# Forecasting extreme trajectories using semi-norm representations

**Séminaire FDD-FiME**


Arthur THOMAS

Gilles DE TRUCHIS and Sébastien FRIES


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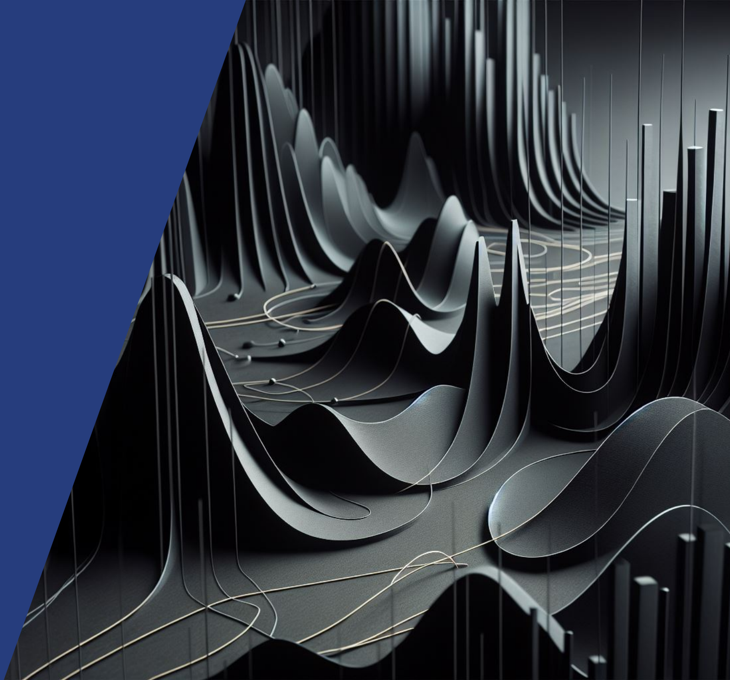
Dauphine | PSL
UNIVERSITÉ PARIS



# Motivations

**Extreme values** and sharp reversals are at the heart of **prediction challenges**

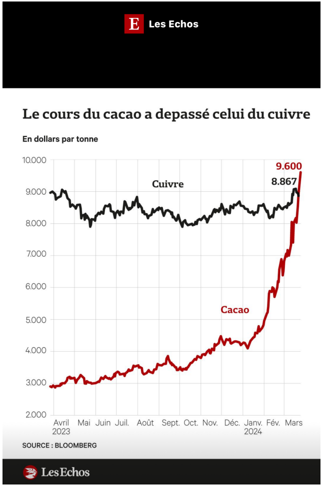
- **Traditional time series** often relies on “the best predictor”

$$\widehat{X}_{t+h} := \mathbb{E}(X_{t+h}|\mathcal{I}_t), \quad h > 0$$

with $\mathcal{I}_t$ the past information

- **However**, future realizations far from central values lead to

**huge prediction errors !**



## $X_t$ is **anticipative**

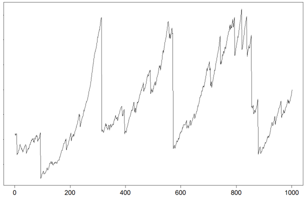
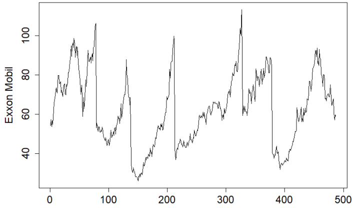



**Figure:** $X_t = \rho X_{\boldsymbol{t+1}} + \varepsilon_t,\ \varepsilon_t \sim$ i.i.d. heavy-tailed

## What do you choose?




## $X_t$ is **causal**

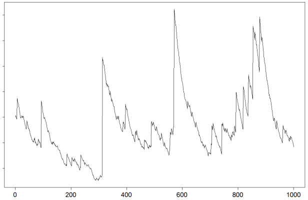
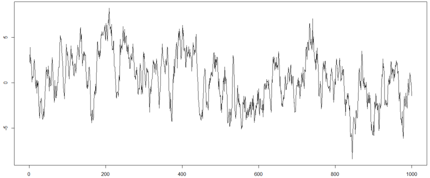
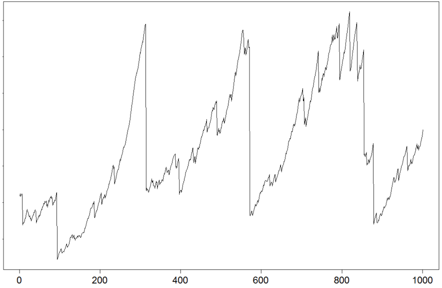



**Figure:** $X_t = \rho X_{\boldsymbol{t-1}} + \varepsilon_t,\ \varepsilon_t \sim$ i.i.d. heavy-tailed




Just for fun **causal** and **Gaussian** $X_t = \rho X_{\boldsymbol{t-1}} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2),\ (\rho = 0.95)$

$$X_t = \rho X_{t+1} + \varepsilon_t$$




$\Rightarrow$ **The heavy elephant in the room: Forecasting!**

- **Econometrics/Statistics** literature:
  - **Rosenblatt (2000), Lanne and Saikkonen (2011), Gouriéroux and Zakoian (2017)**
  - Theoretically mimics bubble data[1]
  - Estimation is well covered[2]
- **Applications:**
  - Economic: Macroeconometrics, Financial data, Bitcoins, Commodities prices, Portfolio management[3]
  - **Climate variables:** Global sea level, GHG emissions, global temperature, sea ice area, and some natural oscillation indices[4]
  - Physics, astronomy, engineering...

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[1] Gouriéroux et al. (2020)

[2] Cavaliere et al. (2017), Fries and Zakoian (2019), Hecq et al. (2016), Hecq et al. (2017b), Hecq et al. (2020), Andrews et al. (2009), Lanne and Saikkonen (2011), Lanne & Saikkonen (2013), Gourieroux & Jasiak (2023)

[3] Lanne and Saikkonen (2011), Lanne & Luoto (2013), Moussa & Thomas (2023), Hecq et al. (2023), Hecq et al. (2024), Fries and Zakoian (2019), Fries (2021), Hencic & Gourieroux (2015), Hecq et al. (2017a), Hecq et al. (2017b), Friedrich et al. (2020), Hecq & Voisin (2021), de Truchis et al. (2024)

[4] Blasques et al (2023), Giancaterini et al. (2022)

# Outline

- We use $\alpha$-**stable linear time series** and discuss a new **semi-norm** representation

  $\Rightarrow$ this naturally leads to the concept of **past-representability**

- We focus on **extreme** trajectories of past-representable processes and show that

  $\Rightarrow$ to some extent, the stochastic nature of the trajectories vanishes

  ... to give way to deterministic features related to MA($\infty$) coefficients

- We suggest two **forecasting procedures for asymptotically extreme trajectories**
- We use a **Monte-Carlo** study to evaluate our results in a non-asymptotic framework
- We illustrate the empirical relevance of our results on **climatic data**[5]

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[5]We develop a web app to replicate the results and play with other time series (emprical data and simulated one)

# Introduction to stable moving-averages

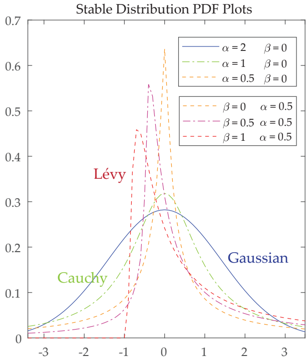
*Anticipative v.s. causal processes*

# Large shocks are non Gaussian

- **Stable laws** are natural candidates

$$\varepsilon_t \sim \mathcal{S}(\alpha, \beta, \sigma, \mu)$$

$$\begin{aligned} \alpha \in (0,2) &: \quad \text{tail index} \\ \beta \in [-1,1] &: \quad \text{asymmetry} \\ \sigma > 0 &: \quad \text{scale} \\ \mu \in \mathbb{R} &: \quad \text{location} \end{aligned}$$




- $\varepsilon_1$ and $\varepsilon_2$ are Stable random variables if $a\varepsilon_1 + b\varepsilon_2$ is Stable
- To simplify the slides we focus on $\beta = 0 \Rightarrow \varepsilon_t \sim \mathcal{S}\alpha\mathcal{S}$
- Unconditional moments exist up to the tail index $\alpha$

$$\mathbb{E}(|\varepsilon_1|^u) < \infty \Rightarrow \mathbb{E}(|X_t|^u) < \infty, \quad u < \alpha$$

# Two-sided stable moving-averages

We consider linear **strictly stationary** processes driven by $\varepsilon_t \overset{iid}{\sim} \mathcal{S}\alpha\mathcal{S}$

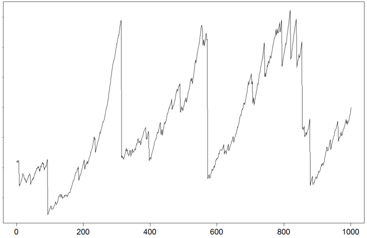



**Figure:** $X_t$ is anticipative: $k \in \mathbb{Z}_+$

$$X_t = \sum_{k=-\infty}^{+\infty} d_k \varepsilon_{t+k} \qquad (1)$$

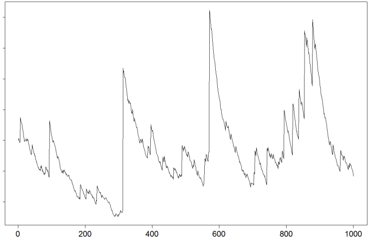



**Figure:** $X_t$ is causal: $k \in \mathbb{Z}_-$

# Causal v.s. anticipative processes

We remain agnostic while developing our theory even if

- the **anticipative** profile is **visually more familiar**

$$X_t = \rho X_{t+1} + \varepsilon_t \Rightarrow d_k = \rho^k, \ \ k \geq 0, \ \ |\rho| < 1$$

However, on the **empirical side**,

- **causal processes** are massively more considered

$$X_t = \rho X_{t-1} + \varepsilon_t \Rightarrow d_k = \rho^k, \ \ k \leq 0, \ \ |\rho| < 1$$

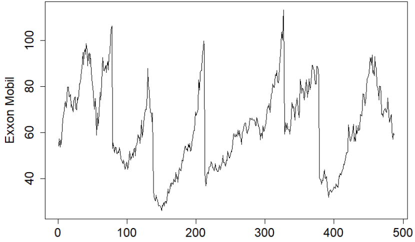
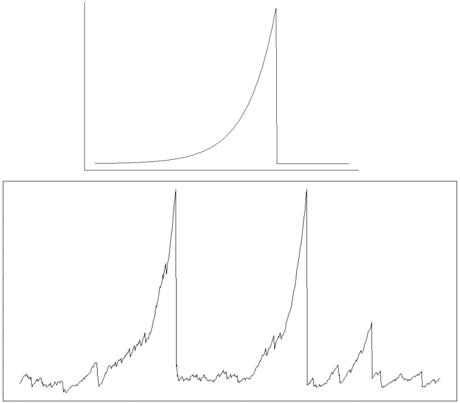
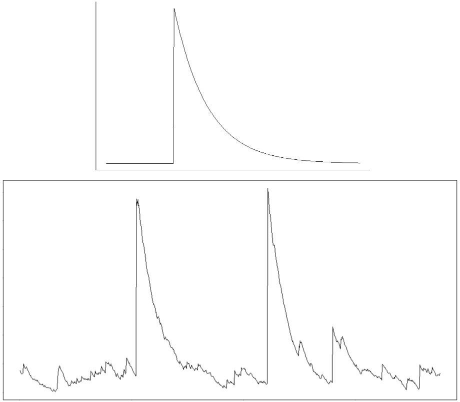



**Figure:** On financial markets $k \in \mathbb{Z}_+$ or $k \in \mathbb{Z}$

# Some trajectories/processes are more predictable than others



$X_t = \rho X_{t+1} + \varepsilon_t$



$Y_t = \rho Y_{t-1} + \varepsilon_t$

# Forecasting stable-MA($\infty$) with conditional moments

- Backward conditional moments as well

$$\mathbb{E}(|X_t|^b|X_{t+1}) < \infty, \;\; b < \alpha$$

- **Forward conditional moments** are more promising

$$X_{t+1} = \mathbb{E}(|X_t|^f|X_{t-1}) < \infty, \;\; f < 2\alpha + 1$$

as the conditional expectation always exists

# Forecasting with $\alpha$-stable vectors

- Fries (2022) suggests a new strategy based on

$$\boldsymbol{X}_t = (X_t, X_{t+h})',$$

that is an $\alpha$-stable vector, as its characteristic function alway exists

$$\mathbb{E}\left[e^{i\langle \boldsymbol{u}, \boldsymbol{X}\rangle}\right] = \exp\left\{ -\int_{S_d} |\langle \boldsymbol{u}, \boldsymbol{s}\rangle|^{\alpha} \left(1 - i, \text{sign}(\langle \boldsymbol{u}, \boldsymbol{s}\rangle)\text{tg}(\pi\alpha/2)\right)\Gamma(d\boldsymbol{s})\right\} \tag{2}$$

and relies on a finite **spectral measure** $\Gamma$ defined on the unit sphere $S_d \in \mathbb{R}^d$

N.B. Any norm can be used to define the unit sphere: hereafter we retain the Euclidean one

$$S_d = \{\boldsymbol{s} \in \mathbb{R}^d : ||\boldsymbol{s}||_e = 1\}$$

# The spectral measure

- For $d = 2$, $S_d$ is a circle and $\Gamma$ acts as a compass
- Given a particular position on the map (the realization of $X_t$)

… $\Gamma$ charges the mass where $X_{t+1}$ is likely to go

- $x_t$ **close to central values**: “magnetic” perturbations occur

$\Rightarrow$ $\Gamma$ charges numerous mass points

- $x_t$ **far from central values**: some patterns emerge

$\Rightarrow$ $\Gamma$ charges a small number of mass points

# Tail conditional distribution of the AR(1)

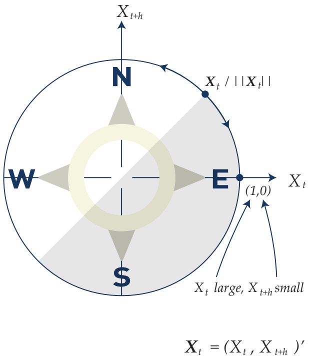
- Typically, for $X_t = \rho X_{t+1} + \varepsilon_t$, with $\varepsilon_t \overset{iid}{\sim} \mathcal{S}\alpha\mathcal{S}$, if $x_t \to +\infty$

$\Rightarrow$ For $\boldsymbol{X}_t = (X_t, X_{t+h})'$, $\Gamma$ points to the "East" coordinates or

$$\frac{\boldsymbol{X}_t}{||\boldsymbol{X}_t||_e} = \frac{(\rho^h, 1)}{\sqrt{1+\rho^{2h}}}$$

- Straightforward interpretation: conditionally to $x_t \to +\infty$
  - Either $X_{t+h}$ crashes to central values with probability $1 - \rho^{\alpha h}$
  - Or $X_{t+h}$ continue to grow with probability $\rho^{\alpha h}$

$\Rightarrow$ when $h$ is also large, the crash probability goes to 1



# Baseline path of stable-MA($\infty$)

- Now consider the general case

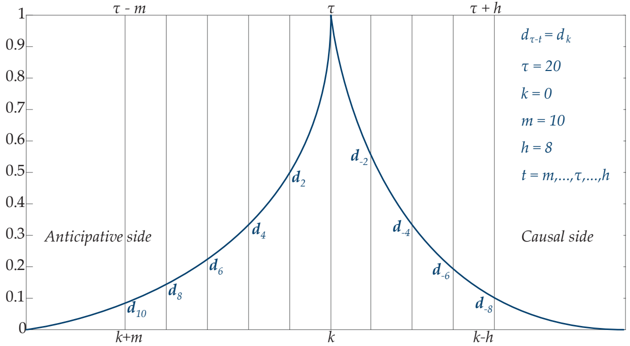
$$X_t = \sum_{k\in\mathbb{Z}} d_k \varepsilon_{t+k}, \quad \varepsilon_t \overset{iid}{\sim} \mathcal{S}(\alpha,\beta,\sigma,0)$$

- Set $\tau = t + k$ such that

$$X_t = \sum_{k\in\mathbb{Z}} d_k \varepsilon_{t+k} = \sum_{\tau\in\mathbb{Z}} d_{\tau-t} \varepsilon_\tau$$

$\Rightarrow$ $X_t$ is a linear combination of deterministic baseline paths scaled by $\varepsilon_\tau$ and shift in time

$$t \longmapsto d_{\tau-t}$$



# Spectral measure of stable-MA($\infty$) vectors

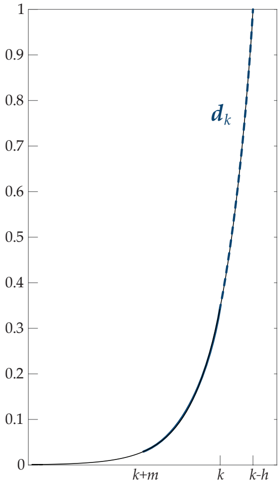
Define $\boldsymbol{X}_t$ an $\alpha$-stable, $m$ and $h$ vector such that

$$\boldsymbol{X}_t = (\underbrace{X_{t-m}, \ldots, X_t}_{\text{observed}}, \underbrace{X_{t+1}, \ldots, X_{t+h}}_{\text{unobserved}})$$

Then $\boldsymbol{X}_t$ has the following spectral measure

$$\Gamma = \sigma^\alpha \sum_{\vartheta \in S_1} \sum_{k \in \mathbb{Z}} w_\vartheta \|\boldsymbol{d}_k\|_e^\alpha \delta_{\left\{ \frac{\vartheta \boldsymbol{d}_k}{\|\boldsymbol{d}_k\|_e} \right\}}, \quad S_1 = \{-1, +1\}$$

where $\boldsymbol{d}_k = (d_{k+m}, \ldots, d_k, d_{k-1}, \ldots, d_{k-h})$ and $w_\vartheta = (1 + \vartheta\beta)/2$.



# Tail conditional distribution of stable-MA($\infty$)

Theorem 1 is a direct application of Theorem 4.4.8 by Samorodnitsky and Taqqu (1994)

## Theorem 1

*For any Borel sets $A$, $B$ of $S_{m+h+1}$,*

$$\mathbb{P}\left(\frac{\boldsymbol{X}_t}{\|\boldsymbol{X}_t\|_e} \in A \,\middle|\, \|\boldsymbol{X}_t\|_e > x, \frac{\boldsymbol{X}_t}{\|\boldsymbol{X}_t\|_e} \in B\right) \underset{x\to+\infty}{\longrightarrow} \frac{\Gamma(A\cap B)}{\Gamma(B)}$$

## Corollary 1

*Let $A \subset S_{m+h+1}$, a Borel set that does not contain any point $\pm\, \boldsymbol{d}_k/\|\boldsymbol{d}_k\|_e$. Then,*

$$\mathbb{P}\left(\frac{\boldsymbol{X}_t}{\|\boldsymbol{X}_t\|_e} \in A \,\middle|\, \|\boldsymbol{X}_t\|_e > x\right) \underset{x\to+\infty}{\longrightarrow} \frac{\Gamma(A)}{\Gamma(S_{m+h+1})} = 0$$

$\Rightarrow$ During extreme events, $\boldsymbol{X}_t$ is necessarily **colinear to** some $\boldsymbol{d}_k$

# Conditioning set

In view of empirical applications, Theorem 1 is not very useful as

$$\frac{\boldsymbol{X}_t}{\|\boldsymbol{X}_t\|_e}, \quad \|\boldsymbol{X}_t\|_e = \|(\underbrace{X_{t-m}, \ldots, X_t}_{\text{observed}}, \underbrace{X_{t+1}, \ldots, X_{t+h}}_{\text{unobserved}})\|_e$$

belongs to the conditioning set and $\boldsymbol{X}_t$ embeds future variables

- An a priori is needed regarding the behavior of $X_{t+1}, \ldots, X_{t+h}$ to choose $B$
- Ideally we would like to **exclude the future from the conditioning set**

# A simple solution with complex implications

A simple solution is to consider, for any sequence $(X_{t-m}, \dots, X_{t+h}) \in \mathbb{R}^{m+h+1}$,

$$\|(X_{t-m}, \dots, X_t, X_{t+1}, \dots, X_{t+h})\| = \|(X_{t-m}, \dots, X_t, 0, \dots, 0)\|$$

However, $\| \cdot \|$ is not positive definite and is actually a **semi-norm**

From a topological point of view, the unit-sphere homeomorphically comes down to

$$C_{m+h+1}^{\|\cdot\|} = \{\boldsymbol{s} \in \mathbb{R}^d : ||\boldsymbol{s}|| = 1\},$$

a **unit-cylinder**

# Two questions naturally arise

1. Can we obtain proper representation of $\alpha$-stable vectors on the unit-cylinder ?

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   Yes but not for all processes you need to have at least one $d_k \neq 0$, with $k > 0$

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   $\Rightarrow$ Causal processes are ruled out

2. Can we derive tail conditional distributions under this semi-norm representation?

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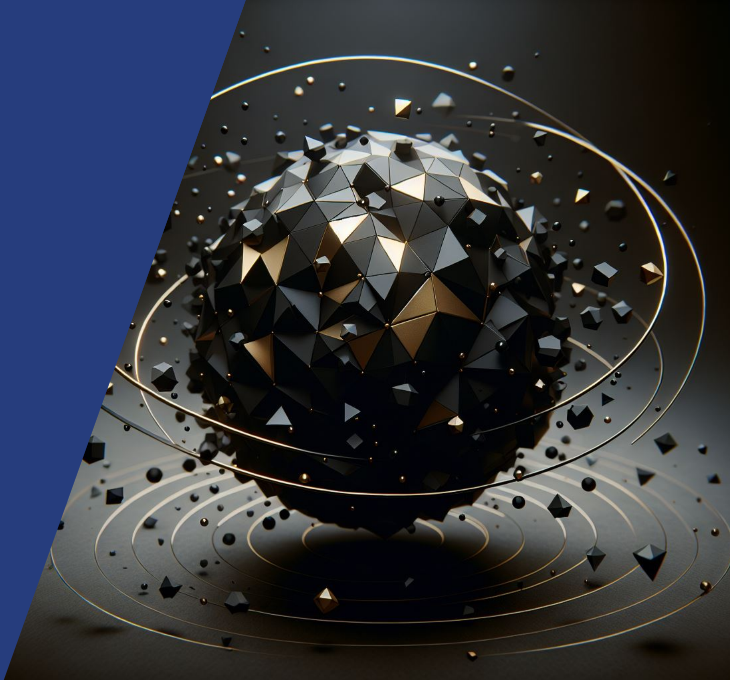
   $\Rightarrow$ Causal processes are ruled out

2. Can we derive tail conditional distributions under this semi-norm representation?
   yes

# Stable vectors on $C_d^{\|\cdot\|}$

*General theoretical results*



# Definition of stable vectors on $C_d^{\|\cdot\|}$

**Definition 1**

*Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be an $\alpha$-stable random vector. For the $S\alpha S$ case, we say that $\boldsymbol{X}$ is representable on $C_d^{\|\cdot\|}$ if there exists a Borel measure $\Gamma^{\|\cdot\|}$ on $C_d^{\|\cdot\|}$ satisfying for all $\boldsymbol{u} \in \mathbb{R}^d$*

$$\int_{C_d^{\|\cdot\|}} |\langle \boldsymbol{u}, \boldsymbol{s} \rangle|^{\alpha} \Gamma^{\|\cdot\|}(d\boldsymbol{s}) < +\infty, \tag{3}$$

*such that the characteristic function of $\boldsymbol{X}$ can be written as in (2) with $(S_d, \Gamma)$ replaced by*

$$(C_d^{\|\cdot\|}, \Gamma^{\|\cdot\|}).$$

- As cylinders are unbounded sets, the integrability condition ensures the sanity of the def.

# Representation of stable vectors on $C_d^{\|\cdot\|}$

**Theorem 2**

*Denote* $K^{\|\cdot\|} = \{x \in S_d : \|x\| = 0\}$ *and let* $\boldsymbol{X}$ *be a* $S\alpha S$ *on* $\mathbb{R}^d$ *with spectral measure* $\Gamma$ *on* $S_d$. *Then,*

$$\boldsymbol{X} \text{ is representable on } C_d^{\|\cdot\|} \iff \Gamma(K^{\|\cdot\|}) = 0.$$

*Moreover, if* $\boldsymbol{X}$ *is representable on* $C_d^{\|\cdot\|}$, *its spectral measure is then given by* $\Gamma^{\|\cdot\|}$ *where*

$$\Gamma^{\|\cdot\|}(d\boldsymbol{s}) = \|\boldsymbol{s}\|_e^{-\alpha}\, \Gamma \circ T_{\|\cdot\|}^{-1}(d\boldsymbol{s})$$

*with* $T_{\|\cdot\|} : S_d \setminus K^{\|\cdot\|} \longrightarrow C_d^{\|\cdot\|}$ *defined by* $T_{\|\cdot\|}(\boldsymbol{s}) = \boldsymbol{s}/\|\boldsymbol{s}\|$

- Unit cylinders do not span all directions of $\mathbb{R}^d$ and encode less information
- The representation exists if these directions are irrelevant to characterize the distribution

# Toward tail conditional distribution on $C_d^{\|\cdot\|}$

**Lemma 1**

*Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be an $\alpha$-stable random vector and let $\|\cdot\|$ be a seminorm on $\mathbb{R}^d$. If $\boldsymbol{X}$ is representable on $C_d^{\|\cdot\|}$, then for every Borel sets $A, B \subset C_d^{\|\cdot\|}$ with $\Gamma^{\|\cdot\|}\Big(\partial(A\cap B)\Big) = \Gamma^{\|\cdot\|}(\partial B) = 0$, and $\Gamma^{\|\cdot\|}(B) > 0$,*

$$\mathbb{P}_x^{\|\cdot\|}(\boldsymbol{X}, A|B) := \mathbb{P}\left(\frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \in A \,\middle|\, \|\boldsymbol{X}\| > x, \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \in B\right) \underset{x\to+\infty}{\longrightarrow} \frac{\Gamma^{\|\cdot\|}(A\cap B)}{\Gamma^{\|\cdot\|}(B)},$$

*where $\partial B$ (resp. $\partial(A\cap B)$) denotes the boundary of $B$ (resp. $A\cap B$)*

- Under our representation Theorem, the result of Taqqu (1994) can be recovered

# Semi-norm representation of stable moving averages

*Theoretical results for trajectories*

# Representation of stable moving averages on $C_d^{\|\cdot\|}$

**Lemma 2**

*Let $\boldsymbol{X}_t = (X_{t-m}, \dots, X_{t+h}) \in \mathbb{R}^{m+h+1}$ and $\|\cdot\|$ a semi-norm on $\mathbb{R}^{m+h+1}$. In the $S\alpha S$ case, $\boldsymbol{X}_t$ is representable on $C_{m+h+1}^{\|\cdot\|}$ if and only if*

$$\forall k \in \mathbb{Z}, \quad \Big[(d_{k+m}, \dots, d_k) = \mathbf{0} \quad \Longrightarrow \quad \forall \ell \leq k-1, \quad d_\ell = 0\Big].$$

$\Rightarrow$ If a piece of the past trajectory of $X_t$ is null, the whole future trajectory has to be

$$(d_{k+m}, \dots, d_k, \dots, d_\ell) = \mathbf{0}$$

- At this stage, this results is quite intriguing and not necessarily clear-cut

# The past-representability property

The past-representability condition **fails** if for some $m$

$$(d_{k+m}, \ldots, d_k) = \mathbf{0}$$

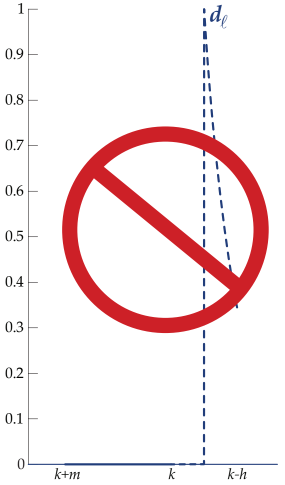
and for some $\ell \in \mathbb{Z}$ we have $d_\ell \neq 0$ such that

$$X_{t+1} = \underbrace{d_\ell}_{\neq 0} \varepsilon_{t+1+\ell} + \sum_{k \neq \ell} d_k \varepsilon_{t+1+k},$$

thereby implying that $\varepsilon_{t+1+\ell}$ is independent of $X_{t-m}, \ldots, X_t$

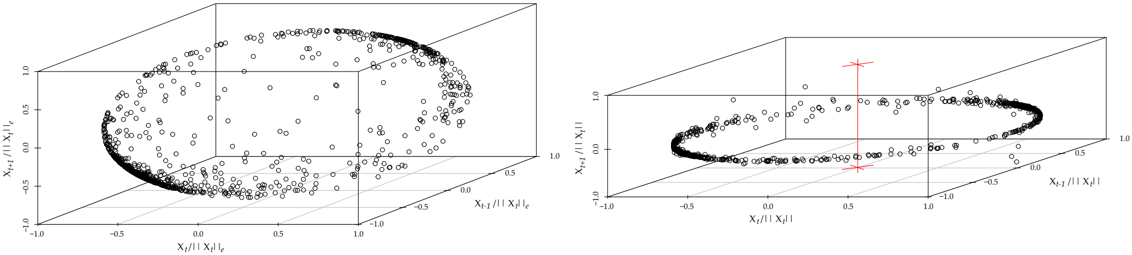
$\Rightarrow$ Observed path is uninformative about extreme events in $X_{t+1}$

$\Rightarrow$ Non-anticipative processes are ruled-out



# Application to anticipative stable-AR(2)

**Figure:** Unit cylinder and unit sphere representations of $X_t = 0.7X_{t+1} + 0.1X_{t+2} + \varepsilon_t$



$\Rightarrow$ $\Gamma^{\|\cdot\|}$ successfully encodes all information contained in $S_2$ sphere on $C_2^{\|\cdot\|}$

$\Rightarrow$ This confirms that the directions of $\mathbb{R}^3$ not spanned by the unit-cylinder are irrelevant

$\Rightarrow$ Extreme realizations of $X_{t+1}$ never occur conditionally to small realisations of $X_{t-1}$ and $X_t$.

# Tail conditional distribution of stable MA($\infty$)

## Proposition 1

*Let $\boldsymbol{X}_t \in \mathbb{R}^{m+h+1}$ be a piece of trajectory of a past-representable stable MA($\infty$), then*

$$\mathbb{P}_x^{\|\cdot\|}\left(\boldsymbol{X}_t, A \middle| B(V)\right) \underset{x\to+\infty}{\longrightarrow} \frac{\Gamma^{\|\cdot\|}\left(\left\{\frac{\vartheta \boldsymbol{d}_k}{\|\boldsymbol{d}_k\|} \in A : \quad \frac{\vartheta f(\boldsymbol{d}_k)}{\|\boldsymbol{d}_k\|} \in V\right\}\right)}{\Gamma^{\|\cdot\|}\left(\left\{\frac{\vartheta \boldsymbol{d}_k}{\|\boldsymbol{d}_k\|} \in C^{\|\cdot\|}_{m+h+1} : \quad \frac{\vartheta f(\boldsymbol{d}_k)}{\|\boldsymbol{d}_k\|} \in V\right\}\right)},$$

*for any Borel sets $A \subset C^{\|\cdot\|}_{m+h+1}$, $V \subseteq S^{\|\cdot\|}_{m+1}$ such that $\left\{\frac{\vartheta \boldsymbol{d}_k}{\|\boldsymbol{d}_k\|} \in C^{\|\cdot\|}_{m+h+1} : \quad \frac{\vartheta f(\boldsymbol{d}_k)}{\|\boldsymbol{d}_k\|} \in V\right\} \neq \emptyset$,*
*$\Gamma^{\|\cdot\|}\left(\partial(A \cap B(V))\right) = \Gamma^{\|\cdot\|}(\partial B(V)) = 0$, where $B(V) = V \times \mathbb{R}^h$ and $f$ is a transformation function.*

# Toward path prediction

**Remark**

Setting $V = S_{m+1}^{\|\cdot\|} \implies B(V) = C_{m+1}^{\|\cdot\|}$ and $A$ a small closed neighborhood of $(\vartheta \boldsymbol{d}_k / \|\boldsymbol{d}_k\|)$

$$\lim_{x \to +\infty} \mathbb{P}\left( \boldsymbol{X}_t / \|\boldsymbol{X}_t\| \in A \,\middle|\, \|\boldsymbol{X}_t\| > x \right) = 1$$

$\Rightarrow$ Far from central values, the **observed path**

$$(X_{t-m}, \ldots, X_t, X_{t+1}) / \|\boldsymbol{X}_t\|$$

necessarily **features patterns of the same shape** as some finite piece

$$\vartheta(d_{k+m}, \ldots, d_k) / \|\boldsymbol{d}_k\|$$

- $k$ points to which piece of the moving average's coefficient it corresponds
- $\vartheta \in \{-1, +1\}$ indicates whether the pattern is flipped upside down if $\varepsilon_\tau < 0$, $\tau > t$

# Path prediction strategy

## Forecasting procedure

($\iota$) Carefully define the Borel sets $A$ and $B(V)$

($\iota\iota$) When $(X_{t-m},\dots,X_{t-1},X_t)$ is large with respect to the semi-norm, use the fact that

$$(X_{t-m},\dots,X_{t-1},X_t)/\|\boldsymbol{X}_t\| = \vartheta(d_{k+m},\dots,d_{k+1},d_k)/\|\boldsymbol{d}_k\|$$

to identify to wich finite piece

$$\vartheta_0(d_{k_0+m},\dots,d_{k_0+1},d_{k_0})/\|\boldsymbol{d}_{k_0}\|$$

of the MA($\infty$) sequence, $\boldsymbol{X}_t$ corresponds

($\iota\iota\iota$) Then, for $V_0$ any small closed neighbourhood of $\vartheta_0 f(\boldsymbol{d}_{k_0})/\|\boldsymbol{d}_{k_0}\|$, compute

$$\mathbb{P}_x^{\|\cdot\|}\left(\boldsymbol{X}_t, A \middle| B(V_0)\right)$$

# Path prediction and uncertainty

- In practice, only noisy observations are available and we can only achieve

$$(X_{t-m}, \ldots, X_{t-1}, X_t)/\|\boldsymbol{X}_t\| \approx \vartheta(d_{k+m}, \ldots, d_{k+1}, d_k)/\|\boldsymbol{d}_k\|$$

  on a realised trajectory

- Even if the observed path can be confidently identified with a particular pattern in

$$\vartheta \boldsymbol{d}_k/\|\boldsymbol{d}_k\|,$$

  **in general, uncertainty** regarding the future trajectory **remains**

  $\Rightarrow$ several patterns can coincide on their first $m+1$ components, but differ by the last $h$

- The tail conditional distribution is obtained as the semi-norm of $\boldsymbol{X}_t$ grows to $\infty$

  $\Rightarrow$ only an approximation of the true dynamics during extreme events

# Application to some particular stable MA($\infty$)

*Path prediction in particular cases*

# The tail conditional distribution of anticipative AR(1)

**Proposition 2**

*Let* $X_t = \rho X_{t+1} + \varepsilon_t$*. Then, the following hold when* $m \geq 1$ *and if* $0 \leq k_0 \leq h$

$$\mathbb{P}_x^{\|\cdot\|}\left(\boldsymbol{X}_t, A_{\vartheta,k} \middle| B(V_0)\right) \underset{x\to\infty}{\longrightarrow} \begin{cases} |\rho|^{\alpha k}(1-|\rho|^{\alpha})\delta_{\vartheta_0}(\vartheta), & 0 \leq k \leq h-1, \\ |\rho|^{\alpha h}\delta_{\vartheta_0}(\vartheta), & k = h. \end{cases}$$

*with* $A_{\vartheta,k}$ *a closed neighborhood of* $\frac{\vartheta \boldsymbol{d}_k}{\|\boldsymbol{d}_k\|}$ *which does not contain any other charged point of* $\Gamma^{\|\cdot\|}$

- The crash date is not observed and can happen either in the next $h-1$ periods, or after $h$
- The probability that the bubble will crash in $k$ periods is $|\rho|^{\alpha k}(1-|\rho|^{\alpha})$
- The probability that the bubble will last at least $h$ more periods is $|\rho|^{\alpha h}$

# The anticipative AR(2)

The anticipative AR(2) is the strictly stationary solution of

$$(1-\lambda_1 F)(1-\lambda_2 F)X_t = \varepsilon_t, \qquad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0), \qquad X_t F^j = X_{t+j}$$

and admits the moving average representation $X_t = \sum_{k\in\mathbb{Z}} d_k \varepsilon_{t+k}$ with

$$d_k = \begin{cases} \dfrac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2} \mathbb{1}_{\{k\geq 0\}}, & \text{if} \quad \lambda_1 \neq \lambda_2, \\ (k+1)\lambda^k \mathbb{1}_{\{k\geq 0\}}, & \text{if} \quad \lambda_1 = \lambda_2 = \lambda. \end{cases}$$

where $0 < |\lambda_i| < 1$ for $i = 1, 2$

# The tail conditional distribution of anticipative AR$(2)$

**Proposition 3**

*Let $\boldsymbol{X}_t$ an anticipative AR(2), $m \geq 1$, $h \geq 1$, and $\boldsymbol{d}_k = (d_{k+m}, \ldots, d_k, d_{k-1}, \ldots, d_{k-h})$. For some $\vartheta_0 \in S_1$, $k_0 \geq -m$, and $B(V_0) = V_0 \times \mathbb{R}^h$, then,*

$$\mathbb{P}_x^{\|\cdot\|}\left(\boldsymbol{X}_t, A \middle| B(V_0)\right) \underset{x \to \infty}{\longrightarrow} \begin{cases} 1, & \text{if } \dfrac{\vartheta_0 \boldsymbol{d}_{k_0}}{\|\boldsymbol{d}_{k_0}\|} \in A, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

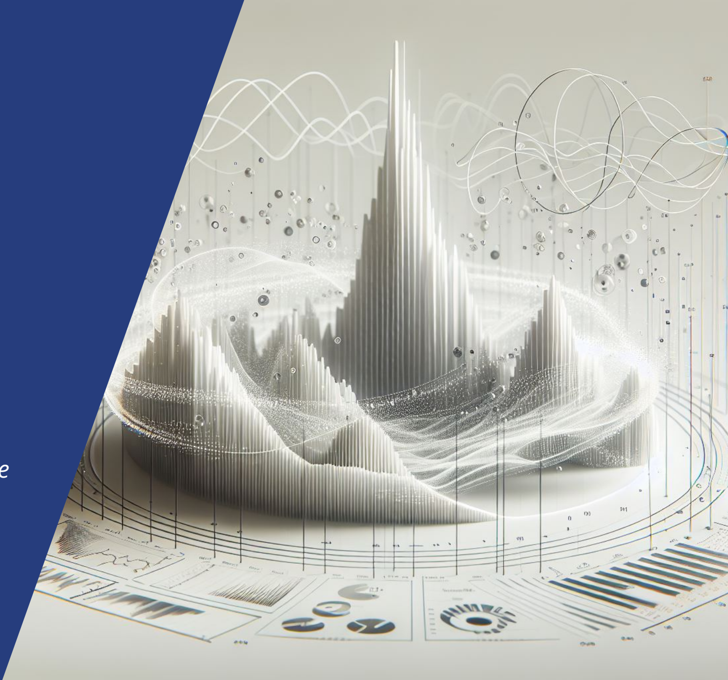
*for any closed neighbourhood $A \subset C_{m+h+1}^{\|\cdot\|}$ such that*

$$\partial A \cap \{\vartheta \boldsymbol{d}_k / \|\boldsymbol{d}_k\| : \ \vartheta \in S_1, \ k \geq -m\} = \emptyset.$$

- When $X_t$ is **anticipative enough**, one can **infer in advance** the peak and crash dates with very high confidence, in principle, **with certainty !**

# Monte Carlo study

*Forecasting procedures in practice*



# Forecasting crash probabilities

We first investigate a crash-probability forecasting procedure

- We generate 1000 trajectories of

$$X_t = 0.7X_{t+1} + 0.1X_{t+2} = \varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.5, 1, 0.5, 0), \quad t = 1, \ldots, 10^6$$

- We consider $h = \{1, 5, 10\}$ and $m = 1$ such that $\boldsymbol{X}_t = (X_{t-1}, X_t, X_{t+h})$ and

$$A = B(V_0) \times [-\delta, \delta], \quad B(V_0) = \left\{ \frac{\vartheta_0 \boldsymbol{d}_{k_0-1}}{\|\boldsymbol{d}_{k_0}\|} \pm 0.1 \right\} \times \left\{ \frac{\vartheta_0 \boldsymbol{d}_{k_0}}{\|\boldsymbol{d}_{k_0}\|} \pm 0.1 \right\}, \quad \delta = 0.3$$

- The semi-norm is defined as $\|\boldsymbol{X}_t\| = \sqrt{X_t^2 + X_{t-1}^2}$ and said large when $\|\boldsymbol{X}_t\| \geq 2q_\alpha$
- $q_\alpha \in \{0.9, 0.99, 0.999, 0.9999\}$, is a theoretical quantile of the marginal distribution of $X_t$

# Simulation results

- The "empirical" probability on the left-hand side of Proposition 4 is compute as

$$\widehat{p}_q = \frac{\sum_{t=1}^{N-h} \mathbb{1}\left(\left\{\frac{(X_{t-1},X_t)}{\|\boldsymbol{X}_t\|} \in B(V_0)\right\} \cap \left\{\frac{X_{t+h}}{\|\boldsymbol{X}_t\|} \leq \delta\right\} \cap \{\|\boldsymbol{X}_t\| > 2q\}\right)}{\sum_{t=1}^{N-h} \mathbb{1}\left(\left\{\frac{(X_{t-1},X_t)}{\|\boldsymbol{X}_t\|} \in B(V_0)\right\} \cap \{\|\boldsymbol{X}_t\| > 2q\}\right)} \quad (5)$$

and the "theoretical" one (right-hand side) similarly but using $\frac{d_{k_0+h}}{\|\boldsymbol{d}_{k_0}\|} \leq \delta$ instead of $\frac{X_{t+h}}{\|\boldsymbol{X}_t\|} \leq \delta$

**Table:** Comparison of theoretical and empirical crash probabilities of bubbles generated by the anticipative AR(2)

| | $h=1$ | $h=5$ | $h=10$ |
|---|---|---|---|
| $p_{0.9}\backslash\hat{p}_{0.9}$ | 84.09\22.75 (22.39-23.14) | 93.00\39.70 (39.27-40.14) | 98.45\46.76 (46.32-47.21) |
| $p_{0.99}\backslash\hat{p}_{0.99}$ | 91.56\89.11 (87.90-90.36) | 95.55\94.63 (93.67-95.56) | 95.71\96.85 (96.11-97.56) |
| $p_{0.999}\backslash\hat{p}_{0.999}$ | 99.50\98.75 (97.04-100) | 99.40\99.40 (98.21-100) | 99.72\99.67 (98.63-100) |
| $p_{0.9999}\backslash\hat{p}_{0.9999}$ | 99.96\99.86 (96.42-100) | 99.91\99.92 (99.90-100) | 99.97\99.98 (100-100) |

# Forecasting crash dates

In this second numerical analysis, we study a crash-date forecasting procedure

- We generate 1000 trajectories of the following $\mathcal{S}\alpha\mathcal{S}$ AR(2)

$$X_t = 0.7X_{t+1} + 0.1X_{t+2}\varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, 0, 0.1, 0), \quad t = 1, \ldots, N$$

- We identify a positive bubble peak as $max(x_t)$, consider $m = \{1, 3, 5, 7, 9, 11\}$ and

... treat as unobserved all future values and $\lceil n \times 0.01 \rceil$ periods preceding the bubble burst

$$k_0 = \{3, 5, 10\}$$

$\Rightarrow$ This impacts to which quantile $X_t$ is likely to belong to

| $N/\alpha$ | 0.9 | 1.2 | 1.5 | 1.8 |
|---|---|---|---|---|
| 250 $\Rightarrow$ $q_{X_t}$ Very High | 0.99 | 0.99 | 0.99 | 0.94 |
| 500 $\Rightarrow$ $q_{X_t}$ High | 0.98 | 0.98 | 0.94 | 0.89 |
| 1000 $\Rightarrow$ $q_{X_t}$ Moderately | 0.97 | 0.96 | 0.91 | 0.78 |

- We compute the bias as the difference between the true crash date and the predicted one

# Simulation results

**Table:** Bias for the crash date predictor

| | $m=1$ | | | | $m=3$ | | | |
|---|---|---|---|---|---|---|---|---|
| $q_{X_t}/\alpha$ | 0.9 | 1.2 | 1.5 | 1.8 | 0.9 | 1.2 | 1.5 | 1.8 |
| Very High | -0.9785 | -0.3985 | 0.0262 | 0.2199 | -0.7320 | -0.2420 | -0.0073 | 0.2815 |
| High | 0.7174 | 1.3771 | 1.9292 | 2.2544 | 0.9421 | 1.6189 | 2.0938 | 2.2914 |
| Moderately High | 5.8112 | 6.6166 | 7.1317 | 7.4263 | 6.0680 | 6.8229 | 7.1698 | 7.3565 |
| | $m=5$ | | | | $m=7$ | | | |
| $q_{X_t}/\alpha$ | 0.9 | 1.2 | 1.5 | 1.8 | 0.9 | 1.2 | 1.5 | 1.8 |
| Very High | -0.5457 | -0.2076 | 0.0483 | 0.2300 | -0.5043 | -0.1256 | 0.1099 | 0.2715 |
| High | 1.2378 | 1.7442 | 2.1075 | 2.3412 | 1.2978 | 1.8118 | 2.0987 | 2.2571 |
| Moderately High | 6.2749 | 6.9284 | 7.2065 | 7.3582 | 6.3193 | 6.9760 | 7.2655 | 7.3763 |
| | $m=9$ | | | | $m=11$ | | | |
| $q_{X_t}/\alpha$ | 0.9 | 1.2 | 1.5 | 1.8 | 0.9 | 1.2 | 1.5 | 1.8 |
| Very High | -0.4079 | -0.0811 | 0.1556 | 0.2976 | -0.4200 | -0.0480 | 0.1891 | 0.3300 |
| High | 1.3407 | 1.8471 | 2.1537 | 2.2857 | 1.3633 | 1.8568 | 2.1417 | 2.3565 |
| Moderately High | 6.3805 | 7.0095 | 7.2599 | 7.4021 | 6.4407 | 7.0253 | 7.3097 | 7.4745 |

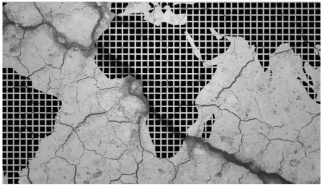
# Empirical illustration

*Forecasting climate anomalies*

# Forecasting climate anomalies

- A growing literature highlights the impact of climate variables on economic performance
- Weather shocks have an impact on growth, inflation, energy and commodity agricultural returns
- A common proxy is the Southern Oscillation Index (SOI)
- Forecasting El Niño/La Niña anomalies is of primary interest from extreme weather warnings to agricultural planning

THE CONVERSATION



En 2015-2016, El Niño avait causé une crise alimentaire touchant 40 millions de personnes en Afrique australe.

**Le retour d'El Niño apporte insécurité alimentaire et instabilité macroéconomique en Afrique australe**

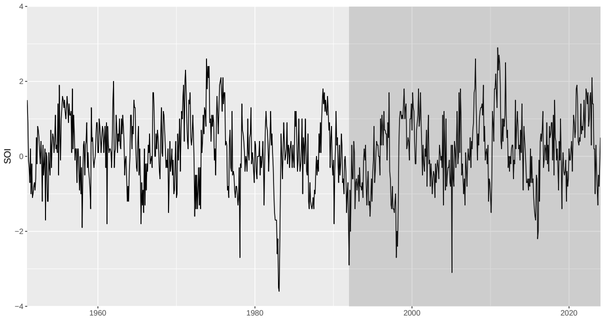
Publié: 28 mars 2024, 17:57 CET

Florian Morvillier, Erica Perego, Fanny Schaeffer, *CEPII*

# Data

- $SOI_t$ is a monthly variable based on air-pressure differentials in the South Pacific
- El Niño (resp. La Niña) anomalies: $SOI_t > 1$ (resp. $SOI_t < -1$) during at least the 3 periods

**Figure:** Southern Oscillation Index (SOI)



# Estimation results

- We estimate a stable anticipative AR(2): $X_t = \varphi_1 X_{t+1} + \varphi_2 X_{t+2} + \varepsilon$for the $SOI_t$ over the period 01/1951 - 12/1991

**Table:** AR(2) estimation for SOI

| $\varphi_1$ | $\varphi_2$ | $\alpha$ | $\beta$ | $\sigma$ | $\mu$ |
|---|---|---|---|---|---|
| 0.44$^{***}$ | 0.30$^{***}$ | 1.88$^{***}$ | -0.48$^{*}$ | 0.46$^{***}$ | 0.01$^{*}$ |
| (1.50E-04) | (1.00E-04) | (0.06) | (0.27) | (3.00E-03) | (0.02) |

*Notes:* Standard deviations are in parentheses. Asterisks $^*$, $^{**}$, and $^{***}$ indicate significance at the 90%, 95% and 99% level, respectively.

# The anticipative AR$(2)$

The anticipative AR(2) is the strictly stationary solution of

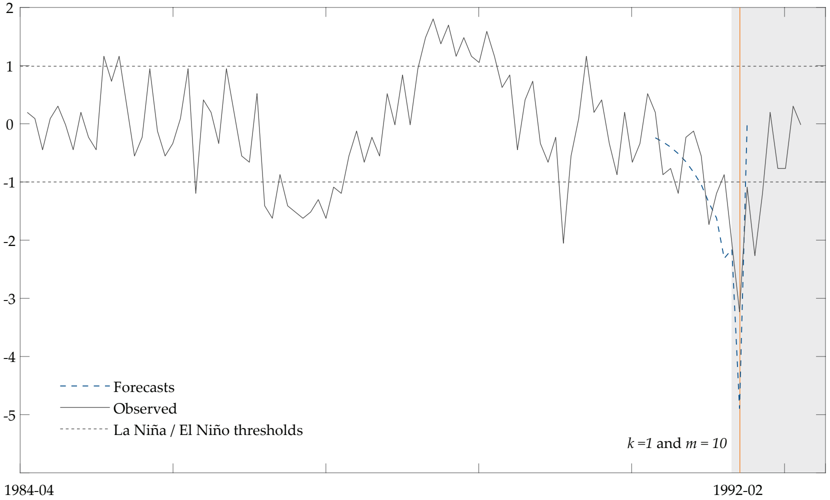
$$(1-\lambda_1 F)(1-\lambda_2 F)X_t = \varepsilon_t, \qquad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0), \qquad X_t F^j = X_{t+j}$$

and admits the moving average representation $X_t = \sum_{k\in\mathbb{Z}} d_k \varepsilon_{t+k}$ with

$$d_k = \begin{cases} \dfrac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2} \mathbb{1}_{\{k\geq 0\}}, & \text{if} \quad \lambda_1 \neq \lambda_2, \\ (k+1)\lambda^k \mathbb{1}_{\{k\geq 0\}}, & \text{if} \quad \lambda_1 = \lambda_2 = \lambda. \end{cases}$$

where $0 < |\lambda_i| < 1$ for $i = 1, 2$

# Forecast of the first out-of-sample La Niña reversal



# Forecast of the first out-of-sample La Niña reversal

**Table:** Forecasting out-of-sample El Niño and La Niña anomalies

| Type of anomaly | El Niño | El Niño | La Niña | El Niño | La Niña | La Niña | El Niño | La Niña | La Niña | La Niña | La Niña |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Start date | 12/1991 | 07/1994 | 11/2007 | 12/2009 | 07/2010 | 11/2010 | 07/2015 | 11/2021 | 02/2022 | 08/2022 | 11/2022 |
| Peak date | 01/1992 | 09/1994 | 02/2008 | 02/2010 | 09/2010 | 12/2010 | 10/2015 | 01/2021 | 03/2022 | 10/2022 | 12/2022 |
| End date | 04/1992 | 10/1994 | 03/2008 | 03/2010 | 11/2010 | 04/2011 | 11/2015 | 03/2021 | 05/2022 | 11/2022 | 02/2023 |
| Forecasted Peak | 01/1992 | 09/1994 | 02/2008 | 03/2010 | 08/2010 | 01/2011 | 09/2015 | 01/2021 | 04/2022 | 10/2022 | 01/2023 |
| Forecasted End | 02/1992 | 10/1994 | 03/2008 | 04/2010 | 09/2010 | 02/2011 | 10/2015 | 02/2021 | 05/2022 | 11/2022 | 02/2023 |
| Peak forecast error | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 0 | 1 | 0 | -1 |
| End forecast error | -2 | 0 | 0 | 1 | -1 | -2 | -1 | -1 | 0 | 0 | 0 |
| $k_0$ | 1 | 2 | 3 | 3 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| $m$ | 10 | 10 | 10 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

For the 14 El Niño/La Niña occurrences, our procedure leads to

- an average error of 0.42 months in finding the peak date
- an average error of 0.57 months in finding the end date

# Extensions

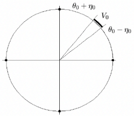
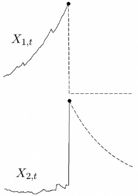
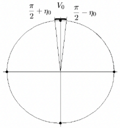
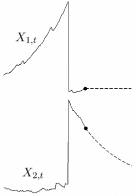
# A step toward multivariate

A simple bi-dimensional process define $(\boldsymbol{X}_t)$ for all $t \in \mathbb{Z}$ as

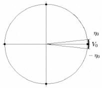
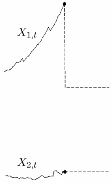
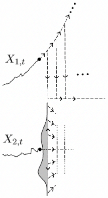
$$\begin{cases} \boldsymbol{X}_t &= (X_{1,t}, X_{2,t})', \\ X_{1,t} &= \rho_1 X_{1,t+1} + \varepsilon_{1,t}, \\ X_{2,t} &= \rho_2 X_{2,t-1} + \varepsilon_{2,t}, \\ \boldsymbol{\varepsilon}_t &= (\varepsilon_{1,t}, \varepsilon_{2,t})' \end{cases}$$

i.i.d. $S\alpha S$ with spectral measure $\Gamma_2$ on $S_2$ and zero shift vector

# A step toward multivariate

| Observation $\frac{(X_{1,t},X_{2,t})}{\sqrt{X_{1,t}^2+X_{2,t}^2}} \in V_0$ | Potential outcomes (neighbourhood of) | Conditional probability | Trajectorial interpretation |
|---|---|---|---|
|   $X_{1,t}$ and $X_{2,t}$ both extreme | $(\cos u, \sin u, 0, \rho_2 \sin u)$ $u \in [\theta_0-\eta_0, \theta_0+\eta_0]$ | 1 |   Bubble peak signaled by jump |
|   $\lvert X_{1,t}\rvert << \lvert X_{2,t}\rvert$ | $(0, 1, 0, \rho_2)$ | 1 |   Post crash, jump decaying |

# A step toward multivariate

| Observation $\frac{(X_{1,t},X_{2,t})}{\sqrt{X_{1,t}^2+X_{2,t}^2}} \in V_0$ | Potential outcomes (neighbourhood of) | Conditional probability | Trajectorial interpretation |
|---|---|---|---|
|   $\lvert X_{1,t}\rvert >> \lvert X_{2,t}\rvert$ | $(1,0,0,0)$ | $\frac{\omega_1}{\omega_1+\omega_2}$ |   Bubble peak, no jump signal |
| | $(1,0,\rho_1^{-1},z)$ for $z \in \mathbb{R}$ | $\frac{\omega_2}{\omega_1+\omega_2}$ |   Pre-peak bubble inflation, potential peak/jump at $t+1$ |

Thank for your attention!

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