Stochastic correlated equilibrium with terminal commitment

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Introduction to Correlated Equilibrium

Correlated Equilibrium is a generalization of Nash Equilibrium in Game Theory, introduced by Robert Aumann.

- In a Nash Equilibrium, players make independent decisions based on their own payoffs.
- In a Correlated Equilibrium, players follow a recommendation from a trusted source (a mediator), which correlates their choices.

Mechanism of Correlated Equilibrium

- A mediator sends private signals to players.
- Each player receives a recommendation on what strategy to choose.
- Players follow the recommendation if they have no incentive to deviate given the signals.

Formally, a strategy profile is in Correlated Equilibrium if:

$$\text{for all players } i, \quad \mathbb{E}\left[u_i(s_i, s_{-i})\right] \geq \mathbb{E}\left[u_i({\color{red}\phi_i}(s_i), s_{-i})\right], \quad \forall \phi_i$$

where s_i is the recommended action and ϕ_i is a function of deviation.

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All Nash equilibria are Correlated equilibria. In correlated equilibria, the players are only responding to the private signals.

Example of Correlated Equilibrium: Traffic Lights

Example: Traffic Light Coordination

- Two drivers are approaching an intersection from different directions.
- If both proceed, they may crash. If one waits while the other goes, both will avoid a collision.
- The traffic light acts as a mediator, signaling "Green" to one driver and "Red" to the other.
- If each driver follows the light's recommendation, they both avoid a collision and minimize waiting time.

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Each driver responds solely to the traffic lights, without considering the actions or position of the other vehicle.

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Classical Calculus of Variations: A Quick Recap

In classical calculus of variations, we seek to find the deterministic path $x(t)$ that minimizes the action functional:

$$S[x] = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

where $L(x, \dot{x}, t)$ is the Lagrangian. The path that minimizes the action satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

Yasue's approach = Stochastic correlated equilibrium

Yasue's Calculus of Variations is an extension of classical calculus of variations to diffusion processes. The objective is to find a signal process in the form of diffusion:

$$dX_t = b(X_t, t)dt + \sigma dW_t,$$

from which the player does NOT want to deviate in order to minimize his action:

$$X = \underset{Z_t=\phi_t(X_t),\ Z_T=X_T}{\arg\min} \mathbb{E}\left[\int_0^T L(Z_t, D_t^+ Z_t, D_t^- Z_t, t)dt\right],$$

where

$$D_t^+ Z_t := \lim_{h\to 0} \frac{1}{h}\mathbb{E}[Z_{t+h} - Z_t | X_t], \quad D_t^+ Z_t := \lim_{h\to 0} \frac{1}{h}\mathbb{E}[Z_{t-h} - Z_t | X_t].$$

Stochastic Euler-Lagrange equation

The key of Yasue's calculus of variations lies on the following integral by parts formula:

$$\frac{d}{dt}\mathbb{E}[X_t Y_t] = \mathbb{E}\Big[Y_t D_t^+ X_t + X_t D_t^- Y_t\Big].$$

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Then, by computing the variations of the action:

$$\mathbb{E}\left[\int_0^T L(Z_t, D_t^+ Z_t, D_t^- Z_t, t)dt\right],$$

we simply obtain the stochastic Euler Lagrange equation:

$$\big(\partial_1 L - D_t^- \partial_2 L - D_t^+ \partial_3 L\big)(X_t, D_t^+ X_t, D_t^- X_t, t) = 0$$

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Derivation of PDEs: example of Schrödinger equation

For the action $L(X, D^+X, D^-X) = V(X) + \frac{1}{4}|D^+X|^2 + \frac{1}{4}|D^-X|^2$, the stochastic Euler Lagrange equation states:

$$\nabla V = \frac{1}{2}D_t^- D_t^+ X_t + \frac{1}{2}D_t^+ D_t^- X_t, \quad \text{Nelson's Newton's law.}$$

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To find the PDEs governing the correlated equilibrium, we need to note

$$D_t^+ f(t, X_t) = \mathcal{L} f(t, X_t) = \big(\partial_t f + \frac{1}{2}\sigma^2 \Delta^2 f + b \cdot \nabla f\big)(t, X_t),$$

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Assuming $\exists R, S$ s.t. $R = \frac{1}{2}\ln m$, $\nabla S = b$, then $\Psi := e^{R+iS}$ satisfies the Schrödinger equation:

$$-i\partial_t \Psi = \frac{1}{2}\Delta\Psi - V\Psi, \quad \text{Nelson's stochastic mechanics (1965).}$$

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Without backward kinetic energy

Consider the action $L(X, D^+X, D^-X) = V(X) + \frac{1}{2}|D^+X|^2$ and $\sigma = 1$. The corresponding stochastic Euler-Lagrange equation reads:

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where m satisfies the Fokker-Planck equation:

$$\partial_t m = -\nabla \cdot \big((b - \nabla \log m)m\big).$$

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Denote by $\eta := \log m$. We find a system of forward PDEs.

Wellposedness and verification

Borrowing the elements from Hao Xing and Gordan Zitković (AOP '18) and Joe Jackson (ECP '23), we can prove that the system of PDEs admits a unique classical solution on a torus, provided that $b_0,\ \eta_0$ are both bounded and ∇V is Lipschitz.

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Given a classical solution of the system on $\mathbb{R}^d$, by further assuming that V is convex, we can verify that as the signal process

$$dX_t = b(t, X_t)dt + dW_t$$

is a stochastic correlated equilibrium.

A possible long-time behavior on torus

Recall the system of parabolic equations:

$$
\begin{aligned}
&-\partial_t \eta + \frac{1}{2}\triangle \eta - b\cdot\nabla\eta + \frac{1}{2}|\nabla\eta|^2 - \nabla\cdot b = 0, \quad \eta(0,\cdot) = \eta_0 \\
&-\partial_t b + \frac{1}{2}\triangle b + \nabla\eta\cdot\nabla b - b\nabla b + \nabla V = 0.
\end{aligned}
$$

We realize that there is a stationary solution such that $\nabla\eta^* = 2b^*$ and

$$
\frac{1}{2}\Delta\eta^* + \frac{1}{4}|\nabla\eta^*|^2 + 2V = \lambda.
$$

It is possible that $(b_t, \eta_t) \to (b^*, \eta^*)$ as $t \to \infty$.

No stationary state on $\mathbb{R}^d$

Suppose there is a stationary pair $(m = e^{\eta}, b)$, and necessarily we have $\nabla \cdot \big((b - \frac{1}{2}\nabla \log m)m\big) = 0$. Using the equations, we can obtain

$$\begin{aligned} d\int \frac{1}{2}|b_t|^2 m_t &= \int (b\partial_t b + b^2\nabla b - \frac{1}{2} b\nabla b\nabla \ln m)m \\ &= \int (b\nabla V - \frac{1}{2}|\nabla b|^2)m \\ &= -\int V\nabla\cdot(bm) - \int \frac{1}{2}|\nabla b|^2 m \\ &= -\frac{1}{2}\int (\Delta V + |\nabla b|^2)m < 0. \end{aligned}$$

1-d Linear-quadratic case

Assume $V(x) = D_0 x^2$, and initial conditions of η and b are quadratic and linear. By considering the solution in the form:

$$\eta(t,x) = -A(t)x^2 + a(t)x + c(t),$$
$$b(t,x) = B(t)x + \alpha(t).$$

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Then we may obtain the system of Riccati equations:

$$\dot{A}(t) = -2A(t)B(t) - 2A(t)^2,$$
$$\dot{B}(t) = -2B(t)A(t) - B(t)^2 + 2D_0.$$

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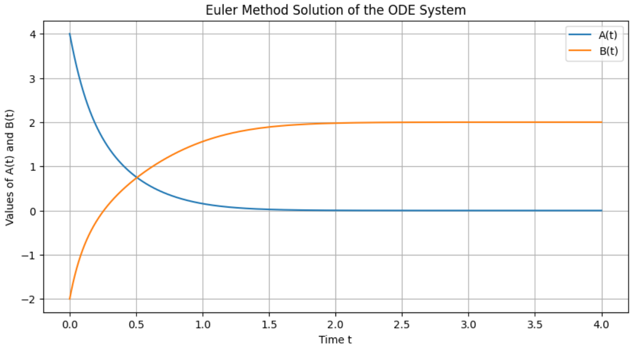

Some remarks/mysteries...

In this definition of stochastic correlated equilibrium, we need to highlight:

- $\eta_0 = \log m_0$: The initial distribution m_0 cannot be a Dirac measure, that is, the initial signal is inherently random. This randomness could arise from uncertainty in measuring the player's initial state;

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- The terminal commitment $Z_T = X_T$ could arise from the verification of the social planer. Since the solution to the PDE system does not depend on T, the verification time T does not need to be determined at the beginning.
- Note that the randomness of X_t consists of two components: the initial randomness and the randomness introduced by W. If we restart the game at time T, the randomness associated with W should be filtered out, as its realization has already been observed (analogous to a quantum collapse);

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Stochastic correlated equilibrium for N players

In a game with N players, each player receive a personal signal process X^i in the form:

$$dX_t^i = b(X_t^i, t)dt + \sigma dW_t^i.$$

At the correlated equilibrium the signal processes satisfy:

$$X^i = \arg\min_{Z_t^i = \phi_t(X_t),\ Z_T^i = X_T^i} \mathbb{E}\left[\int_0^T L(\mu_t^N, Z_t^i, D_t^+ Z_t^i, t)dt\right], \quad \mu_t^N := \frac{1}{N}\sum_{j\neq i}^N \delta_{X_t^j}.$$

Here we shall modify the definition:

$$D_t^+ Z_t^i := \lim_{h\to 0} \frac{1}{h}\mathbb{E}[Z_{t+h}^i - Z_t^i | X_t].$$

Intuition of mean field limit

For $L(\mu, X^i, D^+X^i) = V(\mu, X^i) + \frac{1}{2}|D^+X^i|^2$, the corresponding stochastic Euler-Lagrange equation for fixed μ reads:

$$\partial_x V = D_t^- D_t^+ X_t^i = D_t^- b(X_t^i, t).$$

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Recall that

$$\begin{aligned} D_t^- f(t, X_t^i) &= (\mathcal{L}_{x^i} f - \sigma^2 \nabla_{x^i} \ln m \cdot \nabla f)(t, X_t^i), \quad m_t = \mathrm{Law}(X_t) \\ &\approx (\mathcal{L}_{x^i} f - \sigma^2 \nabla_{x^i} \ln m^i \cdot \nabla f)(t, X_t^i), \quad m_t = \mathrm{Law}(X_t^i). \end{aligned}$$

Mean field equilibrium: a system of three equations

As we see, the Euler-Lagrange equation evolves the triple (X^i, m^i, μ). Therefore, the mean field equilibrium is governed by the system of the three equations ($\eta = \log m^i, \xi = \log \mu$):

$$-\partial_t \xi + \frac{1}{2}\triangle \xi - b \cdot \nabla \xi + \frac{1}{2}|\nabla \xi|^2 - \nabla \cdot b = 0, \quad \xi(0,\cdot) = \xi_0$$

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Note that the initial distribution of the population (μ_0) can be different from that of the individual (m_0^i).

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Note that the initial distribution of the population (μ_0) can be different from that of the individual (m_0^i).

Very different from the correlated mean field game studied by Campi, Fisher, Elie, Lauriere...

Thank you for your attention!