Rough volatility, path-dependent PDEs and weak rates of convergence

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Volatility modelling

The prices of financial derivatives at time $t \in [0, T]$ are given by

$$\mathbb{E}\big[\phi(S_T)|\mathcal{F}_t\big],$$

where the underlying $(S_t)_{t\in[0,T]}$ is a martingale:

$$S_t = S_0 \exp\left(\int_0^t \sigma_s \mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma_s^2 \,\mathrm{d}s\right).$$

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¿ How does one choose the volatility $(\sigma_t)_{t\in[0,T]}$ such that:

(a) We can compute numerically and/or analytically the derivatives' prices

(b) Those prices are consistent with the market's

$\triangleright$ Set $\sigma_s = \psi_s(W_s)$ where W is a Brownian motion
$\hookrightarrow$ Markov, semimartingale, Itô calculus, Monte Carlo, PDE...

Going rough

Main idea: Replace BM W with fractional BM V

$$V_t := \int_0^t (t-r)^{H-\frac{1}{2}} \mathrm{d}W_r, \quad H \in (0, \tfrac{1}{2})$$

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Rationale:

- $\not\perp\!\!\!\perp$ Past dependence
- $\mathbb{P}$ Statistical estimation indicates $H \ll 0.5$
- $\mathbb{Q}$ Term-structure of implied volatility skew $\sim T^{H-\frac{1}{2}}$
- $\odot$ Microstructural fundations
- $\downdownarrows$ Mean-reversion at different time scales

All with one additional parameter H

Simulating rough volatility

♠ Let us consider a rough volatility model with log-price

$$X_t := x + \int_0^t \psi(V_r)\mathrm{d}B_r - \frac{1}{2}\int_0^t \psi(V_r)^2\,\mathrm{d}r, \quad V_t := \int_0^t K(t,r)\mathrm{d}W_r,$$

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with $t \in [0,T]$ and the singular kernel $K(t,r) := (t-r)^{H-\frac{1}{2}}$, $H \in (0, \frac{1}{2})$.

$\simeq$ Let $N \in \mathbb{N}$, set $\Delta_t := \frac{T}{N}$. and $t_i := i\Delta_t$ for $i = 0, \dots, N$ and define the Euler approximation:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + \psi(V_{t_i})\Delta B_{t_i} - \frac{1}{2}\psi(V_{t_i})^2\Delta_t, \qquad \overline{X}_{t_0} = x,$$
$$\overline{X}_t = x + \int_0^t \psi(V_{\kappa_r})\mathrm{d}B_r - \frac{1}{2}\int_0^t \psi(V_{\kappa_r})^2\,\mathrm{d}r, \quad \kappa_r := \frac{\lfloor rN \rfloor}{N},$$

where the Gaussian process V is sampled exactly (e.g. Cholesky).

Strong rates VS weak rates

- **Strong rate:** V is only H-Hölder continuous hence by Itô's formula

$$\mathbb{E}\left[\left|X_t - \overline{X}_t\right|^2\right] \lesssim \int_0^t \mathbb{E}\left[|\psi(V_r) - \psi(V_{\kappa_r})|^2\right] \mathrm{d}r \lesssim \Delta_t^{2H}$$

▶ To divide the error by 2 one needs to multiply the number of time points by $2^{1/H}$. If $H \approx 0$ it essentially doesn't converge.

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▶ To divide the error by 2 one needs to multiply the number of time points by $2^{1/H}$. If $H \approx 0$ it essentially doesn't converge.

• **Weak rate:** More relevant to option pricing, take a test function ϕ:

$$\mathcal{E}^N := \mathbb{E}[\phi(X_T)] - \mathbb{E}\left[\phi\left(\overline{X}_T\right)\right]$$

Example

For Markovian processes ($H = \frac{1}{2}$), $\mathcal{E}^N = \mathcal{O}\left(N^{-1}\right)$

▶ For rough volatility, is it $2H$, $1/2 + H$, 1 or something else ?

A difficult and open problem

Challenges (when $H \neq 1/2$):

- $\not\perp\!\!\!\perp$ No Markov property $\Rightarrow$ no PDE
- $\not{d}f$ No semimartingale property $\Rightarrow$ no Itô calculus

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◇ Exhaustive literature review for weak rates of rough volatility models:

Authors	Weak rate	Assumptions
Bayer, Hall, Tempone (2020)	$1/2 + H$	Linear vol. ($\psi(x) = x$)
Bayer, Fukasawa, Nakahara (2022)	$1/2 + H$	Linear vol. ($\psi(x) = x$)
Gassiat (2022)	$1/2 + 3H$	Linear vol. or cubic payoff ($\phi(x) = x^3$)
Friz, Salkeld, Wagenhofer (2022)	$1/2 + 3H$	Polynomial payoff ($\phi(x) = x^n$)

▶ All of them rely on the structure for explicit computations or induction

Today's talk

(1) Establishes path-dependent PDEs for rough volatility models

Theorem (based on [Viens & Zhang, 2019], [Wang, Yong & Zhang, 2022])

If $\boldsymbol{X}$ solves a Stochastic Volterra Equation, then $u(t,\omega) := \mathbb{E}[\phi(\boldsymbol{X}_T^{t,\omega})]$ is the unique classical solution to a path-dependent PDE.
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$\hookrightarrow$ *This applies in particular to rough volatility models.*

(2) Applies them to weak rates of convergence

Theorem

If $\phi, \psi \in C^\infty$ with suitable growth then we get a weak rate $1/2 + H$, i.e.

$$\mathcal{E}^N = \mathcal{O}(N^{-1/2-H}).$$

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A motivating example [Viens, Zhang 2019]

For $0 \leq t \leq s \leq T$, a natural decomposition is $V_s = V_t + [V_s - V_t]$.

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$\models$ Instead, Viens and Zhang introduce

$$V_s = \int_0^s K(s,r)\mathrm{d}W_r = \underbrace{\int_0^t K(s,r)\mathrm{d}W_r}_{=:\Theta_s^t} + \underbrace{\int_t^s K(s,r)\mathrm{d}W_r}_{=:I_s^t}$$

- Orthogonal decomposition: $\Theta_s^t \hat{\in} \mathcal{F}_t$ and $I_s^t \perp\!\!\!\perp \mathcal{F}_t$;
- $t \to \Theta^t$ is a martingale on $[0, s]$;
- (X, Θ) recovers a flow or Markov property;
- $\Theta_s^t = \mathbb{E}[V_s|\mathcal{F}_t]$ is related to the forward variance.

Markov representation

- Option prices take the form $\mathbb{E}[\phi(X_T)|\mathcal{F}_t]$ but how do we express this path-dependent process as a function?

Markov representation

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$$\begin{aligned}
\mathbb{E}[\phi(X_T)|\mathcal{F}_t] &= \mathbb{E}\left[\phi\left(X_t + \int_t^T \sigma_s(V_s)\mathrm{d}B_s - \frac{1}{2}\int_t^T \sigma_s(V_s)^2\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\phi\left(X_t + \int_t^T \sigma_s(\Theta_s^t + I_s^t)\mathrm{d}B_s - \frac{1}{2}\int_t^T \sigma_s(\Theta_s^t + I_s^t)^2\,\mathrm{d}s\right)\Big|X_t, \Theta_{[t,T]}^t\right] \\
&= u(t, X_t, \Theta_{[t,T]}^t),
\end{aligned}$$

where $u : [0, T] \times \mathbb{R} \times C([t, T]) \to \mathbb{R}$ is defined as

$$u(t, x, \omega) := \mathbb{E}\Big[\phi(X_T)\,\Big|\,X_t = x,\ \Theta^t = \omega\Big]$$

Stochastic Volterra Equations

◎ SVEs encompass rough volatility models; here without drift:

$$\boldsymbol{X}_s = x + \int_0^s K(s,r)\sigma(\boldsymbol{X}_r)\,\mathrm{d}W_r =: \boldsymbol{\Theta}_s^t + \int_t^s K(s,r)\sigma(\boldsymbol{X}_r)\,\mathrm{d}W_r$$

$$\boldsymbol{X}_s^{t,\omega} = \omega_s + \int_t^s K(s,r)\sigma(\boldsymbol{X}_r^{t,\omega})\,\mathrm{d}W_r, \qquad \text{(flow property)}$$

then if pathwise uniqueness holds: $\boldsymbol{X}_s = \boldsymbol{X}_s^{t,\Theta^t}$, for all $s \in [t, T]$.

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Markovian representation

Let $Y_t^{s,\omega} := \mathbb{E}[\Phi(\boldsymbol{X}_T^{s,\omega})|\mathcal{F}_t]$ and $u(t,\omega) := Y_t^{t,\omega}$, then

$$\mathbb{E}[\Phi(\boldsymbol{X}_T)|\mathcal{F}_t] = \mathbb{E}\left[\Phi\left(\boldsymbol{X}_T^{t,\Theta^t}\right)|\mathcal{F}_t\right] = Y_t^{t,\Theta^t} = u(t,\Theta^t)$$

The functional Itô formula for Volterra processes

The Fréchet derivative $\partial_\omega u$ is a linear map in the direction η:

$$u(t,x,\omega+\eta\mathbb{1}_{[t,T]}) - u(t,x,\omega) = \langle \partial_\omega u(t,x,\omega), \eta \rangle + o(\|\eta\mathbb{1}_{[t,T]}\|),$$

$\looparrowright$ The direction of interest is $K^t(s) := K(s,t)$, $s > t$.

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Theorem (Viens and Zhang (2019))

Suppose $u \in C^{1,2}_{+,\alpha}([0, T] \times C([t, T]))$ *and denote* $u_t = u(t, \Theta^t)$*:*

$$\mathrm{d}u_t = \left\{ \partial_t u_t + \frac{\sigma(\boldsymbol{X}_t)^2}{2} \langle \partial^2_\omega u_t, (K^t, K^t) \rangle \right\} \mathrm{d}t + \sigma(\boldsymbol{X}_t) \langle \partial_\omega u_t, K^t \rangle \, \mathrm{d}W_t,$$

Relation with Dupire's:

- Corresponds to $H = \frac{1}{2}$ hence $K = 1$
- The path is only perturbed after t but not frozen

PPDE for rough volatility

Recall our model

$$\begin{cases} X_t = \int_0^t \sigma_s(V_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma_s^2(V_s)\,\mathrm{d}s, \\ V_s = \int_0^s K(s,r)\,\mathrm{d}W_r, \end{cases}$$

and $u(t,x,\omega) := \mathbb{E}\Big[\phi(X_T)\,\Big|\,X_t = x,\ \Theta^t = \omega\Big]$.

Theorem (Bonesini, Jacquier, P. 2023+, based on many people's works)

Under regularity conditions on ϕ and σ, u is the unique $C^{1,2,2}_{+,\alpha}$ solution to

$$\Big\{\partial_t + \frac{1}{2}\sigma_t(\omega_t)^2(\partial_x^2 - \partial_x) + \frac{1}{2}\big\langle \partial_\omega^2 \cdot, (K^t, K^t)\big\rangle + \rho\sigma_t(\omega_t)\big\langle \partial_\omega(\partial_x \cdot), K^t\big\rangle\Big\} u(t,x,\omega) = 0,$$

with terminal condition $u(T,x,\omega) = \phi(x)$.

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Back to weak rates

↻ We denote $\overline{u}_t := u(t, \overline{X}_t, \Theta^t_{[t,T]})$ (recall V is sampled exactly)

$$\mathcal{E}^N = \mathbb{E}\left[\phi(X_T) - \phi\left(\overline{X}_T\right)\right] = \mathbb{E}\left[\overline{u}_{t_0}\right] - \mathbb{E}\left[\overline{u}_{t_N}\right] = -\sum_{i=0}^{N-1} \underbrace{\left(\mathbb{E}[\overline{u}_{t_{i+1}}] - \mathbb{E}[\overline{u}_{t_i}]\right)}_{=:\mathfrak{A}_i},$$

where, by application of the functional Itô formula and the PPDE,

$$\begin{aligned}\mathfrak{A}_i :=& \frac{1}{2}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\partial^2_{xx}\overline{u}_t\left(\psi(V_t)^2 - \psi(V_{t_i})^2\right)\right] \mathrm{d}t \\ &+ \rho\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\left\langle \partial_\omega(\partial_x \overline{u}_t), K^t\right\rangle\left(\psi(V_t) - \psi(V_{t_i})\right)\right] \mathrm{d}t.\end{aligned}$$

- How to obtain the rate from those differences?
- How to recover $\mathbb{E}[\psi(V_t) - \psi(V_{t_i})]$?

Approaches that do not work

For any $g \in C^1$ and $F \in \mathbb{D}^{1,2}$,

- Cauchy-Schwarz yields the strong rate

$$\mathbb{E}\left[\left(g(V_t) - g(V_{t_i})\right)F\right] \leq \sqrt{\mathbb{E}[|g(V_t) - g(V_{t_i})|^2]}\sqrt{\mathbb{E}[F^2]} \sim \Delta_t^H$$

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- Taylor & integration by parts: let $\Delta K(r) := K(t,r) - K(t_i,r)$ then

$$\begin{aligned}\mathbb{E}\left[\big(g(V_t) - g(V_{t_i})\big)F\right] &= \mathbb{E}\left[(V_t - V_{t_i})F \int_0^1 g'(\lambda V_t + (1-\lambda)V_{t_i})\mathrm{d}\lambda\right] \\ &\lesssim \int_0^t |\Delta K(r)| \big(K(t,r) + K(t_i,r)\big)\,\mathrm{d}r\end{aligned}$$

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$$\lesssim \int_0^t |\Delta K(r)|\left(K(t,r) + K(t_i,r)\right)\mathrm{d}r$$

- Itô's formula on $\Theta_t^{\cdot}$ because V is not a local martingale

$$g(V_t) = g(\Theta_t^t) = g(\Theta_t^{t_i}) + \int_{t_i}^t K(t,r)g'(\Theta_t^r)\mathrm{d}W_r + \frac{1}{2}\int_{t_i}^t K(t,r)^2 g''(\Theta_t^r)\,\mathrm{d}r$$

- Clark-Ocone

$$g(V_t) = \mathbb{E}[g(V_t)] + \int_0^t K(t,r)\mathbb{E}[g(V_t)|\mathcal{F}_r]\,\mathrm{d}W_r$$

Joint chaos expansion

Lemma

Let $\boldsymbol{s}_n = (s_1, \cdots, s_n)$. *For* $g \in C^\infty$ *and* $F \in \mathbb{D}^{\infty,2}$, *it holds*

$$\mathbb{E}[F g(V_t)] = \mathbb{E}[F]\,\mathbb{E}[g(V_t)] + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \mathbb{E}[\mathrm{D}_{\boldsymbol{s}_n} F]\,\mathbb{E}[\mathrm{D}_{\boldsymbol{s}_n} g(V_t)] \mathrm{d}\boldsymbol{s}_n.$$

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In the quadratic case $\phi(x) = x^2$, $\mathrm{D}_s \partial^2 \overline{u}_t = 0$ thus it boils down to

$$\mathfrak{A}_i = \int_{t_i}^{t_{i+1}} \mathbb{E}[\partial^2 \overline{u}_t]\mathbb{E}[g(V_t) - g(V_{t_i})]\,\mathrm{d}t \lesssim \int_{t_i}^{t_{i+1}} \left(t^\gamma - t_i^\gamma\right)\,\mathrm{d}t.$$

♣ Turns out that instead of giving rate γ,

$$\sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} (t^\gamma - t_i^\gamma)\,\mathrm{d}t \lesssim \Delta_t \quad \Longrightarrow \quad \text{rate one!}$$

A touch of combinatorics (extension of Faà di Bruno's formula)

For a function $f \in C^\infty$ and a random variable $X \in \mathbb{D}^{\infty,2}$,

$$\begin{aligned}
\mathrm{D}_{1,2} f(X) &= f'(X)\mathrm{D}_{1,2}X + f''(X)\mathrm{D}_1 X\, \mathrm{D}_2 X \\
\mathrm{D}_{1,2,3} f(X) &= f'(X)\mathrm{D}_{1,2,3}X + f'''(X)\mathrm{D}_1 X\, \mathrm{D}_2 X\, \mathrm{D}_3 X \\
&\quad + f''(X)\Big\{\mathrm{D}_{1,2}X\, \mathrm{D}_3 X + \mathrm{D}_{1,3}X\, \mathrm{D}_2 X + \mathrm{D}_{2,3}X\, \mathrm{D}_1 X\Big\}
\end{aligned}$$

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$$\mathrm{D}_{1,\cdots,n} f(X) = \sum_{k=1}^{n} f^{(k)}(X) \sum_{\mathcal{D} \in \boldsymbol{\mathcal{D}}_k^n} \mathcal{D}X.$$

▷ $\boldsymbol{\mathcal{D}}_k^n$ is the set of partitions of a set of n objects into k non-empty subsets;

▷ $\mathcal{D}$ is a product of k Malliavin derivatives.

∴ This permits to write

$$\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_n} \partial_x^2 \overline{u}_t\right] = \mathbb{E}[\mathrm{D}_{\boldsymbol{s}_n} \phi''(X_T)] = \mathbb{E}\left[\sum_{k=1}^{n} \phi^{(k+2)}(X_T) \sum_{\mathcal{D} \in \boldsymbol{\mathcal{D}}_k^n} \mathcal{D}X_T\right]$$

A kernel story

Let $\mathcal{I} \subset \mathbb{N}$ and define $\mathbf{K}(t, \boldsymbol{s}_{\mathcal{I}}) := \prod_{k \in \mathcal{I}} \mathbf{K}(t, s_k)$.

- Recall $X_T = x + \int_t^T \psi(V_r)\, \mathrm{d}B_r$ then if $s_1 < s_2 < \cdots < s_k$ we have

$$\mathrm{D}_{\boldsymbol{s}_k} X_T = \int_{s_k}^T \psi^{(k)}(V_r)\mathbf{K}(r, \boldsymbol{s}_k)\mathrm{d}B_r + \rho\psi^{(k-1)}(V_{s_k})\mathbf{K}(s_k, \boldsymbol{s}_{k-1})$$

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- By triangles,

$$\begin{aligned}\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_n}(g(V_t) - g(V_{t_i}))\right] &= \mathbb{E}\left[g^{(n)}(V_t)\mathbf{K}(t, \boldsymbol{s}_n) - g^{(n)}(V_{t_i})\mathbf{K}(t_i, \boldsymbol{s}_n)\right] \\ &= \mathbf{K}(t, \boldsymbol{s}_n)\mathbb{E}\left[g^{(n)}(V_t) - g^{(n)}(V_{t_i})\right] \\ &\quad + \mathbb{E}\left[g^{(n)}(V_{t_i})\right]\sum_{l=1}^{n} \Delta K(t, t_i, s_l)\mathbf{K}(t, \boldsymbol{s}_{[\![l+1,n]\!]})\mathbf{K}(t_i, \boldsymbol{s}_{[\![1,l-1]\!]})\end{aligned}$$

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▷ We end up with integrals of the form (at least in spirit)

$$\int_{[0,t]^n} \mathbf{K}(t_i, \boldsymbol{s}_{[\![1,l-1]\!]})^2 \left|\Delta K(t, t_i, s_l)\right| \mathbf{K}(t, \boldsymbol{s}_{[\![l+1,n]\!]})^2 \mathrm{d}\boldsymbol{s}_n \lesssim \int_0^t \left|\Delta K(t, t_i, s_l)\right| \mathrm{d}s_l \lesssim \Delta_t^{H+\frac{1}{2}}$$

Bring everything together

⊲ Overall, this entails

$$\mathcal{E}^N \lesssim \Delta_t^{H+\frac{1}{2}} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\sum_{n=1}^{\infty} \frac{C_0^n}{n!} \sum_{k=1}^{n} \mathrm{BDG}_k |\mathcal{D}_k^n| \right) \mathrm{d}t,$$

where $C_0 = C_0(H, T, \phi, \psi)$.

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- The Stirling number of the second kind: $|\mathcal{D}_k^n| \leq \frac{1}{2}\binom{n}{k} k^{n-k}$;
- The BDG constant: $\mathrm{BDG}_k \leq (4k)^{k/2}$.

$$\Longrightarrow \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{C_0^n}{n!} \mathrm{BDG}_k |\mathcal{D}_k^n| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{C_1^n}{n!} k^{k/2} \frac{n!}{k!(n-k)!} k^{n-k}$$

$$= \sum_{k=1}^{\infty} (C_1 \mathrm{e}^{C_1})^k \frac{k^{k/2}}{k!} < \infty$$

Conclusion

Summary:

(1) We established a PPDE theory for SVEs and rough volatility models, based on previous works by Viens, Wang, Yong, Zhang.

(2) We obtained a weak rate of convergence $1/2 + H$ for a (relatively) wide range of models

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Summary:

(1) We established a PPDE theory for SVEs and rough volatility models, based on previous works by Viens, Wang, Yong, Zhang.

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Outlook:

- How to obtain $1/2 + 3H$ which seems to be optimal?
- Extensions to other discretisations, drift
- Extension to fully implicit SVEs, what could be the rate?
- Further applications: control, numerics, regularisation, path-dependent payoffs...

Merci !