From Theoretical Results to Real-World Applications in Bonds, FX, Commodities and Cryptocurrencies: An Overview on Market Making

Olivier Guéant

FDD & FiME seminar – Sep. 2024

A personal note

Fond memories from 2009 or 2010.

What is market making?

Market makers

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:
  - Holding inventory

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:
  - Holding inventory
    $\rightarrow$ face the risk that prices move adversely without them being able to unwind their position.

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:
  - Holding inventory
    $\rightarrow$ face the risk that prices move adversely without them being able to unwind their position.
  - Information asymmetry / adverse selection by informed traders

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:
  - Holding inventory
    $\rightarrow$ face the risk that prices move adversely without them being able to unwind their position.
  - Information asymmetry / adverse selection by informed traders
    $\rightarrow$ face the risk of buying / selling when the price will go down / up.

What is market making?

Market makers

- Activity: providing bid and ask/offer prices to other market participants.
- The way they make money: capturing part of the bid-ask spreads.
- Risks:
  - Holding inventory
    $\rightarrow$ face the risk that prices move adversely without them being able to unwind their position.
  - Information asymmetry / adverse selection by informed traders
    $\rightarrow$ face the risk of buying / selling when the price will go down / up.

Remark: I mainly focused on market making in OTC markets. Not market making in limit order books (no tick size, no queue, no priority).

The early literature

The early literature

Models regarding inventory cost / management

- Stoll (1978)
- Ho and Stoll (1981, 1983)
- Amihud and Mendelson (1980)
- O'Hara and Oldfield (1986)

The early literature

Models regarding inventory cost / management

- Stoll (1978)
- Ho and Stoll (1981, 1983)
- Amihud and Mendelson (1980)
- O'Hara and Oldfield (1986)

Models regarding adverse selection

- Copeland and Galai (1983)
- Easley and O'Hara (1987)
- Glosten and Milgrom (1985)

The early literature

Models regarding inventory cost / management

- Stoll (1978)
- Ho and Stoll (1981, 1983)
- Amihud and Mendelson (1980)
- O'Hara and Oldfield (1986)

Models regarding adverse selection

- Copeland and Galai (1983)
- Easley and O'Hara (1987)
- Glosten and Milgrom (1985)

An economic literature about the determinants of bid-ask spreads in the 1980s and 1990s: Hasbrouck, Huang and Stoll, MRR, etc.

From economists to mathematicians

The financial mathematics community only got interested in market making from 2008 following the paper by Avellaneda and Stoikov.

From economists to mathematicians

The financial mathematics community only got interested in market making from 2008 following the paper by Avellaneda and Stoikov.

High-frequency trading in a limit order book

MARCO AVELLANEDA and SASHA STOIKOV*

Mathematics, New York University, 251 Mercer Street, New York, NY 10012, USA

Overview of my journey

Overview of my journey

The roots

Overview of my journey

The roots

- Post-PhD inspiration (2010): met C.-A. Lehalle (Crédit Agricole Cheuvreux) through J.-M. Lasry. Charles put in my hands Avellaneda-Stoikov's paper.

Overview of my journey

The roots

- Post-PhD inspiration (2010): met C.-A. Lehalle (Crédit Agricole Cheuvreux) through J.-M. Lasry. Charles put in my hands Avellaneda-Stoikov's paper.
  $\rightarrow$ initially more of a theoretical / mathematical exercise.

Overview of my journey

The roots

- Post-PhD inspiration (2010): met C.-A. Lehalle (Crédit Agricole Cheuvreux) through J.-M. Lasry. Charles put in my hands Avellaneda-Stoikov's paper.
  $\rightarrow$ initially more of a theoretical / mathematical exercise.

Multiple interactions with the industry

Overview of my journey

The roots

- Post-PhD inspiration (2010): met C.-A. Lehalle (Crédit Agricole Cheuvreux) through J.-M. Lasry. Charles put in my hands Avellaneda-Stoikov's paper.
  $\rightarrow$ initially more of a theoretical / mathematical exercise.

Multiple interactions with the industry

- OTC trading (neglected area of academic research): Contacted by bond dealers and FX+commodity dealers in London and NYC, for adapting models to match real-world trading environments.

Overview of my journey

The roots

- Post-PhD inspiration (2010): met C.-A. Lehalle (Crédit Agricole Cheuvreux) through J.-M. Lasry. Charles put in my hands Avellaneda-Stoikov's paper.
  $\rightarrow$ initially more of a theoretical / mathematical exercise.

Multiple interactions with the industry

- OTC trading (neglected area of academic research): Contacted by bond dealers and FX+commodity dealers in London and NYC, for adapting models to match real-world trading environments.
- DeFi: More recently contacted by decentralized finance players to build new Automated Market Makers.

Setup of the model (I)

One asset: reference price process ("mid"-price) $(S_t)_t$

Brownian dynamics

$$dS_t = \sigma dW_t.$$

$\rightarrow$ Can be the CBBT / CP+ for corporate bonds or a homemade reference price.
$\rightarrow$ Can be EBS / Refinitiv mid price or a homemade composite.

Setup of the model (II)

Bid and ask prices proposed by the MM

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

Setup of the model (II)

Bid and ask prices proposed by the MM

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

Dynamics of the inventory $(q_t)_t$

$$dq_t = \Delta dN_t^b - \Delta dN_t^a,$$

for two point processes N^b and N^a.

Setup of the model (II)

Bid and ask prices proposed by the MM

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

Dynamics of the inventory $(q_t)_t$

$$dq_t = \Delta dN_t^b - \Delta dN_t^a,$$

for two point processes N^b and N^a.

Competition and demand are modeled indirecty through the probability / intensity of jumps.

Setup of the model (III)

Intensities $(\lambda_t^b)_t$ and $(\lambda_t^a)_t$ of N^b and N^a

$$\lambda_t^b = \Lambda^b(\delta_t^b) 1_{q_{t-} < Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a) 1_{q_{t-} > -Q}.$$

They depend on the distance to the reference price: Λ^b, Λ^a decreasing (of course!)

Setup of the model (III)

Intensities $(\lambda^b_t)_t$ and $(\lambda^a_t)_t$ of N^b and N^a

$$\lambda^b_t = \Lambda^b(\delta^b_t) 1_{q_{t-} < Q} \text{ and } \lambda^a_t = \Lambda^a(\delta^a_t) 1_{q_{t-} > -Q}.$$

They depend on the distance to the reference price: Λ^b, Λ^a decreasing (of course!)

Cash process $(X_t)_t$

$$dX_t = \Delta S^a_t dN^a_t - \Delta S^b_t dN^b_t = -S_t dq_t + \delta^a_t \Delta dN^a_t + \delta^b_t \Delta dN^b_t.$$

Setup of the model (III)

Intensities $(\lambda_t^b)_t$ and $(\lambda_t^a)_t$ of N^b and N^a

$$\lambda_t^b = \Lambda^b(\delta_t^b) 1_{q_{t-} < Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a) 1_{q_{t-} > -Q}.$$

They depend on the distance to the reference price: Λ^b, Λ^a decreasing (of course!)

Cash process $(X_t)_t$

$$dX_t = \Delta S_t^a dN_t^a - \Delta S_t^b dN_t^b = -S_t dq_t + \delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b.$$

Three state variables: X (cash), q (inventory), and S (price).

PnL and objective function

PnL at time T of a market maker

$$PnL_T = X_T + q_T S_T = X_0 + q_0 S_0$$

$$+ \int_0^T \underbrace{\delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b}_{\text{spread capture}} + \underbrace{\sigma q_t dW_t}_{\text{inventory+price risk}}$$

PnL and objective function

PnL at time T of a market maker

$$PnL_T = X_T + q_T S_T = X_0 + q_0 S_0$$
$$+ \int_0^T \underbrace{\delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b}_{\text{spread capture}} + \underbrace{\sigma q_t dW_t}_{\text{inventory+price risk}}$$

The original Avellaneda-Stoikov's model considers a CARA utility (Model A):

CARA objective function

$$\sup_{(\delta_t^a)_t, (\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T)\right)\right],$$

where γ is the absolute risk aversion parameter, and $\mathcal{A}$ the set of predictable processes bounded from below.

HJB equation

In what follows, u is a candidate for the value function.

Hamilton-Jacobi-Bellman

$$\text{(HJB)} \qquad 0 = \partial_t u(t,x,q,S) + \frac{1}{2}\sigma^2 \partial^2_{SS} u(t,x,q,S)$$

$$+1_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b)\left[u(t, x - \Delta S + \Delta\delta^b, q + \Delta, S) - u(t,x,q,S)\right]$$

$$+1_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a)\left[u(t, x + \Delta S + \Delta\delta^a, q - \Delta, S) - u(t,x,q,S)\right]$$

with final condition:

$$u(T,x,q,S) = -\exp\left(-\gamma(x + qS)\right)$$

Change of variables

Change of variables

Ansatz

$$u(t, x, q, S) = -\exp(-\gamma(x + qS + \theta(t, q)))$$

Change of variables

Ansatz

$$u(t,x,q,S) = -\exp(-\gamma(x + qS + \theta(t,q)))$$

New equation

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2$$
$$+1_{q<Q} \sup_{\delta^b} \frac{\Lambda^b(\delta^b)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^b + \theta(t,q+\Delta) - \theta(t,q)\right)\right)\right)$$
$$+1_{q>-Q} \sup_{\delta^a} \frac{\Lambda^a(\delta^a)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^a + \theta(t,q-\Delta) - \theta(t,q)\right)\right)\right)$$

with final condition $\theta(T,q) = 0$.

A new transform

$$H_\xi^b(p) = \sup_\delta \frac{\Lambda^b(\delta)}{\xi\Delta}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

$$H_\xi^a(p) = \sup_\delta \frac{\Lambda^a(\delta)}{\xi\Delta}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

Equation for θ

A new transform

$$H_\xi^b(p) = \sup_\delta \frac{\Lambda^b(\delta)}{\xi\Delta}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

$$H_\xi^a(p) = \sup_\delta \frac{\Lambda^a(\delta)}{\xi\Delta}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

New equation

$$0 = \partial_t\theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q}\Delta H_\gamma^b\left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta}\right)$$

$$+1_{q>-Q}\Delta H_\gamma^a\left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

Another objective function

Variant (Cartea, Jaimungal *et al.*) with a running penalty:

Risk-neutral with running penalty (Model B)

$$\sup_{(\delta^a_t)_t,(\delta^b_t)_t\in\mathcal{A}} \mathbb{E}\left[X_T + q_T S_T - \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt\right]$$

i.e.

$$\sup_{(\delta^a_t)_t,(\delta^b_t)_t\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left(\Delta\delta^a_t \Lambda^a(\delta^a_t) 1_{q_{t-}>-Q} + \Delta\delta^b_t \Lambda^b(\delta^b_t) 1_{q_{t-}<Q} - \frac{\gamma}{2}\sigma^2 q_t^2\right) dt\right]$$

where γ is a kind of absolute risk aversion parameter.

Another objective function

Variant (Cartea, Jaimungal *et al.*) with a running penalty:

Risk-neutral with running penalty (Model B)

$$\sup_{(\delta^a_t)_t, (\delta^b_t)_t \in \mathcal{A}} \mathbb{E}\left[X_T + q_T S_T - \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt \right]$$

i.e.

$$\sup_{(\delta^a_t)_t, (\delta^b_t)_t \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left(\Delta\delta^a_t \Lambda^a(\delta^a_t) 1_{q_{t-} > -Q} + \Delta\delta^b_t \Lambda^b(\delta^b_t) 1_{q_{t-} < Q} - \frac{\gamma}{2}\sigma^2 q_t^2 \right) dt \right]$$

where γ is a kind of absolute risk aversion parameter.

$\rightarrow$ Optimal control on a very simple finite graph (truncated $\Delta\mathbb{Z}$)

Value function θ (Model B)

Value function θ (Model B)

Hamilton-Jacobi equation (Model B)

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} \Delta H_0^b \left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta} \right)$$
$$+ 1_{q>-Q} \Delta H_0^a \left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta} \right)$$

with final condition $\theta(T,q) = 0$.

Value function θ (Model B)

Hamilton-Jacobi equation (Model B)

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q}\Delta H_0^b\left(\frac{\theta(t,q)-\theta(t,q+\Delta)}{\Delta}\right)$$
$$+1_{q>-Q}\Delta H_0^a\left(\frac{\theta(t,q)-\theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

Same kind of transform

$$H_0^b(p) = \sup_\delta \Lambda^b(\delta)(\delta - p)$$

$$H_0^a(p) = \sup_\delta \Lambda^a(\delta)(\delta - p)$$

Analysis

Analysis

- Both equations look like a classical Hamilton-Jacobi PDE of order 1.

Analysis

- Both equations look like a classical Hamilton-Jacobi PDE of order 1.
- A system of $2Q/\Delta + 1$ non-linear ODEs.

Analysis

- Both equations look like a classical Hamilton-Jacobi PDE of order 1.
- A system of $2Q/\Delta + 1$ non-linear ODEs.

Light assumptions of the intensity functions

1. $\Lambda^{b/a}$ is C^2.
2. $\Lambda^{b/a\prime} < 0$.
3. $\lim_{\delta \to +\infty} \Lambda^{b/a}(\delta) = 0$.
4. The intensity functions $\Lambda^{b/a}$ satisfy:

$$\sup_{\delta} \frac{\Lambda^{b/a}(\delta)\Lambda^{b/a\prime\prime}(\delta)}{\left(\Lambda^{b/a\prime}(\delta)\right)^2} < 2.$$

The functions H_ξ^b and H_ξ^a

The functions H_ξ^b and H_ξ^a

Proposition

- $\forall \xi \geq 0$, $H_\xi^{b/a}$ is a decreasing function of class C^2.
- In the definition of $H_\xi^{b/a}(p)$, the supremum is attained at a unique $\tilde{\delta}_\xi^{b/a*}(p)$ characterized by

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi \Delta H_\xi^{b/a}(p) - {H_\xi^{b/a}}'(p)\right).$$

- The function $p \mapsto \tilde{\delta}_\xi^{b/a*}(p)$ is increasing.

Existence and uniqueness

Existence and uniqueness

Results for θ

There exists a unique C^1 (in time) solution $t \mapsto (\theta(t,q))_{|q|\leq Q}$ to

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q}\Delta H^b_\xi\left(\frac{\theta(t,q)-\theta(t,q+\Delta)}{\Delta}\right)$$

$$+1_{q>-Q}\Delta H^a_\xi\left(\frac{\theta(t,q)-\theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q)=0$.

Solution of the initial problem (verification argument)

Solution of the initial problem (verification argument)

By using a verification argument, the function u is the value function associated with the problem.

Optimal quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \tilde{\delta}_\xi^{b*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + \Delta)}{\Delta}\right)$$

$$\delta_t^{a*} = \tilde{\delta}_\xi^{a*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - \Delta)}{\Delta}\right)$$

where

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi \Delta H_\xi^{b/a}(p) - {H_\xi^{b/a}}'(p)\right).$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (I)

The functions $H_\xi^{b/a}$ and $\tilde{\delta}_\xi^{b/a*}$

If $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$, then $H_\xi^{b/a}(p) = \frac{A}{k} C_\xi \exp(-kp)$, with

$$C_\xi = \begin{cases} \left(1 + \frac{\xi\Delta}{k}\right)^{-\frac{k}{\xi\Delta}-1} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0. \end{cases}$$

and

$$\tilde{\delta}_\xi^{b/a*}(p) = \begin{cases} p + \frac{1}{\xi\Delta}\log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (II)

The system of ODEs

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 +$$

$$+\frac{A\Delta}{k} C_\xi \left(1_{q<Q} e^{k\frac{\theta(t,q+\Delta)-\theta(t,q)}{\Delta}} + 1_{q>-Q} e^{k\frac{\theta(t,q-\Delta)-\theta(t,q)}{\Delta}} \right),$$

with final condition $\theta(T,q) = 0$.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (II)

The system of ODEs

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 +$$

$$+\frac{A\Delta}{k} C_\xi \left(1_{q<Q} e^{k\frac{\theta(t,q+\Delta)-\theta(t,q)}{\Delta}} + 1_{q>-Q} e^{k\frac{\theta(t,q-\Delta)-\theta(t,q)}{\Delta}} \right),$$

with final condition $\theta(T,q) = 0$.

Change of variables: $v_q(t) = \exp\left(\frac{k\theta(t,q)}{\Delta}\right)$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (III)

A linear system of ODEs

$$v_q'(t) = \alpha q^2 v_q(t) - \eta_\xi \left(1_{q<Q} v_{q+\Delta}(t) + 1_{q>-Q} v_{q-\Delta}(t) \right),$$

with

$$\alpha = \frac{k}{2\Delta}\gamma\sigma^2, \qquad \eta_\xi = AC_\xi$$

and the terminal condition $v(T,q) = 1$.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (III)

A linear system of ODEs

$$v_q'(t) = \alpha q^2 v_q(t) - \eta_\xi \left(1_{q<Q} v_{q+\Delta}(t) + 1_{q>-Q} v_{q-\Delta}(t)\right),$$

with

$$\alpha = \frac{k}{2\Delta}\gamma\sigma^2, \qquad \eta_\xi = AC_\xi$$

and the terminal condition $v(T, q) = 1$.

This simplifies a lot the equations of Avellaneda and Stoikov. See the paper Guéant-Lehalle-Fernandez-Tapia (2013) (when $\Delta = 1$ and $\xi = \gamma$).

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (IV)

Optimal quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \delta^{b*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln \left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}+\Delta}(t)} \right)$$

$$\delta_t^{a*} = \delta^{a*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln \left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}-\Delta}(t)} \right)$$

$$D_\xi = \begin{cases} \frac{1}{\xi\Delta} \log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (IV)

Optimal quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \delta^{b*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln\left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}+\Delta}(t)}\right)$$

$$\delta_t^{a*} = \delta^{a*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln\left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}-\Delta}(t)}\right)$$

$$D_\xi = \begin{cases} \frac{1}{\xi\Delta} \log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

The optimal quotes are made of two components:

- D_ξ corresponds to the static trade-off.
- $\frac{1}{k} \ln\left(\frac{v_q(t)}{v_{q+\Delta}(t)}\right)$ or $\frac{1}{k} \ln\left(\frac{v_q(t)}{v_{q-\Delta}(t)}\right)$: dynamic aspects.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (V)

The optimal quote functions far from T only depend on q:

Asymptotics

$$\delta_\infty^{b*}(q) = \lim_{T\to\infty} \delta^{b*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+\Delta}^0}\right)$$

$$\delta_\infty^{a*}(q) = \lim_{T\to\infty} \delta^{a*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-\Delta}^0}\right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (V)

The optimal quote functions far from T only depend on q:

Asymptotics

$$\delta_\infty^{b*}(q) = \lim_{T\to\infty} \delta^{b*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+\Delta}^0}\right)$$

$$\delta_\infty^{a*}(q) = \lim_{T\to\infty} \delta^{a*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-\Delta}^0}\right)$$

$f^0 \in \mathbb{R}^{2Q+1}$ is characterized by:

$$\underset{\|f\|_2=1}{\operatorname{argmin}} \sum_{|q|\le Q} \alpha q^2 f_q^2 + \eta_\xi \left(\sum_{q=-Q}^{Q-\Delta} (f_{q+\Delta} - f_q)^2 + (f_Q)^2 + (f_{-Q})^2\right).$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (VI)

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2\right) dx.$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (VI)

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2 \right) dx.$$

$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta} \sqrt{\frac{\alpha}{\eta_\xi}} x^2 \right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (VI)

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\operatorname*{argmin}_{\|\tilde{f}\|_{L^2(\mathbb{R})}=1} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2\right) dx.$$

$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}} x^2\right)$$

Hence, we get an approximation of the form:

$$f_q^0 \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}} q^2\right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (VII)

Using the continuous counterpart, we get:

Closed-form approximations: optimal quotes (Model A: $\xi = \gamma$)

$$\delta^{b*}_{\infty}(q) \simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

$$\delta^{a*}_{\infty}(q) \simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

Remark: these formulas are used by many practitioners in Europe and Asia on quote-driven markets.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (VIII)

Using the continuous counterpart, we get:

Closed-form approximations: optimal quotes (Model B: $\xi = 0$)

$$\delta_\infty^{b*}(q) \simeq \frac{1}{k} + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$
$$\delta_\infty^{a*}(q) \simeq \frac{1}{k} - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (IX)

A good way to analyze the result is to consider the spread $\psi = \delta^b + \delta^a$ and the skew $\zeta = \delta^b - \delta^a$.

Closed-form approx.: spread and skew (Model A, $\xi = \gamma$)

$$\psi^*_\infty(q) \simeq \frac{2}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) + \Delta\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

$$\zeta^*_\infty(q) \simeq 2q\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$ (X)

Closed form approx.: spread and skew (Model B, $\xi = 0$)

$$\psi^*_\infty(q) \simeq \frac{2}{k} + \Delta\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

$$\zeta^*_\infty(q) \simeq 2q\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

Extensions

Basic ideas

Extensions

Basic ideas

- Other objective functions.

Basic ideas

- Other objective functions.
- Including a drift and / or price jumps (easy).

Basic ideas

- Other objective functions.
- Including a drift and / or price jumps (easy).
- Including stoch. vol. models (easy but lead to a system of parabolic PDEs in dimension depending of the number of factors).

Extensions

Basic ideas

- Other objective functions.
- Including a drift and / or price jumps (easy).
- Including stoch. vol. models (easy but lead to a system of parabolic PDEs in dimension depending of the number of factors).
- Modeling price by microstructural models / point processes:
  $\rightarrow$ lead to a system of PDEs with nonlocal terms.

Extensions

Important practical considerations

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.

Extensions

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.
- Multiple tiers of clients (quotes per tier): does not change the class of equations
  $\rightarrow$ can also be useful to get signal/drift.

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.
- Multiple tiers of clients (quotes per tier): does not change the class of equations
  $\rightarrow$ can also be useful to get signal/drift.
- Asymmetric long/short penalties (important for bonds).

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.
- Multiple tiers of clients (quotes per tier): does not change the class of equations
  $\rightarrow$ can also be useful to get signal/drift.
- Asymmetric long/short penalties (important for bonds).
- Better modelling of adverse selection $\rightarrow$ trades convey information depending on id and price (a fair deal with a standard client may be better than a seemingly good deal with an informed client).

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.
- Multiple tiers of clients (quotes per tier): does not change the class of equations
  $\rightarrow$ can also be useful to get signal/drift.
- Asymmetric long/short penalties (important for bonds).
- Better modelling of adverse selection $\rightarrow$ trades convey information depending on id and price (a fair deal with a standard client may be better than a seemingly good deal with an informed client).
- Price signalling: need to model the impact of streamed prices.

Important practical considerations

- Multiple sizes of transactions (one quote per size): does not change the class of equations.
- Multiple tiers of clients (quotes per tier): does not change the class of equations
  $\rightarrow$ can also be useful to get signal/drift.
- Asymmetric long/short penalties (important for bonds).
- Better modelling of adverse selection $\rightarrow$ trades convey information depending on id and price (a fair deal with a standard client may be better than a seemingly good deal with an informed client).
- Price signalling: need to model the impact of streamed prices.
- D2C vs. D2D (internalization vs. externalization) + market impact on the D2D segment.

Regarding extensions

Regarding extensions

Many effects can be taken into account in the one-asset case

Regarding extensions

Many effects can be taken into account in the one-asset case

- Important in itself.

Regarding extensions

Many effects can be taken into account in the one-asset case

- Important in itself.
- Important for decentralized finance
  $\rightarrow$ In 2021, 3 days before Christmas, I received an email from David Bouba who co-founded an AMM (Swaap): a whitepaper, interesting remarks, and an invitation to discuss with him.

Regarding extensions

Many effects can be taken into account in the one-asset case

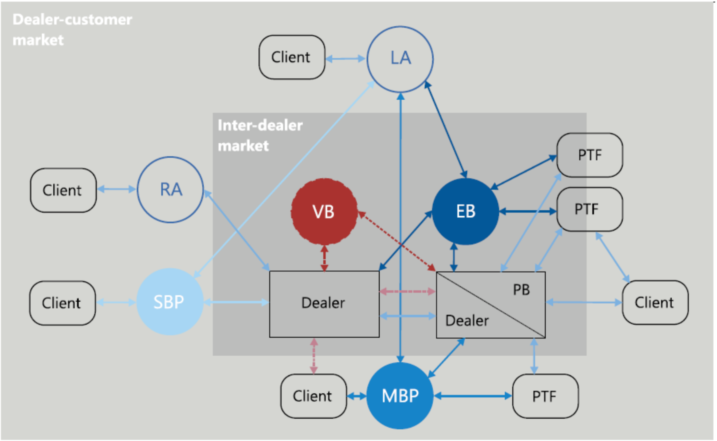
- Important in itself.
- Important for decentralized finance
  $\rightarrow$ In 2021, 3 days before Christmas, I received an email from David Bouba who co-founded an AMM (Swaap): a whitepaper, interesting remarks, and an invitation to discuss with him.
  $\rightarrow$ Less than a year later, we released a paper to build AMM based on price oracles or offchain reference prices (Bergault, Bertucci, Bouba, Guéant).

Regarding extensions

Many effects can be taken into account in the one-asset case

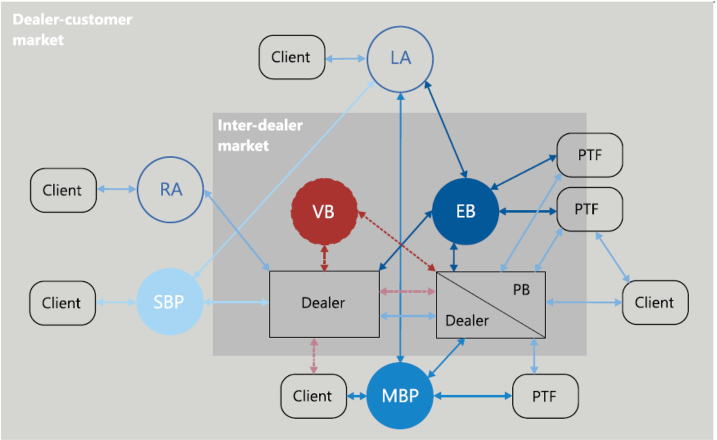
- Important in itself.
- Important for decentralized finance
  $\rightarrow$ In 2021, 3 days before Christmas, I received an email from David Bouba who co-founded an AMM (Swaap): a whitepaper, interesting remarks, and an invitation to discuss with him.
  $\rightarrow$ Less than a year later, we released a paper to build AMM based on price oracles or offchain reference prices (Bergault, Bertucci, Bouba, Guéant).
- Not really satisfying for FX (correlations + triplets) or corporate bonds (many securities for one issuer).
  $\rightarrow$ diversification and liquidity differences must be taken into account.

Organization of the FX market (Schrimpf-Sushko)



EB = electronic broker; LA = liquidity aggregator; MBP = multi-bank platform; PB = prime broker; PTF = principal trading firm; RA = retail aggregator; SBP = single-bank platform; VB = voice broker. Dashed lines indicate voice execution; solid lines indicate electronic execution.

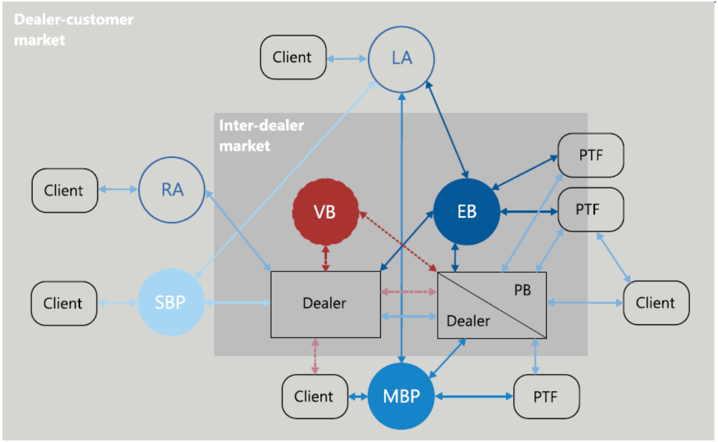
Organization of the FX market (Schrimpf-Sushko)



EB = electronic broker; LA = liquidity aggregator; MBP = multi-bank platform; PB = prime broker; PTF = principal trading firm; RA = retail aggregator; SBP = single-bank platform; VB = voice broker. Dashed lines indicate voice execution; solid lines indicate electronic execution.

RFSs and RFQs (D2C) and access to multiple platforms (D2D and all-to-all)

Organization of the FX market (Schrimpf-Sushko)



EB = electronic broker; LA = liquidity aggregator; MBP = multi-bank platform; PB = prime broker; PTF = principal trading firm; RA = retail aggregator; SBP = single-bank platform; VB = voice broker. Dashed lines indicate voice execution; solid lines indicate electronic execution.

RFSs and RFQs (D2C) and access to multiple platforms (D2D and all-to-all) $\rightarrow$ dealers can internalize or externalize the flow.

Numerical approximation

Numerical approximation

- Equations are there but no grid in high dimension

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
  - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
    - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.
    - For HJ equations on graphs, the easy case is that of exponential Hamiltonians (hence the special case of exponential intensities).

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
  - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.
  - For HJ equations on graphs, the easy case is that of exponential Hamiltonians (hence the special case of exponential intensities).
  - But...

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
  - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.
  - For HJ equations on graphs, the easy case is that of exponential Hamiltonians (hence the special case of exponential intensities).
  - But...
  - Quadratic approximations of the H functions (makes sense for not too asymmetric flows)...

Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
  - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.
  - For HJ equations on graphs, the easy case is that of exponential Hamiltonians (hence the special case of exponential intensities).
  - But...
  - Quadratic approximations of the H functions (makes sense for not too asymmetric flows)...
  - Gives an approximate value function $\tilde{\theta}$: Polynomial of degree 2 in y with coefficients solving a Riccati-like equation (no curse of dimensionality).

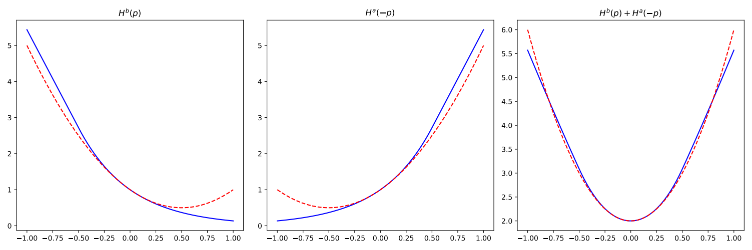
Numerical approximation

- Equations are there but no grid in high dimension
- Simple structure for the risk with a few risk factors
- (Deep) Reinforcement Learning (slow and the danger of NN).
- Trick in Bergault, Evangelista, Guéant, Vieira:
  - HJB equations associated with continuous state space optimal control problems boil down to Riccati equations when Hamiltonians are quadratic.
  - For HJ equations on graphs, the easy case is that of exponential Hamiltonians (hence the special case of exponential intensities).
  - But...
  - Quadratic approximations of the H functions (makes sense for not too asymmetric flows)...
  - Gives an approximate value function $\tilde{\theta}$: Polynomial of degree 2 in y with coefficients solving a Riccati-like equation (no curse of dimensionality).
  - $\tilde{\theta}$ is plugged in the above equations to get great pricing and hedging strategies.

Approximation of the Hamiltonians

$$H^b\left(\frac{\theta(t,q)-\theta(t,q+\Delta)}{\Delta}\right)+H^a\left(\frac{\theta(t,q)-\theta(t,q-\Delta)}{\Delta}\right)$$

$$\rightsquigarrow \quad H^b(p)+H^a(-p)$$



Example (video)

Time for a short animation

The problem with precious metals

The problem with precious metals

- In many cases, market making on the spot market and hedging on both the (illiquid) spot market and the (liquid) futures market.
- Futures hedging is imperfect $\rightarrow$ the remaining (basis) risk cannot be modelled with a Brownian motion: it is stationary.

The problem with precious metals

- In many cases, market making on the spot market and hedging on both the (illiquid) spot market and the (liquid) futures market.
- Futures hedging is imperfect $\to$ the remaining (basis) risk cannot be modelled with a Brownian motion: it is stationary.

The dynamics of prices (Nested OU)

- Spot: $dS_t = \sigma_S dW_t^S, \qquad \sigma_S > 0.$

The problem with precious metals

- In many cases, market making on the spot market and hedging on both the (illiquid) spot market and the (liquid) futures market.
- Futures hedging is imperfect $\to$ the remaining (basis) risk cannot be modelled with a Brownian motion: it is stationary.

The dynamics of prices (Nested OU)

- Spot: $dS_t = \sigma_S dW_t^S, \qquad \sigma_S > 0.$
- Futures: $F_t = S_t + E_t$.

The problem with precious metals

- In many cases, market making on the spot market and hedging on both the (illiquid) spot market and the (liquid) futures market.
- Futures hedging is imperfect $\rightarrow$ the remaining (basis) risk cannot be modelled with a Brownian motion: it is stationary.

The dynamics of prices (Nested OU)

- Spot: $dS_t = \sigma_S dW_t^S, \qquad \sigma_S > 0.$
- Futures: $F_t = S_t + E_t$.
  We want prices with linear dynamics to stay in the quadratic value case:

$$\begin{aligned} dE_t &= -k_E\left(E_t - D_t\right)dt + \sigma_E dW_t^E, \qquad k_E, \sigma_E > 0, \\ dD_t &= -k_D\left(D_t - \bar{D}\right)dt + \sigma_D dW_t^D, \qquad k_D, \sigma_D \geq 0, \quad \bar{D} \in \mathbb{R}, \end{aligned}$$

  where $(W_t^S, W_t^E, W_t^D)_t$ is a three-dimensional Brownian motion with correlation matrix R (covariance matrix: Σ).

Other state variables

Inventories

- Spot – jumps (trade with clients) and execution:

$$dq_t^S = \int_{z=0}^{\infty} zJ^b(dt, dz) - \int_{z=0}^{\infty} zJ^a(dt, dz) + v_t^S dt.$$

- Futures – execution: $dq_t^F = v_t^F dt$.

Other state variables

Inventories

- Spot – jumps (trade with clients) and execution:

$$dq_t^S = \int_{z=0}^{\infty} zJ^b(dt, dz) - \int_{z=0}^{\infty} zJ^a(dt, dz) + v_t^S dt.$$

- Futures – execution: $dq_t^F = v_t^F dt.$

where the intensities are

$$\Lambda^b(z, \delta) = \Lambda^a(z, \delta) = \Lambda(z, \delta) = \lambda(z) f(\delta) \quad \text{with} \quad f(\delta) = \frac{1}{1 + e^{\alpha + \beta\delta}}.$$

Other state variables

Cash

The resulting cash process $(X_t)_t$ follows:

$$
\begin{aligned}
dX_t &= \int_{z=0}^{\infty} S^a(t,z) z J^a(dt,dz) - \int_{z=0}^{\infty} S^b(t,z) z J^b(dt,dz) \\
&\quad - v_t^S S_t dt - L^S(v_t^S)dt - v_t^F F_t dt - L^F(v_t^F)dt
\end{aligned}
$$

where $L^S(v_t^S)$ and $L^F(v_t^F)$ account for execution costs upon externalizing.

Stochastic optimal control

Objective function

The goal is now to maximize

$$\mathbb{E}\left[-\exp\left(-\gamma\left(X_T + q_T^S S_T + q_T^F F_T - K\left((q_T^S)^2 + (q_T^F)^2\right)\right)\right)\right]$$

by selecting δ^b, δ^a, v^S and v^F optimally.

Hamilton-Jacobi-Bellman equation

$$
\begin{aligned}
0 &= \partial_t u - k_E (E - D)\, \partial_E u - k_D \left(D - \bar{D}\right) \partial_D u \\
&+ \frac{1}{2} \mathrm{Tr}(\Sigma \nabla^2_{SED} u) + \mathcal{L}^b u + \mathcal{L}^a u \\
&+ \sup_{v^S} \left(v^S \partial_{q^S} u - \left(L^S(v^S) + v^S S \right) \partial_x u \right) \\
&+ \sup_{v^F} \left(v^F \partial_{q^F} u - \left(L^F(v^F) + v^F (S + E) \right) \partial_x u \right),
\end{aligned}
$$

with terminal condition

$$
\begin{aligned}
& u(T, x, q^S, q^F, S, E, D) \\
= & -\exp\left(-\gamma \left(x + q^S S + q^F (S + E) - K \left((q^S)^2 + (q^F)^2 \right) \right) \right).
\end{aligned}
$$

Hamilton-Jacobi-Bellman equation

Nonlocal jump operators:

$$
\begin{aligned}
&\mathcal{L}^b u(t, x, q^S, q^F, S, E, D) \\
= &\int_0^\infty \sup_{\delta^b} f(\delta^b) \left(u(t, x - z(S - \delta^b), q^S + z, q^F, S, E, D) \right. \\
&\qquad\qquad \left. - u(t, x, q^S, q^F, S, E, D) \right) \lambda(z)\, dz
\end{aligned}
$$

Hamilton-Jacobi-Bellman equation

Nonlocal jump operators:

$$\mathcal{L}^b u(t,x,q^S,q^F,S,E,D)$$
$$= \int_0^\infty \sup_{\delta^b} f(\delta^b)\left(u(t,x-z(S-\delta^b),q^S+z,q^F,S,E,D)\right.$$
$$\left. - u(t,x,q^S,q^F,S,E,D)\right)\lambda(z)\,dz$$

$$\mathcal{L}^a u(t,x,q^S,q^F,S,E,D)$$
$$= \int_0^\infty \sup_{\delta^a} f(\delta^a)\left(u(t,x+z(S+\delta^a),q^S-z,q^F,S,E,D)\right.$$
$$\left. - u(t,x,q^S,q^F,S,E,D)\right)\lambda(z)\,dz$$

Change of variables

Ansatz

$$
\begin{aligned}
& u(t, x, q^S, q^F, S, E, D) \\
= \; & -\exp\left(-\gamma\left(x + q^S S + q^F(S+E) + \theta(t, q^S, q^F, E, D)\right)\right)
\end{aligned}
$$

Change of variables

Ansatz

$$
\begin{aligned}
& u(t, x, q^S, q^F, S, E, D) \\
= \quad & -\exp\left(-\gamma\left(x + q^S S + q^F (S+E) + \theta(t, q^S, q^F, E, D)\right)\right)
\end{aligned}
$$

New equation for θ

The equation for θ becomes:

$$
\begin{aligned}
0 \quad = \quad & \partial_t \theta - k_E\,(E - D)\left(q^F + \partial_E \theta\right) - k_D\left(D - \bar{D}\right)\partial_D \theta + \frac{1}{2}\mathrm{Tr}(\widetilde{\Sigma}\nabla^2_{ED}\theta) \\
& -\frac{\gamma}{2}\begin{pmatrix} q^S + q^F \\ q^F + \partial_E \theta \\ \partial_D \theta \end{pmatrix}^{\intercal} \Sigma \begin{pmatrix} q^S + q^F \\ q^F + \partial_E \theta \\ \partial_D \theta \end{pmatrix} + \mathcal{J}_H \theta + \mathcal{H}^S\left(\partial_{q^S}\theta\right) + \mathcal{H}^F\left(\partial_{q^F}\theta\right)
\end{aligned}
$$

Notations

- $\widetilde{\Sigma}$ is the submatrix of Σ obtained by removing the first row and the first column
- $\mathcal{H}^S$ and $\mathcal{H}^F$ are the Hamiltonian functions defined by

$$\mathcal{H}^S : p \in \mathbb{R} \mapsto \sup_{v^S} \left(v^S p - L^S(v^S)\right)$$

$$\mathcal{H}^F : p \in \mathbb{R} \mapsto \sup_{v^F} \left(v^F p - L^F(v^F)\right)$$

- Nonlocal jump operators:

$$\begin{aligned} &\mathcal{J}_H\theta(t, q^S, q^F, E, D) \\ = \; &\int_0^{\infty} zH\left(z, \frac{\theta(t, q^S, q^F, E, D) - \theta(t, q^S + z, q^F, E, D)}{z}\right)\lambda(z)dz \\ &+ \int_0^{\infty} zH\left(z, \frac{\theta(t, q^S, q^F, E, D) - \theta(t, q^S - z, q^F, E, D)}{z}\right)\lambda(z)dz \end{aligned}$$

with $H : (z, p) \in (0, +\infty) \times \mathbb{R} \mapsto \sup_{\delta} \frac{f(\delta)}{\gamma z}(1 - e^{-\gamma z(\delta - p)})$.

Solution

If we approximate the Hamiltonian terms by polynomials of degree 2, the resulting approximation of θ will be a polynomial of degree 2, with coefficients solving simple ODEs (Riccati and linear).

Solution

If we approximate the Hamiltonian terms by polynomials of degree 2, the resulting approximation of θ will be a polynomial of degree 2, with coefficients solving simple ODEs (Riccati and linear).

Optimal strategy

$$
\begin{cases}
\delta^{b*}(t,z) = \bar{\delta}\left(z, \dfrac{\theta(t, q^S_{t-}, q^F_t, E_t, D_t) - \theta(t, q^S_{t-} + z, q^F_t, E_t, D_t)}{z}\right) \\
\delta^{a*}(t,z) = \bar{\delta}\left(z, \dfrac{\theta(t, q^S_{t-}, q^F_t, E_t, D_t) - \theta(t, q^S_{t-} - z, q^F_t, E_t, D_t)}{z}\right) \\
v^{S*}_t = \mathcal{H}^{S\prime}\left(\partial_{q^S}\theta(t, q^S_{t-}, q^F_t, E_t, D_t)\right) \\
v^{F*}_t = \mathcal{H}^{F\prime}\left(\partial_{q^F}\theta(t, q^S_{t-}, q^F_t, E_t, D_t)\right)
\end{cases}
$$

where $\bar{\delta}(z,p) = f^{-1}\left(\gamma z H(z,p) - \partial_p H(z,p)\right)$.

Stochastic filtering

- The spot and futures prices are observable, so is the basis $(E_t)_t$.

Stochastic filtering

- The spot and futures prices are observable, so is the basis $(E_t)_t$.
- $(D_t)_t$ is not observable. It has to be filtered.

Stochastic filtering

- The spot and futures prices are observable, so is the basis $(E_t)_t$.
- $(D_t)_t$ is not observable. It has to be filtered.

The dynamics after filtering – Same Nested OU structure

Assuming $d\langle W^S, W^E\rangle = \rho dt$ and $\langle W^S, W^D\rangle = \langle W^E, W^D\rangle = 0$, we get:

Stochastic filtering

- The spot and futures prices are observable, so is the basis $(E_t)_t$.
- $(D_t)_t$ is not observable. It has to be filtered.

The dynamics after filtering – Same Nested OU structure

Assuming $d\langle W^S, W^E\rangle = \rho dt$ and $\langle W^S, W^D\rangle = \langle W^E, W^D\rangle = 0$, we get:

$$\begin{aligned} dE_t &= -k_E\left(E_t - \widehat{D}_t\right)dt + \sigma_E d\widehat{W}_t^E \\ d\widehat{D}_t &= -k_D\left(\widehat{D}_t - \bar{D}\right)dt + \frac{1}{\sqrt{1-\rho^2}}\frac{k_E}{\sigma_E}\nu_t^2 d\widehat{W}_t^D \end{aligned}$$

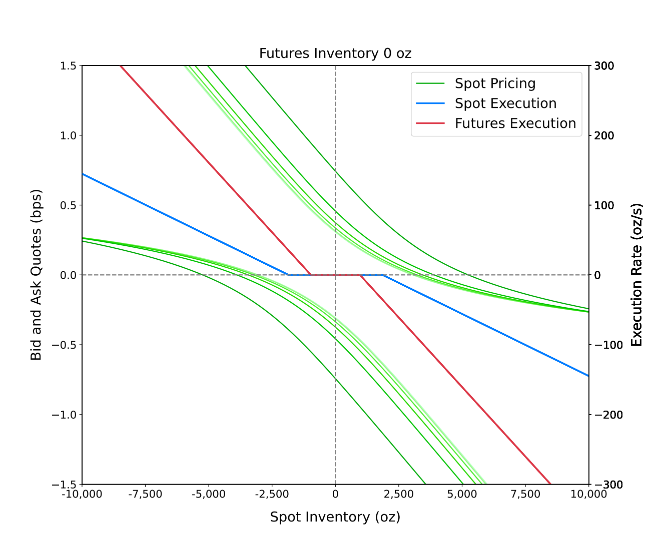
where

$$\widehat{D}_t = \mathbb{E}\left[D_t|(S_s)_{s\le t},(E_s)_{s\le t}\right], \quad \nu_t^2 = \mathbb{V}\left(D_t|(S_s)_{s\le t},(E_s)_{s\le t}\right),$$

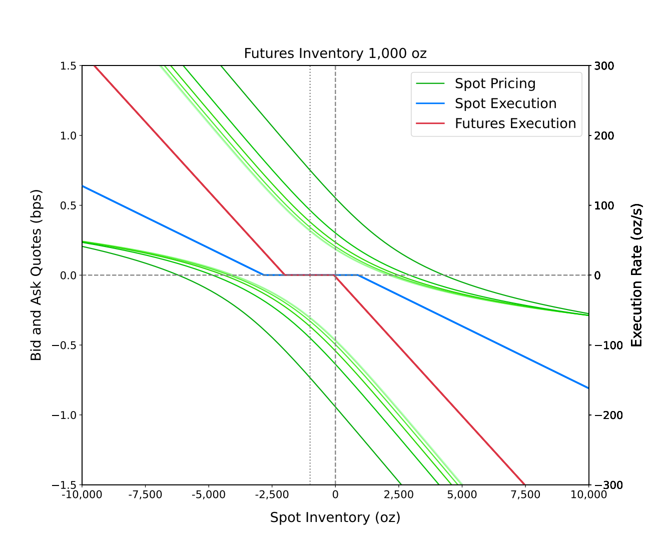
and

$$\widehat{W}_t^E = W_t^E + \frac{k_E}{\sigma_E}\int_0^t (D_s - \widehat{D}_s)ds \quad \text{and} \quad \widehat{W}_t^D = \frac{\widehat{W}_t^E - \rho W_t^S}{\sqrt{1-\rho^2}}$$

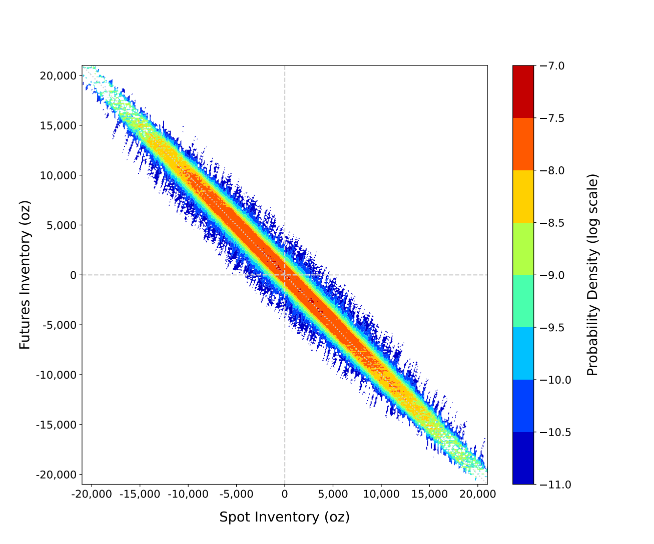
Optimal strategies



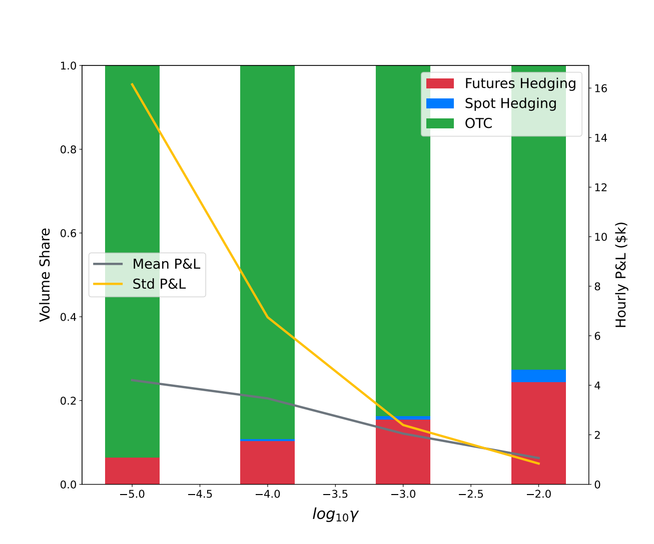
Optimal strategy



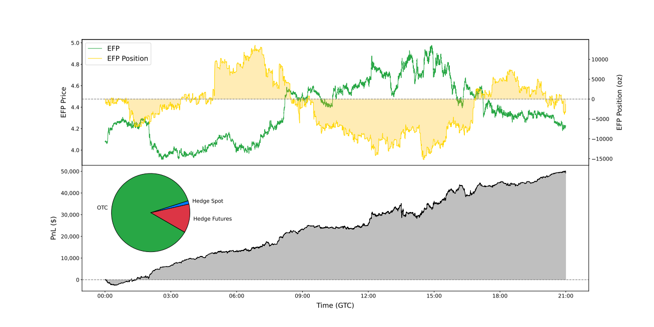
Inventory probability distribution



Volume shares



Performance



Questions

Thanks for your attention.
Questions?