a martingale, with

$$\lambda^k_t = \mu + \int_0^{t-} \sum_{l=1}^{d} \varphi^{lk}(t-s) dN^l_s.$$

- $X_t = N^+_t - N^-_t$ is characterised by $(\lambda^+_t, \lambda^-_t)$. We pick

$$\begin{cases} \lambda^+_t & = \mu + \int_0^{t-} \varphi(t-s) dN^-_s \\ \lambda^-_t & = \mu + \int_0^{t-} \varphi(t-s) dN^+_s. \end{cases}$$

- The model is parametrised by (μ, φ).
- This is a microscopic model, in continuous time over $[0, T]$ with large T. Macroscopic renormalisation :

$$X^{(T)}_t = T^{-1/2}(N^+_{tT} - N^-_{tT}) \quad t \in [0, 1]$$

Microscopic and macroscopic fluctuations

- If $\|\varphi\|_{L^1} < 1$, we have

$$T^{-1}(N_T^+ + N_T^-) = \frac{2\mu}{1 - \|\varphi\|_{L^1}}(1 + O_{\mathbb{P}}(T^{-1/2}))$$

and

$$(X_t^{(T)})_{0\leq t\leq 1} \xrightarrow{(d)} (\sigma W_t)_{1\leq t\leq 1},$$

with

$$\boxed{\sigma^2 = \frac{2\mu}{(1 - \|\varphi\|_{L^1})(1 + \|\varphi\|_{L^1})^2}}.$$

- Empirical trace of microstructure :

$$\langle X_t^{(T)}, X_t^{(T)} \rangle_n = \frac{2\mu}{(1 - \|\varphi\|_{L^1})(1 + \|\varphi\|_{L^1})^2}(1 + O_{\mathbb{P}}(n^{-1/2}))$$

in the macroscopic limit $T \to \infty$.

In dimension 2 : Epps effect

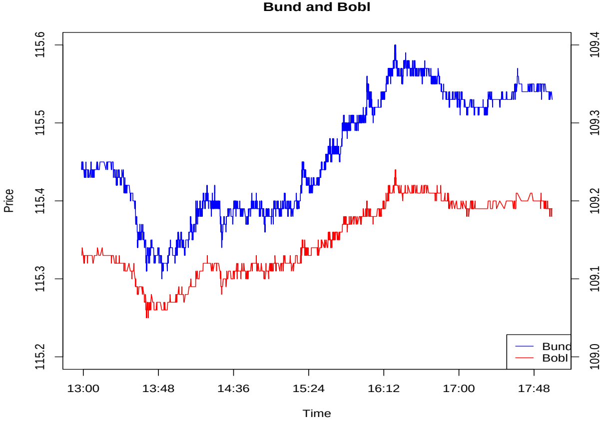

Figure – FGBL/FGBM

Epps effect

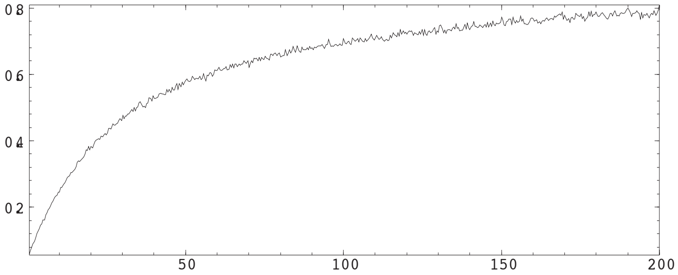

Figure – $D \mapsto \langle X^1, X^2 \rangle_{n,D}$ (normalised) with $X^1 = FGBL$, $X^2 = FGBM$, 40 days, 9-11AM.

Multivariate model

- How to build a multivariate model that reproduces the Epps effect ?
- Pick

$$(X_t^1, X_t^2) = (N_t^{1,+} - N_t^{1,-}, N_t^{2,+} - N_t^{2,-}),$$

where $N = (N_t^{l,\pm})_{1 \le l \le 2}$ is a 4-dimensional Hawkes process, with baseline $(\mu^1, \mu^1, \mu^2, \mu^2)$ and kernel

$$\begin{pmatrix} 0 & \varphi & \psi & 0 \\ \varphi & 0 & 0 & \psi \\ \psi & 0 & 0 & \varphi \\ 0 & \psi & \varphi & 0 \end{pmatrix}$$

Multivariate model

- How to build a multivariate model that reproduces the Epps effect ?
- Pick

$$(X_t^1, X_t^2) = (N_t^{1,+} - N_t^{1,-}, N_t^{2,+} - N_t^{2,-}),$$

where $N = (N_t^{l,\pm})_{1\leq l\leq 2}$ is a 4-dimensional Hawkes process, with baseline $(\mu^1, \mu^1, \mu^2, \mu^2)$ and kernel

$$\begin{pmatrix} \lambda_{N^{1,+}} \\ \lambda_{N^{1,-}} \\ \lambda_{N^{2,+}} \\ \lambda_{N^{2,-}} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \varphi_{-\rightarrow+} & \psi_{+\rightarrow+} & 0 \\ \varphi_{+\rightarrow-} & 0 & 0 & \psi_{-\rightarrow-} \\ \psi_{+\rightarrow+} & 0 & 0 & \varphi_{-\rightarrow+} \\ 0 & \psi_{-\rightarrow-} & \varphi_{+\rightarrow-} & 0 \end{pmatrix} \begin{pmatrix} dN^{1,+} \\ dN^{1,-} \\ dN^{2,+} \\ dN^{2,-} \end{pmatrix}$$

- Beware overfitting !

Abstract asymptotic theory

- $K = (\|\varphi^{kl}\|_{L^1})_{1\le k,l\le d}$ with the assumption : $\rho(K) \overset{!}{<} 1$.
- LLN :

$$\sup_{t\in[0,1]} |T^{-1}N_{tT} - t(\mathrm{Id} - K)^{-1}\mu| \to 0, \quad T \to \infty.$$

- $\Sigma = \mathrm{diag}((\mathrm{Id} - K)^{-1})$. Fluctuations :

$$\Big(T^{1/2}(T^{-1}N_{tT} - t(\mathrm{Id} - K)^{-1}\mu)\Big)_{0\le t\le 1} \xrightarrow{(d)} \Big((\mathrm{Id} - K)^{-1}\Sigma^{1/2}W\Big)_{0\le t\le 1}, \quad T \to \infty,$$

  if moreover $\int_0^\infty t^{1/2}\varphi^{kl}(t)dt < \infty$.
- $\rho(K) \approx 1$: criticality, gateway to stochastic volatility, rough volatility and so on.

From endogenous to exogenous effects

- First ignore microstructure noise and endogenous effects.
- How to build the simplest microscopic model with exogenous effects ?
- Even simpler ! Let us model the microscopic volatility process. If $K = 0$ then

$$V_t^1 = N_t^{1,+} + N_t^{1,-} \sim \mathrm{PP}(2\mu), \quad V_t^2 = N_t^{2,+} + N_t^{2,-} \sim \mathrm{PP}(2\nu),$$

yet both are independent !
- How to couple Poisson-like processes in a smart way ?

Microscopic exogeneous effects : first attempt

- M_t^i : baseline independent Poisson processes.
- Try the common shock model :

$$N_t^1 = M_t^1 + M_t^3, \quad N_t^2 = M_t^2 + M_t^3.$$

- Degenerate situation : common jumps.
- Illuminating idea by Thomas : delayed Poisson (Cox and Lewis 05). Replace in one of the components

$$M_t^3 = \sum_{k \geq 1} \mathbf{1}_{\{T_k^3 \leq t\}}$$

by

$$\boxed{\widetilde{M}_t^3 = \sum_{k \geq 1} \mathbf{1}_{\{T_k^3 + \varepsilon_k \leq t\}}},$$

where ε_k are IID continuous nonnegative delaying random variables.

Delayed Poisson processes

- M^3 Poisson process with intensity μ_3.
- $\widetilde{M}^{3,k}$, $k = 1, 2$ two exponentially delayed versions of M^3 with parameter $a > 0$.
- In their own filtrations, $\widetilde{M}^{3,k}$, $k = 1, 2$ have intensity

$$\mu_3(1 - \exp(-at))$$

  and are asymptotically Poisson.
- By playing on the (asymptotically negligible) parameter a, we have no common jump but a strong dependence between

$$N_t^1 = M_t^1 + \widetilde{M}_t^{3,1}, \ \ N_t^2 = M_t^2 + \widetilde{M}_t^{3,2}.$$

- Yet, we have a surprisingly simple structure !

Delayed Poisson processes

- $(M_t^3, \widetilde{M}_t^{3,1}, \widetilde{M}_t^{3,2})$ is a point process with no common jumps and intensity

$$\begin{cases} \lambda_t^3 & = \mu_3 \\ \widetilde{\lambda}_t^{3,1} & = a(M_{t-}^3 - \widetilde{M}_{t-}^{3,1}) \\ \widetilde{\lambda}_t^{3,2} & = a(M_{t-}^3 - \widetilde{M}_{t-}^{3,2}) \end{cases}$$

- It is a "Hawkes" process with baseline $(\mu_3, 0, 0)$ and kernel

$$a \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

- Asymptotic theory (almost) for free (L^1_{loc} kernel, negative entries).
- Glue this building block in our general framework !

An Epps effect friendly common shock model

- We slightly modify the common shock model by setting

$$N_t^1 = M_t^1 + \widetilde{M}_t^{3,1}, \quad N_t^2 = M_t^2 + \widetilde{M}_t^{3,2}$$

- Supported by 5 independent random Poisson measures.
- Covariance across scales : $T = nD$, time mesh $D > 0$ (the scale) and $\bar{N}_t = \mathbb{E}[N_t]$,

$$\langle N \rangle_{D,T} = T^{-1} \sum_{i=1}^{n} \left(\bar{N}_{iD} - \bar{N}_{(i-1)D} \right) \left(N_{iD} - \bar{N}_{(i-1)D} \right)^{\top}.$$

- For $D = D_T$, if $D_T / T \to 0$ as $T \to \infty$, we have

$$\boxed{\langle \widetilde{M}^{3,1}, \widetilde{M}^{3,2} \rangle_{D_T,T} \xrightarrow{\mathbb{P}} \mu_3 \begin{pmatrix} 1 & 1 - \frac{1-\mathrm{e}^{-aD_T}}{aD_T} \\ 1 - \frac{1-\mathrm{e}^{-aD_T}}{aD_T} & 1 \end{pmatrix}}$$

as $T \to \infty$.

An Epps effect friendly common shock model

- Correlation across scales for $(\widetilde{M}^{3,1}, \widetilde{M}^{3,2})$:

$$1 - \frac{1 - \mathrm{e}^{-aD_T}}{aD_T}$$

- Correlation across scales for (N^1, N^2) :

$$\boxed{\frac{\mu_3}{\sqrt{(\mu_1 + \mu_2)(\mu_1 + \mu_3)}} \left(1 - \frac{1 - \mathrm{e}^{-aD_T}}{aD_T}\right)}$$

- We obtain an Epps effect for Poisson-like processes obtained by delaying a common shock model.
- The may serve as a proxy volatility process.

A general volatility framework

- For simplicity, we first model the microscopic volatility processes as

$$\boxed{(V_t^1, V_t^2) = (N_t^{1,+} + N_t^{1,-}, N_t^{2,+} + N_t^{2,-})}.$$

- Step 1 : delayed shocks (produces exogenous correlation) + microstructure noise on margins.
- Step 2 : combine delayed shocks, margin microstructure noise and endogenous correlation.
- Step 3 : from volatility to prices. Construct

$$\boxed{(X_t^1, X_t^2) = (N_t^{1,+} - N_t^{1,-}, N_t^{2,+} - N_t^{2,-})}$$

in the simplest extended framework.

A general volatility framework

- Step 1 : delayed shocks (produces exogenous correlation) + microstructure noise (on margins).
- We need : a point process supported by 5 random Poisson measures

$$(N^1_t, N^2_t, (N^3_t, \widetilde{N}^{3,1}_t, \widetilde{N}^{3,2}_t))$$

defined via its intensity process

$$\begin{cases} \lambda^1_t & = \mu_1 + \int_0^{t-} \varphi_1(t-s) d(N^1_s + \widetilde{N}^{3,1}_s) \\ \lambda^2_t & = \mu_2 + \int_0^{t-} \varphi_2(t-s) d(N^2_s + \widetilde{N}^{3,2}_s) \\ \lambda^3_t & = \mu_3 \\ \widetilde{\lambda}^{3,1}_t & = a(N^3_{t-} - \widetilde{N}^{3,1}_{t-}) \\ \widetilde{\lambda}^{3,2}_t & = a(N^3_{t-} - \widetilde{N}^{3,2}_{t-}). \end{cases}$$

A general volatility framework

- It is a 5-dimensional "Hawkes process" with baseline $(\mu_1, \mu_2, \mu_3, 0, 0)$ and kernel

$$\begin{pmatrix} \Phi & (0_{1\times 2}, \Phi) \\ 0_{3\times 2} & aP \end{pmatrix},$$

with

$$\Phi = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

- Final volatility model :

$$\boxed{(V_t^1, V_t^2) = (N_t^1 + \widetilde{N}_t^{3,1}, N_t^2 + \widetilde{N}_t^{3,2})}.$$

A general volatility framework

- Microscopic variance-covariance of (V_t^1, V_t^2) :

$$\begin{pmatrix} \frac{\mu_1+\mu_3}{1-\|\varphi_1\|_{L^2}} & 0 \\ 0 & \frac{\mu_1+\mu_3}{1-\|\varphi_1\|_{L^2}} \end{pmatrix}$$

- Macroscopic variance-covariance :

$$\begin{pmatrix} (\mu_1+\mu_3)(1-\|\varphi_1\|_{L^2}) & \frac{\mu_3}{(1-\|\varphi_1\|_{L^1})(1-\|\varphi_2\|_{L^1})} \\ \frac{\mu_3}{(1-\|\varphi_1\|_{L^1})(1-\|\varphi_2\|_{L^1})} & (\mu_1+\mu_3)(1-\|\varphi_1\|_{L^2}) \end{pmatrix}$$

A general volatility framework

- Macroscopic correlation :

$$\sqrt{(1-\|\varphi_1\|_{L^1})(1-\|\varphi_2\|_{L^1})}\frac{\mu_3}{\sqrt{(\mu_1+\mu_3)(\mu_2+\mu_3)}}.$$

- It is the correlation of the Epps-friendly common shock model :

$$\times\sqrt{(1-\|\varphi_1\|_{L^1})(1-\|\varphi_2\|_{L^1})}.$$

- Proportion of endogenous migrants in each component : $(1-\|\varphi_i\|_{L^1})$, $i=1,2$.
- Explicit formula across scales for $\varphi_i(t)=\alpha_i\exp(-\beta_i t)$, $\alpha_i<\beta_i$.

Mixing endogenous and exogenous effects

- So far, we only incorporate exogenous effects in the dependence between V^1 and V^2.
- Step 2 : endogenous dependence $\rightsquigarrow$ classical cross kernels.
- 5-dimensional point process

$$(N_t^1, N_t^2, (N_t^3, \widetilde{N}_t^{3,1}, \widetilde{N}_t^{3,2}))$$

defined via

$$\begin{cases} \lambda_t^1 = \mu_1 + \int_0^{t-} \varphi_1(t-s)d(N_s^1 + \widetilde{N}_s^{3,1}) + \int_0^{t-} \psi_1(t-s)d(N_s^2 + \widetilde{N}_s^{3,2}) \\ \lambda_t^2 = \mu_2 + \int_0^{t-} \varphi_2(t-s)d(N_s^2 + \widetilde{N}_s^{3,2}) + \int_0^{t-} \psi_2(t-s)d(N_s^1 + \widetilde{N}_s^{3,1}) \\ \lambda_t^3 = \mu_3 \\ \widetilde{\lambda}_t^{3,1} = a(N_{t-}^3 - \widetilde{N}_{t-}^{3,1}) \\ \widetilde{\lambda}_t^{3,2} = a(N_{t-}^3 - \widetilde{N}_{t-}^{3,2}). \end{cases}$$

Mixing endogenous and exogenous effects

- It is a 5-dimensional "Hawkes process" with baseline $(\mu_1, \mu_2, \mu_3, 0, 0)$ and kernel

$$\begin{pmatrix} \Phi & (0_{1\times 2}, \Phi) \\ 0_{3\times 2} & aP \end{pmatrix},$$

with

$$\Phi = \begin{pmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

- Final volatility model : $(V_t^1, V_t^2) = (N_t^1 + \widetilde{N}_t^{3,1}, N_t^2 + \widetilde{N}_t^{3,2})$.

Asymptotic theory

- $V_t = (V_t^1, V_t^2)$ and $\Lambda = (\mathrm{Id} - \|\Phi\|_{L^1})^{-1} \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_2 + \mu_3 \end{pmatrix}$.
- LLN :

$$\sup_{t\in[0,1]} |T^{-1} V_{tT} - t\Lambda| \to 0, \quad T \to \infty.$$

- Fluctuations :

$$\Big(T^{1/2}(T^{-1}V_{tT} - t\Lambda\Big)_{0\le t\le 1}$$
$$\xrightarrow{(d)} \Big((\mathrm{Id} - \|\Phi\|_{L^1})^{-1}\mathrm{Diag}\Big(\Lambda - \begin{pmatrix} \mu_3 \\ \mu_3 \end{pmatrix}\Big)\Big)^{1/2}\begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$$
$$+ \mu_3^{1/2}(\mathrm{Id} - \|\Phi\|_{L^1})^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} W^3\Big)_{0\le t\le 1}, \quad T \to \infty,$$

with $(W^i)_{1\le i\le 3}$ standard BM.

Recovering the parameters

- The macroscopic covariance is

$$\boxed{(\mathrm{Id} - \|\Phi\|_{L^1})^{-1}\left(\mathrm{diag}(\Lambda) + \begin{pmatrix} 0 & \mu_3 \\ \mu_3 & 0 \end{pmatrix}\right)(\mathrm{Id} - \|\Phi^\top\|_{L^1})^{-1}}.$$

- Too many parameters (latent components) to disentangle endogenous from exogenous effects by first and second order statistics only.
- Alternative : third order statistics, via nonlinear correlations.

First, second and third order statistics

- $N = (N_t^k)_{0 \le t \le T}$ well-defined d-dimensional Hawkes process.
- First order statistics :
$$\frac{N_T}{T}.$$
- Second order statistics, $n = T/D$:
$$\langle N \rangle_{D,T} = T^{-1} \sum_{i=1}^{n} \left(\bar{N}_{iD} - \bar{N}_{(i-1)D} \right) \left(N_{iD} - \bar{N}_{(i-1)D} \right)^{\top},$$
with $\bar{N}_t = N_t - \mathbb{E}[N_t]$.
- We have a relatively complete picture (LLN and fluctuations). Moving beyond ?

Third order statistics

For $1 \leq j, k, l \leq 2$.

$$\mathcal{M}^{jkl}_{n,D} = T^{-1} \sum_{i=1}^{n} (\bar{N}^j_{iD} - \bar{N}^j_{(i-1)D})(N^k_{iD} - \bar{N}^k_{(i-1)D})(\bar{N}^l_{iD} - \bar{N}^l_{(i-1)D}).$$

- Empirical skewness as a 3-tensor.
- Some history for Hawkes processes cumulants : Jovanovic (2015), Achab *et al.* (2018) for numerical implementation (via GMM-like estimation).
- We have a limit theory (at least for the LNN).

Third order statistics, limit theory

- N with intensity $\mu + \int_0^t \varphi(t-s)dN_s$ in dimension d.
- Limiting objects :

$$R = (\mathrm{Id} - \|\varphi\|_{L^1})^{-1},\ \Lambda = R\mu,\ C = R\,\mathrm{diag}(\Lambda)\,R^\top.$$

- If $D = D_T \to \infty$ is such that $D_T^2/T \to 0$ as $T \to \infty$, then

$$\mathcal{M}_{n,D}^{jkl} \xrightarrow{L^2(\mathbb{P})} \sum_{m=1}^{d} \left(R^{lm}R^{jm}C^{km} + R^{lm}C^{jm}R^{km} + C^{lm}R^{jm}R^{km}\right) - 2\sum_{m=1}^{d} \Lambda^m R^{lm}R^{jm}R^{km}.$$

- Together with first and second order statistics, gateway to GMM methods for recovering μ and φ even if some components are latent.

Final price model

- We set
$$\boxed{(X^1_t, X^2_t) = (N^{1,+}_t - N^{1,-}_t, N^{2,+}_t - N^{2,-}_t)}.$$
- The construction of $(N^{1,+}_t, N^{1,-}_t, N^{2,+}_t, N^{2,-}_t)$ requires 10 $(= 4 + 3 \times 2)$ Poisson random measures via some latent processes for the exogenous part.
- With $\pm$ for upward+downward jumps, the final price process
$$\Big(N^{1,\pm}_t, N^{2,\pm}_t, (N^{3,\pm}_t, \widetilde{N}^{3,1,\pm}_t, \widetilde{N}^{3,2,\pm}_t)\Big),$$
is defined via 10 Poisson random measures.

Final price model

- Re-write $\left((N_t^{1,\pm}, N_t^{2,\pm}), (N_t^{3,\pm}, \widetilde{N}_t^{3,1,\pm}, \widetilde{N}_t^{3,2,\pm})\right)$ as

$$\boxed{\left((N_t^i, i = 1, \ldots, 4), (N_t^{\text{exo},k}, k = 1, 2), (\widetilde{N}_t^j, j = 1, \ldots, 4)\right)}$$

- The final price process is defined via its intensities

$$\begin{cases} \lambda_t^i & = \mu_i + \sum_{j=1}^4 \int_0^{t-} \varphi_{ij}(t-s) d(N_s^j + \widetilde{N}_s^j) & i = 1, \ldots, 4, \\ \lambda_t^{\text{exo},k} & = \nu_k & k = 1, 2 \\ \widetilde{\lambda}_t^j & = a_1(N_{t-}^{\text{exo},1} - \widetilde{N}_{t-}^j) & j = 1, 2 \\ \widetilde{\lambda}_t^j & = a_2(N_{t-}^{\text{exo},2} - \widetilde{N}_{t-}^j) & j = 3, 4. \end{cases}$$

- $4 + 2 + 16 \times (1 \text{ or } 2) + 2$ parameters $\approx$ (for exponential kernels) $8 + 32$ usually reduced to $8 + 8 = 16$ parameters

$$(\mu_i, \nu_k, \varphi_{ij}, a_l)_{1 \le i,j, \le 4, 1 \le k,l \le 2}$$

Disentangling exogenous and endogenous effects

- For simplicity, we work with the – general – volatility model but not the price model.
- $(V^1_t, V^2_t) = (N^1_t + \widetilde{N}^{3,1}_t, N^2_t + \widetilde{N}^{3,2}_t)$, with intensity

$$\begin{cases} \lambda^1_t = \mu_1 + \int_0^{t-} \varphi_1(t-s)d(N^1_s + \widetilde{N}^{3,1}_s) + \int_0^{t-} \psi_1(t-s)d(N^2_s + \widetilde{N}^{3,2}_s) \\ \lambda^2_t = \mu_2 + \int_0^{t-} \varphi_2(t-s)d(N^2_s + \widetilde{N}^{3,2}_s) + \int_0^{t-} \psi_2(t-s)d(N^1_s + \widetilde{N}^{3,1}_s) \\ \lambda^3_t = \mu_3 \\ \widetilde{\lambda}^{3,1}_t = a(N^3_{t-} - \widetilde{N}^{3,1}_{t-}) \\ \widetilde{\lambda}^{3,2}_t = a(N^3_{t-} - \widetilde{N}^{3,2}_{t-}). \end{cases}$$

Disentangling exogenous and endogenous effects

- Basic objects :

$$\Phi = \begin{pmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{pmatrix} \text{ and } \Lambda = (\mathrm{Id} - \|\Phi\|_{L^1})^{-1} \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_2 + \mu_3 \end{pmatrix}.$$

- The macroscopic covariance of (V^1, V^2) is

$$(\mathrm{Id} - \|\Phi\|_{L^1})^{-1} \left(\mathrm{diag}(\Lambda) + \begin{pmatrix} 0 & \mu_3 \\ \mu_3 & 0 \end{pmatrix} \right) (\mathrm{Id} - \|\Phi^\top\|_{L^1})^{-1}$$

$$= (\mathrm{Id} - \|\Phi\|_{L^1})^{-1} \Big((\mathrm{Id} - \|\Phi\|_{L^1})^{-1} \|\Phi\|_{L^1} \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_2 + \mu_3 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & \mu_3 \\ \mu_3 & 0 \end{pmatrix} \Big) (\mathrm{Id} - \|\Phi^\top\|_{L^1})^{-1}$$

- Intricate nonlinear combination of endogenous and exogenous effects on the correlation !

Disentangling exogenous and endogenous effects

- We simplify everything further ! Ignore the delay.
- The model becomes

$$V_t^1 = N_t^1 + N_t^3, \quad V_t^1 = N_t^2 + N_t^3.$$

- With the basic objects

$$R = \begin{pmatrix} (\mathrm{Id} - \|\Phi\|_{L^1})^{-1} & (\mathrm{Id} - \|\Phi\|_{L^1})^{-1}\|\Phi\|_{L^1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

and

$$\Lambda = (\mathrm{Id} - \|\Phi\|_{L^1})^{-1}\begin{pmatrix} \mu_1 + \mu_3 \\ \mu_2 + \mu_3 \end{pmatrix}.$$

we can obtain a simple population interpretation of the mixed endogenous and exogenous effects.

Population interpretation

- We have, with $V_t^1 = N_t^1 + N_t^3, \;\; V_t^1 = N_t^2 + N_t^3,$

$$\mathrm{Cov}(V^a, V^b) = \sum_{i\in\{a,3\}} \sum_{j\in\{b,3\}} \sum_{k=1,2,3} \Lambda^k R^{ik} R^{jk}$$
$$= \sum_{k=1,2,3} \Lambda^k \Big(\sum_{i\in\{a,3\}} R^{ik} \Big) \Big(\sum_{j\in\{b,3\}} R^{jk} \Big)$$

- R^{ij} : mean number of events of type i triggered by one event of type j.
- $\sum_{i\in\{a,3\}} R^{ik}$: mean number of events of V^a triggered by one event of type k.
- Exogenous effect : $\mu_3\times$ (the mean number of events of V^a triggered by one exogenous event + likewise for V^b).

THANK YOU FOR YOUR ATTENTION !