A Mean Field Game between Informed Traders and a Broker

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Overview

1. Introduction
2. The $N-$player game
3. Facing many informed traders
4. The solution
5. Numerical results

Introduction

Market making

What is a market maker?

- A market maker is a liquidity provider. He provides bid and ask prices for a list of assets to other market participants.
- Today, often replaced by algorithms.

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A market maker faces a complex optimization problem

- Makes money out of the bid-ask spread.
- Faces the risk that the price moves adversely without him being able to unwind his position rapidly enough.

From economics to mathematics

Classical literature in economics on market making

- Ho and Stoll. Optimal dealer pricing under transactions and return uncertainty. JoFE, 1981.
- O'Hara and Oldfield. The microeconomics of market making. JoFQA, 1986.
- Grossman and Miller. Liquidity and market structure. JoF, 1988.

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New interest after the crisis

- Avellaneda and Stoikov. High-frequency trading in a limit order book. QF, 2008.
- Guéant, Lehalle, and Fernandez-Tapia. Dealing with the Inventory Risk : A solution to the market making problem. MAFE, 2013.
- Cartea, Jaimungal, and Ricci. Buy Low, Sell High : A High Frequency Trading perspective. SIFIN, 2014.

An interesting research strand

Many extensions of the initial one-asset model

- Multi-asset framework.

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- General intensities *(e.g. logistic).*

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- Mean field game version.

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The problem

On many markets (e.g. FX cash markets), market maker have access to a liquidity pool (e.g. D2D market) were they can unwind part of their inventory.

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Literature

- Barzykin, Bergault, and Guéant. Algorithmic market making in dealer markets with hedging and market impact. MaFi, 2023.
- Cartea and Sánchez-Betancourt. Brokers and Informed Traders: Dealing with Toxic Flow and Extracting Trading Signals. Preprint, 2022.
- Nutz, Webster, and Zhao. Unwinding Stochastic Order Flow: When to Warehouse Trades. Preprint, 2023.

Externalisation: our contribution

Externalisation: our contribution

Goals of this paper

We propose a mean-field version of the paper by Cartea and Sánchez-Betancourt:

- What happens when a broker faces a large number of (informed) traders?
- How should the broker hedge?
- And, on another note, how should each individual trader use its signal?

The $N-$player game

The market

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Reference price process

Under probability $\mathbb{P}$, the price process $(S_t)_t$ is given by

$$\mathrm{d}S_t = \sigma^S \mathrm{d}W_t^S.$$

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Everyone observe a common signal $(\alpha_t)_t$ given by

$$\mathrm{d}\alpha_t = -k^\alpha \alpha_t \mathrm{d}t + \sigma^\alpha \mathrm{d}W_t^\alpha.$$

Informed traders dynamics

Informed traders dynamics

Private signal

Each trader observe a particular signal given for trader n by

$$\mathrm{d}\alpha_t^n = -\bar{k}\alpha_t^n \mathrm{d}t + \bar{\sigma}\mathrm{d}W_t^n.$$

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Inventory

The inventory $(Q_t^n)_t$ of trader n is given by

$$\mathrm{d}Q_t^n = \nu_t^n \mathrm{d}t.$$

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$$\mathrm{d}Q_t^n = \nu_t^n \mathrm{d}t.$$

Cash process

The cash process $(X_t^n)_t$ of trader n is given by

$$\mathrm{d}X_t^n = -\nu_t^n \left(S_t + \eta^I \nu_t^n\right) \mathrm{d}t.$$

Broker's dynamics

Broker's dynamics

Inventory

The inventory $(Q_t^B)_t$ of the broker is given by

$$\mathrm{d}Q_t^B = \left(N\nu_t^B - \sum_{n=1}^{N} \nu_t^n \right) \mathrm{d}t.$$

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Cash process

The cash process $(X_t^B)_t$ of the broker is given by

$$\mathrm{d}X_t^B = \sum_{n=1}^{N} \nu_t^n \left(S_t + \eta^I \nu_t^n \right) \mathrm{d}t - N\nu_t^B \left(S_t + \eta^B \nu_t^n \right) \mathrm{d}t.$$

The problem of the $n-$th informed trader

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Change of probability

We introduce the probability $\mathbb{P}^{n,\nu^B}$ given by

$$\left.\frac{\mathrm{d}\mathbb{P}^{n,\nu^B}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t^n} = \exp\left(\int_0^t \frac{b\,\nu_u^B + \alpha_u^n + \alpha_u}{\sigma^S}\mathrm{d}W_u^S - \frac{1}{2}\int_0^t \left(\frac{b\,\nu_u^B + \alpha_u^n + \alpha_u}{\sigma^S}\right)^2 \mathrm{d}u\right).$$

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Under this probability, the price has dynamics

$$\mathrm{d}S_t = \left(b\,\nu_t^B + \alpha_t^n + \alpha_t\right)\mathrm{d}t + \sigma^S \mathrm{d}\tilde{W}^{S,n}.$$

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For a given $(\nu^B_t)_{t\in[0,T]}$, the $n-$th informed trader maximises the following objective function

$$\mathbb{E}^{n,\nu^B}\left[X^n_T + Q^n_T S_T - \bar{a}\,(Q^n_T)^2 - \bar{\phi}\int_0^T (Q^n_t)^2\,\mathrm{d}t\right].$$

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This amounts to maximizing

$$\mathbb{E}^{n,\nu^B}\left[\int_0^T \left\{Q_t^n\left(b\,\nu_t^B + \alpha_t^n + \alpha_t\right) - \eta^I\,(\nu_t^n)^2 - 2\,\bar{a}Q_t^n\nu_t^n - \bar{\phi}\,(Q_t^n)^2\right\}\mathrm{d}t\right].$$

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The problem of the broker

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Objective function

For a given $(\nu_t^1)_{t\in[0,T]}, \ldots, (\nu_t^N)_{t\in[0,T]}$, the broker wants to maximise the following objective function

$$\mathbb{E}^{B,\nu^B}\left[X_T^B + Q_T^B S_T - \frac{a^B}{N}\left(Q_T^B\right)^2 - \frac{\phi^B}{N}\int_0^T \left(Q_t^B\right)^2 \mathrm{d}t\right].$$

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This amounts to maximizing

$$\mathbb{E}^{B,\nu^B}\left[\int_0^T\left\{Q_t^B\left(b\,\nu_t^B+\alpha_t\right)+\eta^I\sum_{n=1}^N(\nu_t^n)^2-N\eta^B\left(\nu_t^B\right)^2-2\frac{a^B}{N}Q_t^B\left(N\nu_t^B-\sum_{n=1}^N\nu_t^n\right)-\frac{\phi^B}{N}\left(Q_t^B\right)^2\right\}\mathrm{d}t\right].$$

The problem of the broker

The problem of the broker

Objective function

The optimisation problem remains unchanged if we scale the objective function by dividing it by N, in which case the broker maximises

$$\mathbb{E}^{B,\nu^B}\left[\int_0^T\left\{\bar{Q}_t^B\left(b\,\nu_t^B+\alpha_t\right)+\eta^I\frac{1}{N}\sum_{n=1}^{N}(\nu_t^n)^2-\eta^B\left(\nu_t^B\right)^2-2a^B\,\bar{Q}_t^B\left(\nu_t^B-\frac{1}{N}\sum_{n=1}^{N}\nu_t^n\right)-\phi^B\left(\bar{Q}_t^B\right)^2\right\}\mathrm{d}t\right]$$

where $\left(\bar{Q}_t^B\right)_t=\left(\frac{Q_t^B}{N}\right)_t$, that is,

$$\mathrm{d}\bar{Q}_t^B=\left(\nu_t^B-\frac{1}{N}\sum_{n=1}^{N}\nu_t^n\right)\mathrm{d}t.$$

Facing many informed traders

The framework

The framework

Common signal

As before, everyone observe a common signal $(\alpha_t)_t$ given by

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Private signal of the representative informed trader

We consider a representative informed trader who observes a private signal $(\alpha_t^I)_t$ given by

$$\mathrm{d}\alpha_t^I = -\bar{k}\alpha_t^I \mathrm{d}t + \bar{\sigma}\mathrm{d}W_t^I.$$

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Inventory of the representative informed trader

The inventory $(Q_t^I)_t$ of the representative informed trader is given by

$$\mathrm{d}Q_t^I = \nu_t^I \mathrm{d}t.$$

The framework

The framework

A mean-field of informed traders

Let us denote by $(\mu_t)_t$ the process with values in $\mathcal{P}(\mathbb{R})$ representing at time t the distribution of the execution rates of the (other) informed traders conditionally to $\mathcal{F}_t^\alpha$. The mean field execution rate $(\bar{\nu}_t)_t$ is given by

$$\bar{\nu}_t = \int_{\mathbb{R}} x \, \mu_t(\mathrm{d}x).$$

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$$\bar{\nu}_t = \int_{\mathbb{R}} x \, \mu_t(\mathrm{d}x).$$

Inventory of the broker

The (scaled) inventory $(\bar{Q}_t^B)_t$ of the broker is given by

$$\mathrm{d}\bar{Q}_t^B = \left(\nu_t^B - \bar{\nu}_t\right) \mathrm{d}t,$$

Optimisation problems

Optimisation problems

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The representative informed trader wants to solve

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where

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with $b \leq 2a^B, 2\eta^B, 2\eta^I, 4\phi^B, 4\bar{\phi}$.

Optimisation problems

Optimisation problems

Definition

A solution of the above game is given by a probability flow $\mu^\star \in \mathcal{P}(\mathbb{R})$, a control $\nu^{I,\star} \in \mathcal{A}$, and a control $\nu^{B,\star} \in \mathcal{A}$ such that

(i) $H^{I,\nu^{B,\star}}(\nu^{I,\star}) = \sup_{\nu^I \in \mathcal{A}} H^{I,\nu^{B,\star}}(\nu^I)$;

(ii) $H^{B,\mu^\star}(\nu^{B,\star}) = \sup_{\nu^B \in \mathcal{A}} H^{B,\mu^\star}(\nu^B)$;

(iii) $\mu^\star_t$ is the distribution of $\nu^{I,\star}_t$ conditionally to $\mathcal{F}^\alpha_t$ for Lebesgue–almost every $t \in [0,T]$,

where $\mathbb{F}^\alpha := (\mathcal{F}^\alpha_t)_{t\in[0,T]}$ is the $\mathbb{P}-$completed filtration generated by W^α.

The solution

The informed trader's optimality condition

The informed trader's optimality condition

Lemma

Let $\nu^B \in \mathcal{A}$. The functional $H^{I,\nu^B}(\cdot) : \mathcal{A} \to \mathbb{R}$ is strictly concave up to a $\mathbb{P} \otimes \mathrm{d}t-$null set,

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$$H^{I,\nu^B}(\rho\,\zeta + (1-\rho)\,\nu) > \rho\, H^{I,\nu^B}(\zeta) + (1-\rho)\, H^{I,\nu^B}(\nu)\,.$$

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$$H^{I,\nu^B}(\rho\,\zeta + (1-\rho)\,\nu) > \rho\, H^{I,\nu^B}(\zeta) + (1-\rho)\, H^{I,\nu^B}(\nu)\,.$$

Lemma

The functional H^{I,ν^B} is everywhere Gâteaux differentiable in $\mathcal{A}$. The Gâteaux derivative at a point $\nu^I \in \mathcal{A}$ in a direction $w^I \in \mathcal{A}$ is given by

$$\begin{aligned}\left\langle DH^{I,\nu^B}(\nu^I), w^I \right\rangle = \mathbb{E}\Big[\int_0^T w_t^I \Big\{ & -2\,\eta^I \nu_t^I - 2\,a^I Q_T^I \\ & + \int_t^T \Big(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^I \Big)\, \mathrm{d}u \Big\} \mathrm{d}t \Big].\end{aligned}$$

The informed trader's optimality condition

The informed trader's optimality condition

Theorem

We have that

$$\nu^{I,\star} = \underset{\nu^I \in \mathcal{A}}{\arg\max}\; H^{I,\nu^B}(\nu^I)$$

if and only if $\nu^{I,\star}$ is the unique strong solution to the FBSDE

$$\begin{cases} -\mathrm{d}\left(2\,\eta^I \nu_t^{I,\star}\right) & = \left(b\,\nu_t^B + \alpha_t^I + \alpha_t - 2\phi^I Q_t^{I,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^I, \\ 2\,\eta^I \nu_T^{I,\star} & = -2\,a^I Q_T^{I,\star}, \end{cases}$$

where $Z^I \in \mathbb{H}_T^2$ is a martingale.

The informed trader's optimality condition

The informed trader's optimality condition

Proof

Let us first assume that $\left\langle DH^{I,\nu^B}(\nu^{I,\star}), w^I \right\rangle = 0$ for all $w^I \in \mathcal{A}$.

The informed trader's optimality condition

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Let us first assume that $\langle DH^{I,\nu^B}(\nu^{I,\star}), w^I \rangle = 0$ for all $w^I \in \mathcal{A}$. This implies that

$$\mathbb{E}\left[-2\,\eta^I \nu_t^{I,\star} - 2\,a^I Q_T^I + \int_t^T \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u \middle| \mathcal{F}_t\right] = 0$$

almost surely for all $t \in [0, T]$.

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Proof

Let us first assume that $\langle DH^{I,\nu^B}(\nu^{I,\star}), w^I \rangle = 0$ for all $w^I \in \mathcal{A}$. This implies that

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almost surely for all $t \in [0, T]$. Therefore,

$$\begin{aligned}
-2\,\eta^I \nu_t^{I,\star} &= \mathbb{E}\left[2\,a^I Q_T^{I,\star} - \int_t^T \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u \middle| \mathcal{F}_t\right] \\
&= \int_0^t \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u \\
&\qquad + \mathbb{E}\left[2\,a^I Q_T^{I,\star} - \int_0^T \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u \middle| \mathcal{F}_t\right] \\
&= \int_0^t \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u - Z_t^I,
\end{aligned}$$

The informed trader's optimality condition

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Proof

where the process Z^I given by

$$Z_t^I := -\mathbb{E}\left[2\, a^I\, Q_T^{I,\star} - \int_0^T \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I\, Q_u^{I,\star}\right) \mathrm{d}u \middle| \mathcal{F}_t\right]$$

is a martingale, by definition. Hence it is clear that $\nu^{I,\star}$ is solution to the FBSDE.

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is a martingale, by definition. Hence it is clear that $\nu^{I,\star}$ is solution to the FBSDE.

Conversely, assume that $\nu^{I,\star}$ is solution to the FBSDE. Then $\nu^{I,\star}$ can be represented implicitly as

$$2\,\eta^I \nu_t^{I,\star} = \mathbb{E}\left[-2\, a^I Q_T^{I,\star} + \int_t^T \left(b\,\nu_u^B + \alpha_u^I + \alpha_u - 2\phi^I Q_u^{I,\star}\right) \mathrm{d}u \middle| \mathcal{F}_t\right].$$

Plugging this into the expression of the Gâteaux derivative, it is clear that it vanishes almost surely for any $w^I \in \mathcal{A}$.

The broker's optimality condition

The broker's optimality condition

Lemma

Let $(\mu_t)_{t\in[0,T]}$ with values in $\mathcal{P}(\mathbb{R})$ be the distribution of the execution rates of the informed traders conditionally to $\mathcal{F}_t^\alpha$. The functional $H^{B,\mu}(\cdot) : \mathcal{A} \to \mathbb{R}$ is strictly concave up to a $\mathbb{P} \otimes \mathrm{d}t-$null set.

The broker's optimality condition

Lemma

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Lemma

The functional $H^{B,\mu}$ is everywhere Gâteaux differentiable in $\mathcal{A}$. The Gâteaux derivative at a point $\nu^B \in \mathcal{A}$ in a direction $w^b \in \mathcal{A}$ is given by

$$\begin{aligned}\left\langle DH^{B,\mu}(\nu^B), w^B \right\rangle = \mathbb{E}\Big[\int_0^T w_t^B \Big\{ &(b - 2\,a^B)\bar{Q}_T^B - 2\,\eta^B \nu_t^B \\ &+ \int_t^T \Big(b \int_{\mathbb{R}} x\,\mu_u(\mathrm{d}x) + \alpha_u - 2\phi^B \bar{Q}_u^B \Big) \mathrm{d}u \Big\} \mathrm{d}t \Big].\end{aligned}$$

The broker's optimality condition

The broker's optimality condition

Theorem

We have that

$$\nu^{B,\star} = \arg\max_{\nu^B \in \mathcal{A}} H^{B,\mu}(\nu^B)$$

if and only if $\nu^{B,\star}$ is the unique strong solution to the FBSDE

$$\begin{cases} -\mathrm{d}\left(2\,\eta^B \nu_t^{B,\star}\right) & = \left(b\,\bar{\nu}_t + \alpha_t - 2\phi^B\,\bar{Q}_t^{B,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^B, \\ 2\,\eta^B \nu_T^{B,\star} & = (b - 2\,a^B)\bar{Q}_T^{B,\star}, \end{cases}$$

where $Z^B \in \mathbb{H}_T^2$ is a martingale.

Equilibrium condition

The mean field FBSDE system

At equilibrium, we have the following system of FBSDEs

$$\begin{cases} -\mathrm{d}\left(2\,\eta^I \nu_t^{I,\star}\right) & = \left(b\,\nu_t^B + \alpha_t^I + \alpha_t - 2\,\phi^I Q_t^{I,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^I, \\ -\mathrm{d}\left(2\,\eta^B \nu_t^{B,\star}\right) & = \left(b\,\bar{\nu}_t^\star + \alpha_t - 2\phi^B \bar{Q}_t^{B,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^B, \\ 2\,\eta^I \nu_T^{I,\star} & = -2\,a^I Q_T^{I,\star} \\ 2\,\eta^B \nu_T^{B,\star} & = -(2\,a^B - b)\bar{Q}_T^{B,\star}, \\ \bar{\nu}_t^\star & = \mathbb{E}\left[\nu_t^{I,\star}\,|\mathcal{F}_t^\alpha\right]. \end{cases}$$

Optimal strategy of the broker

FBSDE system

At the equilibrium, we solve the system

$$\begin{cases} -\mathrm{d}\left(2\,\eta^I\bar{\nu}_t^\star\right) & = \left(b\,\nu_t^{B,\star} + \alpha_t - 2\,\bar{\phi}\bar{Q}_t^\star\right)\mathrm{d}t - \mathrm{d}\bar{Z}_t^I, \\ -\mathrm{d}\left(2\,\eta^B\nu_t^{B,\star}\right) & = \left(b\,\bar{\nu}_t^\star + \alpha_t - 2\phi^B\bar{Q}_t^{B,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^B, \\ 2\,\eta^I\bar{\nu}_T^\star & = -2\,\bar{a}\bar{Q}_T^\star \\ 2\,\eta^B\nu_T^{B,\star} & = -(2\,a^B - b)\bar{Q}_T^{B,\star}. \end{cases}$$

Optimal strategy of the broker

Ansatz

We look for a solution to the above system in the form

$$\bar{\nu}^{\star}_t = g^a_t \alpha_t + g^b_t\, \bar{Q}^{\star}_t + g^c_t\, \bar{Q}^{B,\star}_t\,,$$
$$\nu^{B,\star}_t = h^a_t \alpha_t + h^b_t\, \bar{Q}^{\star}_t + h^c_t\, \bar{Q}^{B,\star}_t\,,$$

where g^a_t, g^b_t, g^c_t and h^a_t, h^b_t, h^c_t are deterministic $\mathcal{C}^1$ functions, with terminal conditions $g^a_T = h^a_T = g^c_T = h^b_T = 0$, $g^b_T = -\bar{a}/\eta^I$ and $h^c_T = -(2\, a^B - b)/2\,\eta^B$, and where

$$\bar{Q}^{\star}_t = \int_0^t \bar{\nu}^{\star}_u \,\mathrm{d}u\,, \qquad \text{and} \qquad \bar{Q}^{B,\star}_t = \int_0^t \left(\nu^{B,\star}_u - \bar{\nu}^{\star}_u\right)\, \mathrm{d}u\,.$$

Optimal strategy of the broker

A system of ODEs

We observe that the system of equations becomes

$$
\begin{aligned}
0 &= \mathrm{d}g_t^a + \left[-k^\alpha g_t^a + g_t^b\, g_t^a + g_t^c\,(h_t^a - g_t^a) + \frac{b\,h_t^a + 1}{2\,\eta^I}\right]\mathrm{d}t\\
0 &= \mathrm{d}h_t^a + \left[-k^\alpha h_t^a + h_t^b\, g_t^a + h_t^c\,(h_t^a - g_t^a) + \frac{b\,g_t^a + 1}{2\,\eta^B}\right]\mathrm{d}t\\
0 &= \mathrm{d}g_t^b + \left[\left(g_t^b\right)^2 + g_t^c\left(h_t^b - g_t^b\right) + \frac{b\,h_t^b - 2\,\bar{\phi}}{2\,\eta^I}\right]\mathrm{d}t\\
0 &= \mathrm{d}h_t^b + \left[h_t^b\, g_t^b + h_t^c\left(h_t^b - g_t^b\right) + \frac{b\,g_t^b}{2\,\eta^B}\right]\mathrm{d}t\\
0 &= \mathrm{d}g_t^c + \left[g_t^b\, g_t^c + g_t^c\,(h_t^c - g_t^c) + \frac{b\,h_t^c}{2\,\eta^I}\right]\mathrm{d}t\\
0 &= \mathrm{d}h_t^c + \left[h_t^b\, g_t^c + h_t^c\,(h_t^c - g_t^c) + \frac{b\,g_t^c - 2\,\phi^B}{2\,\eta^B}\right]\mathrm{d}t\,,
\end{aligned}
$$

with terminal condition $g_T^a = h_T^a = g_T^c = h_T^b = 0$, $g_T^b = -\bar{a}/\eta^I$ and $h_T^c = -(2\,a^B - b)/2\,\eta^B$. We see that the system for $g_t^b, g_t^c, h_t^b, h_t^c$ is independent of the solution to g_t^a, h_t^a.

Optimal strategy of the broker

A Riccati equation

Let $\boldsymbol{P} : [0, T] \to \mathbb{R}^4$ be given by

$$\boldsymbol{P}_t = -\begin{pmatrix} h_t^c & h_t^b \\ g_t^c & g_t^b \end{pmatrix}$$

and let $\boldsymbol{U}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S} \in \mathbb{R}^{2\times 2}$ be given by

$$\boldsymbol{U} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \boldsymbol{Y} = \begin{pmatrix} 0 & \frac{b}{2\,\eta^B} \\ \frac{b}{2\,\eta^I} & 0 \end{pmatrix}, \boldsymbol{Q} = \begin{pmatrix} -\frac{\phi^B}{\eta^B} & 0 \\ 0 & -\frac{\bar{\phi}}{\eta^I} \end{pmatrix}, \boldsymbol{S} = \begin{pmatrix} \frac{2\,a^B - b}{2\,\eta^B} & 0 \\ 0 & \frac{\bar{a}}{\eta^I} \end{pmatrix}.$$

Optimal strategy of the broker

A Riccati equation

Let $\boldsymbol{P} : [0, T] \to \mathbb{R}^4$ be given by

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The system of ODEs for $g_t^b, g_t^c, h_t^b, h_t^c$ can be written as the following matrix Riccati differential equation

$$\begin{cases} 0 = \frac{\mathrm{d}\boldsymbol{P}_t}{\mathrm{d}t} + \boldsymbol{Y}\,\boldsymbol{P}_t - \boldsymbol{P}_t\,\boldsymbol{U}\,\boldsymbol{P}_t - \boldsymbol{Q}\,, & t \in [0, T)\,, \\ \boldsymbol{P}_T = \boldsymbol{S}\,. \end{cases}$$

Optimal strategy of the broker

Solution of the Riccati ODE (Freiling et al. 2000, Freiling 2002)

The unique solution takes the form

$$\boldsymbol{P}_t = \boldsymbol{T}_t \, \boldsymbol{R}_t^{-1} \, ,$$

where $\boldsymbol{R}_t$, $\boldsymbol{T}_t$ solve the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{T}_t \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{U} \\ -\boldsymbol{Q} & -\boldsymbol{Y} \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{T}_t \end{pmatrix} , \qquad \begin{pmatrix} \boldsymbol{R}_T \\ \boldsymbol{T}_T \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{S} \end{pmatrix} .$$

Optimal strategy of the broker

A linear ODE

Finally, we just have to solve the linear system of ODEs given by:

$$\begin{cases} 0 & = \mathrm{d}g_t^a + \left[-k^\alpha g_t^a + g_t^b \, g_t^a + g_t^c \, (h_t^a - g_t^a) + \frac{b \, h_t^a + 1}{2 \, \eta^I} \right] \mathrm{d}t \\ 0 & = \mathrm{d}h_t^a + \left[-k^\alpha h_t^a + h_t^b \, g_t^a + h_t^c \, (h_t^a - g_t^a) + \frac{b \, g_t^a + 1}{2 \, \eta^B} \right] \mathrm{d}t \, , \end{cases}$$

with terminal conditions $g_T^a = h_T^a = 0$.

Optimal strategy of the broker

A linear ODE

Let

$$\boldsymbol{X}_t = \begin{pmatrix} h^a_t \\ g^a_t \end{pmatrix}, \quad \boldsymbol{A}_t = \begin{pmatrix} -\frac{1}{2\,\eta^B} \\ -\frac{1}{2\,\eta^I} \end{pmatrix}, \quad \boldsymbol{B}_t = \begin{pmatrix} k^\alpha - h^c_t & h^c_t - h^b_t - \frac{b}{2\,\eta^B} \\ -g^c_t - \frac{b}{2\,\eta^I} & k^\alpha + g^c_t - g^b_t \end{pmatrix},$$

then, we have that the system for h^a_t and g^a_t can be written as

$$\mathrm{d}\boldsymbol{X}_t = (\boldsymbol{A}_t + \boldsymbol{B}_t\,\boldsymbol{X}_t)\,\mathrm{d}t\,,$$

with terminal condition $\boldsymbol{X}_T = 0$.

Optimal strategy of the broker

Optimal strategy of the broker

The strategy

The closed-form optimal solution to the FBSDE is then

$$\begin{pmatrix} \nu_t^{B,\star} \\ \bar{\nu}_t^{\star} \end{pmatrix} = \boldsymbol{X}_t \, \alpha_t - \boldsymbol{P}_t \begin{pmatrix} \bar{Q}_t^{B,\star} \\ \bar{Q}_t^{\star} \end{pmatrix} .$$

Optimal strategy of the broker

The strategy

The closed-form optimal solution to the FBSDE is then

$$\begin{pmatrix} \nu_t^{B,\star} \\ \bar{\nu}_t^{\star} \end{pmatrix} = \boldsymbol{X}_t \, \alpha_t - \boldsymbol{P}_t \begin{pmatrix} \bar{Q}_t^{B,\star} \\ \bar{Q}_t^{\star} \end{pmatrix} .$$

Remark

The optimal trading strategy of the broker can be written as

$$\begin{aligned} \nu_t^{B,\star} &= q_t^a \left(\bar{\nu}_t^{\star} - g_t^b \, \bar{Q}_t^{\star} - g_t^c \, \bar{Q}_t^{B,\star} \right) + h_t^b \, \bar{Q}_t^{\star} + h_t^c \, \bar{Q}_t^{B,\star} \\ &= q_t^a \, \bar{\nu}_t^{\star} + \left(h_t^b - q_t^a \, g_t^b \right) \, \bar{Q}_t^{\star} + \left(h_t^c - q_t^a \, g_t^c \right) \, \bar{Q}_t^{B,\star} , \end{aligned}$$

where the externalisation rate q_t^a is defined as

$$q_t^a = \frac{h_t^a}{g_t^a} .$$

Optimal strategy of the informed trader

Optimal strategy of the informed trader

FBSDE of the representative trader

$$\begin{cases} -\mathrm{d}\left(2\,\eta^I \nu_t^{I,\star}\right) & = \left(b\,\nu_t^{B,\star} + \alpha_t^I + \alpha_t - 2\,\phi^I Q_t^{I,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^I, \\ 2\,\eta^I \nu_T^{I,\star} & = -2\,a^I Q_T^{I,\star}. \end{cases}$$

Optimal strategy of the informed trader

FBSDE of the representative trader

$$\begin{cases} -\mathrm{d}\left(2\,\eta^I \nu_t^{I,\star}\right) & = \left(b\,\nu_t^{B,\star} + \alpha_t^I + \alpha_t - 2\,\phi^I Q_t^{I,\star}\right)\mathrm{d}t - \mathrm{d}Z_t^I, \\ 2\,\eta^I \nu_T^{I,\star} & = -2\,a^I Q_T^{I,\star}. \end{cases}$$

Ansatz

As before, we make an ansatz and look for a solution with the form

$$\nu_t^{I,\star} = f_t^a \alpha_t + f_t^{a,I} \alpha_t^I + f_t^b \bar{Q}_t^\star + f_t^{b,I} Q_t^{I,\star} + f_t^c \bar{Q}_t^{B,\star},$$

where $f^a, f^{a,I}, f^b, f^{b,I}, f^c$ are deterministic $\mathcal{C}^1$ functions, with terminal conditions $f_T^a = f_T^{a,I} = f_T^b = f_T^c = 0$ and $f_T^{b,I} = -a^I/\eta^I$, and where

$$Q_t^{I,\star} = \int_0^t \nu_u^{I,\star} \mathrm{d}u.$$

Optimal strategy of the informed trader

A system of ODEs

We observe that the system of equations becomes

$$
\begin{aligned}
0 &= \mathrm{d}f_t^a + \left[-k^\alpha f_t^a + f_t^b g_t^a + f_t^{b,I} f_t^a + f_t^c (h_t^a - g_t^a) + \frac{b h_t^a + 1}{2\eta^I}\right] \mathrm{d}t \\
0 &= \mathrm{d}f_t^{a,I} + \left[-k^I f_t^{a,I} + f_t^{b,I} f_t^{a,I} + \frac{1}{2\eta^I}\right] \mathrm{d}t \\
0 &= \mathrm{d}f_t^b + \left[f_t^b g_t^b + f_t^{b,I} f_t^b + f_t^c (h_t^b - g_t^b) + \frac{b h_t^b}{2\eta^I}\right] \mathrm{d}t \\
0 &= \mathrm{d}f_t^{b,I} + \left[\left(f_t^{b,I}\right)^2 - \frac{\phi^I}{\eta^I}\right] \mathrm{d}t \\
0 &= \mathrm{d}f_t^c + \left[f_t^b g_t^c + f_t^{b,I} f_t^c + f_t^c \left(h_t^c - g_t^c\right) + \frac{b h_t^c}{2\eta^I}\right] \mathrm{d}t,
\end{aligned}
$$

with terminal conditions $f_T^a = f_T^{a,I} = f_T^b = f_T^c = 0$ and $f_T^{b,I} = -a^I/\eta^I$.

Optimal strategy of the informed trader

Optimal strategy of the informed trader

A Riccati ODE

Notice that the equation for $f^{b,I}$ is independent of the others, and is given by

$$\begin{cases} 0 & = \mathrm{d}f_t^{b,I} + \left[\left(f_t^{b,I} \right)^2 - \frac{\phi^I}{\eta^I} \right] \mathrm{d}t, \\ f_T^{b,I} & = -a^I / \eta^I. \end{cases}$$

Optimal strategy of the informed trader

A Riccati ODE

Notice that the equation for $f^{b,I}$ is independent of the others, and is given by

$$\begin{cases} 0 & = \mathrm{d}f^{b,I}_t + \left[\left(f^{b,I}_t \right)^2 - \frac{\phi^I}{\eta^I} \right] \mathrm{d}t, \\ f^{b,I}_T & = -a^I/\eta^I. \end{cases}$$

This is a simple Riccati ODE, and its solution is given by

$$f^{b,I}_t = -\sqrt{\frac{\phi^I}{\eta^I}} \tanh\left(\sqrt{\frac{\phi^I}{\eta^I}} (T-t) \right) - \frac{e^{2\int_t^T y_p(s)\mathrm{d}s}}{\eta^I/a^I + \int_t^T e^{2\int_u^T y_p(s)\mathrm{d}s} \mathrm{d}u}$$

with

$$y_p(t) = -\sqrt{\frac{\phi^I}{\eta^I}} \tanh\left(\sqrt{\frac{\phi^I}{\eta^I}} (T-t) \right).$$

Optimal strategy of the informed trader

Optimal strategy of the informed trader

A linear ODE

Once we have solved the equation for $f^{b,I}$, the equation for $f^{a,I}$ is just a linear ODE given by

$$\begin{cases} 0 & = \mathrm{d}f_t^{a,I} + \left[-k^I f_t^{a,I} + f_t^{b,I} f_t^{a,I} + \frac{1}{2\eta^I}\right] \mathrm{d}t \\ f_T^{a,I} & = 0. \end{cases}$$

Optimal strategy of the informed trader

A linear ODE

Once we have solved the equation for $f^{b,I}$, the equation for $f^{a,I}$ is just a linear ODE given by

$$\begin{cases} 0 & = \mathrm{d}f_t^{a,I} + \left[-k^I f_t^{a,I} + f_t^{b,I} f_t^{a,I} + \frac{1}{2\eta^I} \right] \mathrm{d}t \\ f_T^{a,I} & = 0. \end{cases}$$

Its solution for $t \in [0, T]$ is therefore given by

$$f_t^{a,I} = \frac{1}{2\eta^I} \int_t^T e^{-\int_t^u \left(k^I - f_s^{b,I}\right) \mathrm{d}s} \mathrm{d}u\,.$$

Optimal strategy of the informed trader

Optimal strategy of the informed trader

A linear system of ODEs

Let $\boldsymbol{A}^{b,c} : [0,T] \to \mathbb{R}^4$ and $\boldsymbol{b}^{b,c} : [0,T] \to \mathbb{R}^2$ be given by

$$\boldsymbol{A}^{b,c}_t = -\begin{pmatrix} g^b_t + f^{b,I}_t & h^b_t - g^b_t \\ g^c_t & h^c_t - g^c_t + f^{b,I}_t \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^{b,c}_t = -\frac{b}{2\eta^I}\begin{pmatrix} h^b_t \\ h^c_t \end{pmatrix}.$$

We introduce the function $\boldsymbol{F}^{b,c} : [0,T] \to \mathbb{R}^2$ given by

$$\boldsymbol{F}^{b,c}_t = \begin{pmatrix} f^b_t \\ f^c_t \end{pmatrix}.$$

Optimal strategy of the informed trader

A linear system of ODEs

Let $\boldsymbol{A}^{b,c} : [0, T] \to \mathbb{R}^4$ and $\boldsymbol{b}^{b,c} : [0, T] \to \mathbb{R}^2$ be given by

$$\boldsymbol{A}^{b,c}_t = -\begin{pmatrix} g^b_t + f^{b,I}_t & h^b_t - g^b_t \\ g^c_t & h^c_t - g^c_t + f^{b,I}_t \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^{b,c}_t = -\frac{b}{2\eta^I}\begin{pmatrix} h^b_t \\ h^c_t \end{pmatrix}.$$

We introduce the function $\boldsymbol{F}^{b,c} : [0, T] \to \mathbb{R}^2$ given by

$$\boldsymbol{F}^{b,c}_t = \begin{pmatrix} f^b_t \\ f^c_t \end{pmatrix}.$$

Then $\boldsymbol{F}^{b,c}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F}^{b,c}_t = \boldsymbol{A}^{b,c}_t \boldsymbol{F}^{b,c}_t + \boldsymbol{b}^{b,c}_t$$

with terminal condition $\boldsymbol{F}^{b,c}_T = 0$.

Optimal strategy of the informed trader

Optimal strategy of the informed trader

A linear ODE

Finally, if we define $b^a : [0, T] \to \mathbb{R}$ by

$$b^a_t = -f^b_t g^a_t - f^c_t (h^a_t - g^a_t) - \frac{b h^a_t + 1}{2\eta^I} \qquad \forall t \in [0, T],$$

then the unique solution to the linear Equation for f^a is given by

$$f^a_t = -\int_t^T b^a_u e^{-\int_t^u (k^\alpha - f^{b,I}_s \mathrm{d}s)} \mathrm{d}u$$

for $t \in [0, T]$.

Numerical results

Model parameters

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- Signal volatility: $\sigma^\alpha = 1 \ \$ \cdot \text{day}^{-3/2}$;

Model parameters

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- Initial common signal: $\alpha_0 = 0 \ \$ \cdot \text{day}^{-1}$;
- Signal volatility: $\sigma^\alpha = 1 \ \$ \cdot \text{day}^{-3/2}$;
- Mean-reversion of signal: $k^\alpha = 5 \ \text{day}^{-1}$;

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- Transaction costs of traders: $\eta^I = 10^{-3} \ \$ \cdot \text{day}$;

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- Transaction cost of the broker: $\eta^B = 1.2 \cdot 10^{-3} \ \$ \cdot \text{day}$;

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- Initial price: $S_0 = 100$ \$;
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- Signal volatility: $\sigma^\alpha = 1 \ \$ \cdot \text{day}^{-3/2}$;
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- Transaction cost of the broker: $\eta^B = 1.2 \cdot 10^{-3} \ \$ \cdot \text{day}$;
- Terminal penalties: $a^I = a^B = 1$ \$;

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- Signal volatility: $\sigma^\alpha = 1 \ \$ \cdot \text{day}^{-3/2}$;
- Mean-reversion of signal: $k^\alpha = 5 \ \text{day}^{-1}$;
- Transaction costs of traders: $\eta^I = 10^{-3} \ \$ \cdot \text{day}$;
- Transaction cost of the broker: $\eta^B = 1.2 \cdot 10^{-3} \ \$ \cdot \text{day}$;
- Terminal penalties: $a^I = a^B = 1$ \$;
- Risk aversion: $\phi^I = \phi^B = 10^{-2} \ \$ \cdot \text{day}^{-1}$.

Sample paths of signal and price

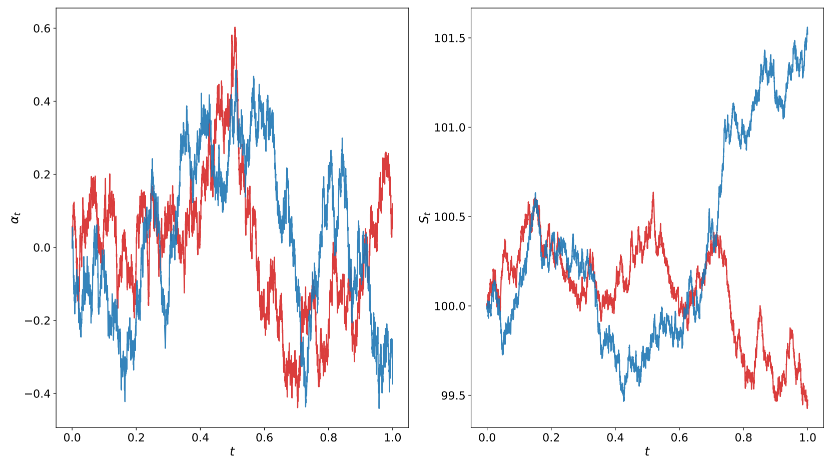

Figure 1: Signal and price.

Sample paths of execution rates

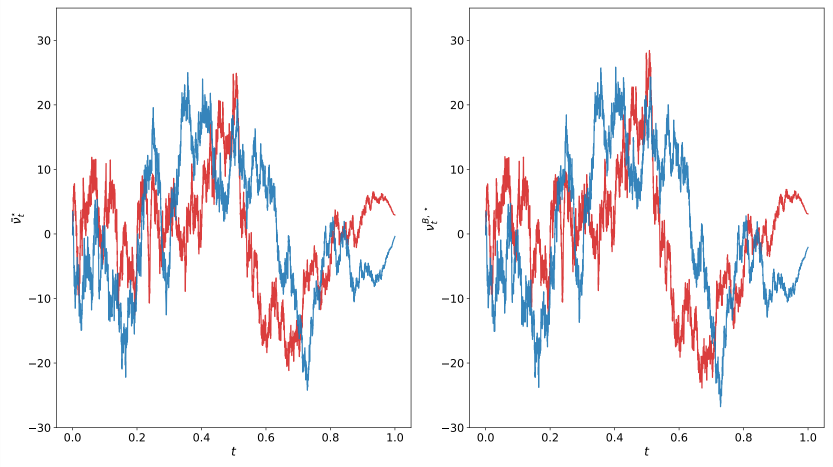

Figure 2: Mean-field execution rate and broker's execution rate.

Sample paths of inventories

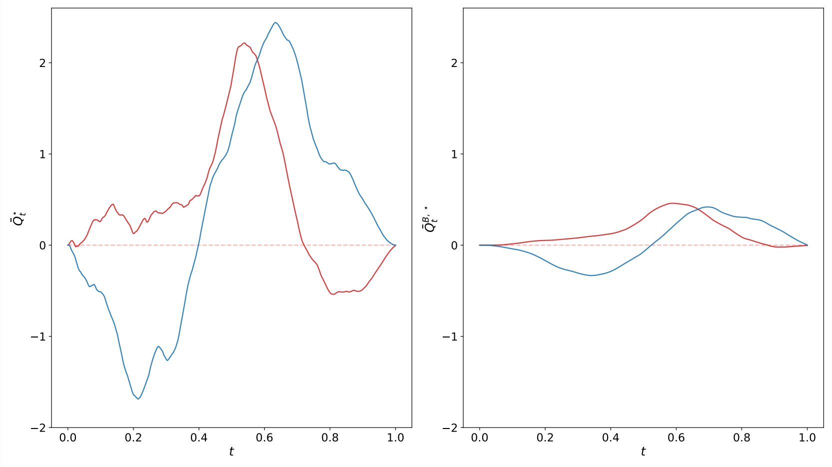

Figure 3: Mean-field inventory and broker's inventory.

Representative trader: model parameters

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- Initial private signal: $\alpha_0^I = 0$ \$ $\cdot$ day^{-1};

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Sample paths of signals

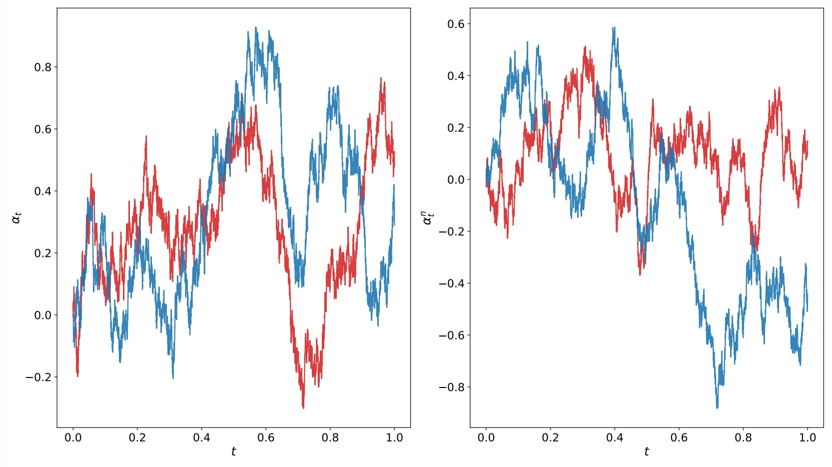

Figure 4: Signals.

Sample paths of trader's execution rates

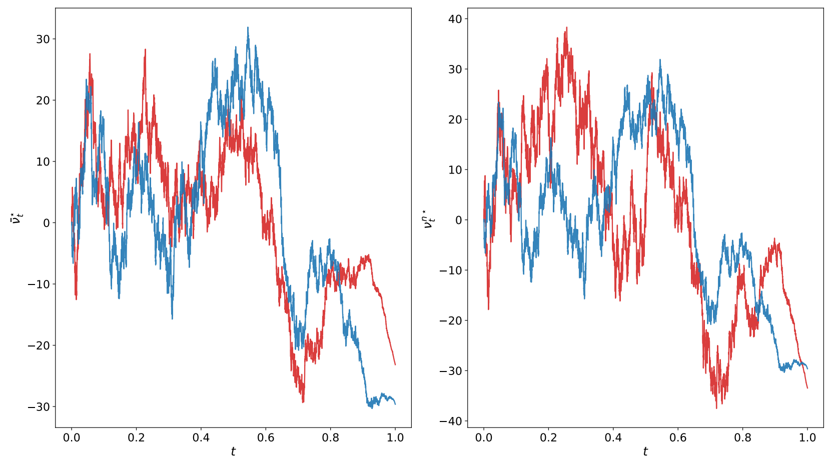

Figure 5: Mean-field execution rate and representative trader's execution rate.

Sample paths of inventories

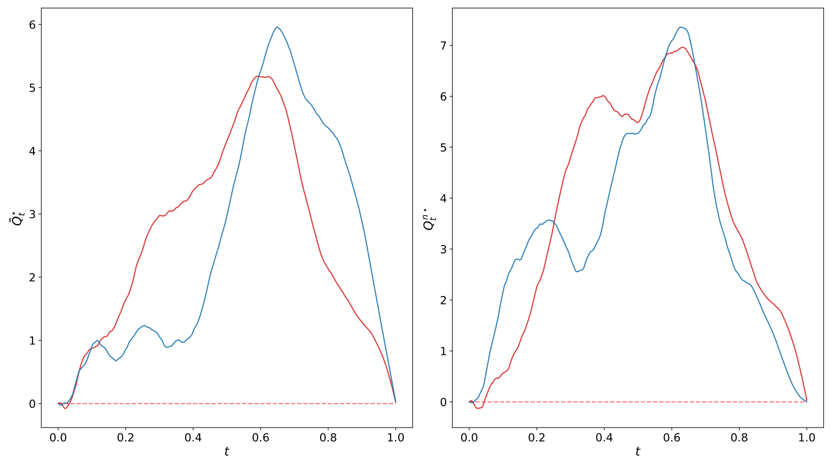

Figure 6: Mean-field inventory and representative trader's inventory.

Sample paths for the broker

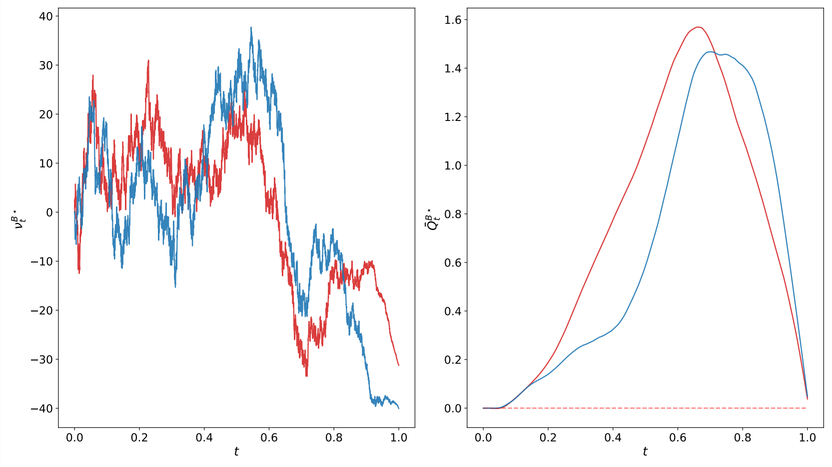

Figure 7: Execution rate and inventory of the broker.

The End

Thank You!