Principal-agent problems with volatility control

A new approach

Emma Hubert[1]

Joint work with Alessandro Chiusolo[1]

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[1]Department of Operations Research and Financial Engineering (ORFE), Princeton University.
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Content

Introduction

- Analyse interactions between economic agents, in particular with asymmetric information (moral hazard).

  The principal (she) initiates a contract for a period $[0, T]$, represented by a terminal payment ξ.

  The agent (he) accepts or not the contract proposed by the principal.

The principal must suggest an optimal contract: maximises her utility, and that the agent will accept (reservation utility R_A).

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- Sannikov [9] (2008): general method for continuous-time principal-agent problems;

Principal-agent problems

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- Holmström and Milgrom [6] (1987): first model in continuous time;
- Sannikov [9] (2008): general method for continuous-time principal-agent problems;
- Cvitanić, Possamaï, and Touzi [4] (2018): extension to volatility control.
  (i) identify a 'nice' class of contracts;
  (ii) prove that this restriction is without loss of generality (using 2BSDE);
  (iii) solve the principal's problem, which is now standard.

- The agent controls both the drift and the volatility of a (one-dimensional) output process X through an adapted control $\nu \in \mathcal{U}$:

$$X_t = X_0 + \int_0^t \sigma(s, X_s, \nu_s)\big(\lambda(s, X_s, \nu_s)ds + dW_s^{\nu}\big), \ t \in [0, T], \ \mathbb{P}^{\nu}\text{-a.s.},$$

where W^{ν} is a d-dimensional $\mathbb{P}^{\nu}$-Brownian Motion.

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$$V_A(\xi) := \sup_{\nu \in \mathcal{U}} J_A(\xi, \nu), \quad \text{with} \quad J_A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\left[U_A\left(\xi - \int_0^T c(t, X_t, \nu_t)dt\right)\right].$$

- Optimal response to a contract ξ: $\mathcal{U}^\star(\xi) := \{\nu^\star \in \mathcal{U} : V_A(\xi) = J_A(\xi, \nu^\star)\}$.

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- Optimal response to a contract ξ: $\mathcal{U}^\star(\xi) := \{\nu^\star \in \mathcal{U} : V_A(\xi) = J_A(\xi, \nu^\star)\}$.

- The goal of the principal is to find the optimal contract, i.e.

$$V_P := \sup_{\xi \in \Xi} \sup_{\nu^\star \in \mathcal{U}^\star(\xi)} \mathbb{E}^{\mathbb{P}^{\nu^\star}}\big[U_P(X_T - \xi)\big],\ \Xi := \big\{\xi\ \mathcal{F}_T\text{-measurable},\ V_A(\xi) \geq R_A\big\}.$$

- Drift control case: if the volatility is uncontrolled (i.e. $\sigma(s, X_s)$), the agent's dynamic value function $(Y_t)_{t\in[0,T]}$ is related to the first component of the solution to a **BSDE**, with terminal condition $Y_T := U_A(\xi)$.

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The **optimal** form of contracts for the agent is $\xi = U_A^{-1}(Y_T^Z)$ where

$$Y_t^Z = y_0 - \int_0^t \sup_{u\in U} \{\sigma(s, X_s)\lambda(s, X_s, u)Z_s - c(s, X_s, u)\}ds + \int_0^t Z_s dX_s,$$

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- Volatility control case: restrict the study to contracts $\xi = U_A^{-1}(Y_T^{Z,\Gamma})$ where

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$$\text{with}\quad \mathcal{H}_A(t,x,z,\gamma) := \sup_{u\in U}\Big\{[\sigma\lambda](t,x,u)z + \frac{1}{2}\big[\sigma\sigma^\top\big](t,x,u)\gamma - c(t,x,u)\Big\},$$

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where $y_0 \in \mathbb{R}$ and Z, Γ optimally chosen by the principal.

▶ The proof that the previous form of contracts is without loss of generality relies on 2BSDE.

- Applications:
  - Finance: Cvitanić, Possamaï, and Touzi [3] (2017), ...
  - Energy-related: Aïd, Possamaï, and Touzi [1] (2022), Élie, Hubert, Mastrolia, and Possamaï [5] (2021), Aïd, Kemper, and Touzi [2] (2023), ...

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- Extensions:
  - Multi-agent problems: Hubert [7] (2023), Aïd, Kemper, and Touzi [2] (2023) $\Rightarrow$ Multidimensional 2BSDEs;
  - Continuum of agents: Élie, Hubert, Mastrolia, and Possamaï [5] (2021), ... $\Rightarrow$ Mean-field 2BSDEs;
  - Output process with jumps? $\Rightarrow$ 2BSDEs with jumps.

Towards a new approach

Short detour via the 'first-best'

- Principal-agent problems with moral hazard ('second-best') are usually compared to their 'first-best' counterpart.

- In the first-best case, the principal directly chooses the agent's controls:

$$V_P^\circ := \sup_{\xi \in \Xi} \sup_{\nu \in \mathcal{U}} \mathbb{E}^{\mathbb{P}^\nu}\big[U_P\big(X_T - \xi\big)\big].$$

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- If the principal can observe the agent's controls, then SB = FB if one can find a 'penalisation/forcing contract', i.e. a contract offered by the principal which allows her to achieve her value in the first-best case.

- Main idea: if the principal observes the agent's controls, she can strongly penalise him whenever he deviates from a 'recommended effort'.

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- Usually the case if there are no 'strong' assumptions on the set of admissible contracts (may not work in a framework with limited liability).

A first illustrative example

- One-dimensional output process with one-dimensional noise and volatility control only:

$$dX_t = \nu_t(\lambda dt + dW_t), \ \lambda > 0, \ t \in [0, T].$$

- The principal observes X in continuous-time, and can therefore deduce its quadratic variation $\langle X \rangle_t$, $t \in [0, T]$.

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- Two possible approaches:

(i) Apply the results from [4], by considering contracts indexed on $\langle X \rangle$ through the parameter Γ.

(ii) Notice that since the principal observes $\langle X \rangle$, she can (almost) deduce the agent's effort $\nu_t = \pm\sqrt{\langle X \rangle_t}$, $t \in [0, T]$ $\Rightarrow$ FB!

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- In the following, we will take for $\gamma_A, \gamma_P > 0$,

$$J_A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\big[- \exp(-\gamma_A \xi)\big], \quad J_P(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\big[- \exp\big(- \gamma_P(X_T - \xi)\big)\big].$$

- **Step 1.** We consider a slightly different problem: the principal chooses a contract ξ and controls the quadratic variation of X through a process Σ:

$$d\langle X\rangle_t = \Sigma_t dt, \ t \in [0, T].$$

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- The original Hamiltonian can be decomposed as follows:

$$\begin{aligned}\mathcal{H}_A(t, x, z, \gamma) &= \sup_{u\in\mathbb{R}} \left\{ u\lambda z + \frac{1}{2}\gamma u^2 \right\} \\ &= \sup_{S\in\mathbb{R}_+} \left\{ \sup_{u\in\mathbb{R}:\ u^2=S} \{u\lambda z\} + \frac{1}{2}\gamma S \right\}\end{aligned}$$

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▶ For fixed Σ, we shall consider the 'constrained' Hamiltonian:

$$\mathcal{H}_A^\circ(t,x,z,S) = \sup_{u\in\mathbb{R}\ :\ u^2=S} \{u\lambda z\} \ \Rightarrow\ u^\circ(z,S) := \operatorname{sgn}(z)\sqrt{S},\ S \in \mathbb{R}_+.$$

- **Step 2.** Using BSDEs, one can show that the optimal form of contracts is:

$$\xi = \xi_0 - \int_0^T \sup_{u \in \mathbb{R} \,:\, u^2 = \Sigma_t} \{u \lambda Z_t\} dt + \int_0^T Z_t dX_t + \frac{\gamma_A}{2} \int_0^T Z_t^2 \Sigma_t dt.$$

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- The principal's problem becomes:

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$$\text{with} \quad dX_t = \mathrm{sgn}(Z_t)\sqrt{\Sigma_t}(\lambda dt + dW_t),$$

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- It is clear that the 'first-best' value V_P° is higher than the value V_P of the original problem... We want to show equality!

- **Step 3.** We consider the contract's form in [4]:

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- By restricting the study to contracts of the previous form, we get

$$V_P \geq \sup_{Z,\Gamma} \mathbb{E}\Big[- \exp\big(- \gamma_P (X_T - \xi) \big) \Big]$$

$$\text{with} \quad dX_t = -\lambda \frac{Z_t}{\Gamma_t} (\lambda dt + dW_t),$$

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- Proving the equality above requires 2BSDEs...

A 'first-best' alternative problem: conclusion

- 'First-best' solution:

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- The two problems are the same, and there is a one-to-one correspondence between the processes Σ and Γ:

$$d\langle X \rangle_t = \Sigma_t dt \quad \Rightarrow \quad \Sigma_t = \lambda^2 \frac{Z_t^2}{\Gamma_t^2}, \quad \text{or equivalently} \quad \Gamma_t = -\lambda \frac{Z_t}{\sqrt{\Sigma_t}}.$$

The new approach: general framework

The general framework

- The agent controls both the drift and the volatility of a one-dimensional output process X through an adapted control $\nu \in \mathcal{U}$:

$$X_t = X_0 + \int_0^t \sigma(s, X_s, \nu_s)\big(\lambda(s, X_s, \nu_s)\mathrm{d}s + \mathrm{d}W_s\big),\ t \in [0, T],\ \mathbb{P}^\nu\text{-a.s.},$$

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Problem 1 (Original problem)

$$V_P := \sup_{\xi \in \Xi} \sup_{\nu^\star \in \mathcal{U}^\star(\xi)} \mathbb{E}^{\mathbb{P}^{\nu^\star}}\big[U_P(X_T - \xi)\big],$$

where

$$\Xi := \big\{\xi\ \mathcal{F}_T\text{-measurable},\ V_A(\xi) \geq R_A\big\},$$
$$\mathcal{U}^\star(\xi) := \{\nu^\star \in \mathcal{U} : V_A(\xi) = J_A(\xi, \nu^\star)\} \neq \varnothing \text{ for } \xi \in \Xi.$$

$$V_A(\xi) := \sup_{\nu \in \mathcal{U}} J_A(\xi, \nu), \quad \text{with} \quad J_A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\left[U_A\left(\xi - \int_0^T c(t, X_t, \nu_t)dt\right)\right].$$

- The principal observes X in continuous-time, and can therefore deduce its quadratic variation $\langle X \rangle$ (defined pathwise by Karandikar [8] (1995)), with density Σ with respect to the Lebesgue measure:

$$d\langle X \rangle_t = \Sigma_t dt, \quad t \in [0, T].$$

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- Sort of 'first-best' on the quadratic variation!

- Step 1. Write an alternative problem:

(i) the principal chooses both a contract ξ and a process $\Sigma \in \mathcal{S}$,
(ii) the agent computes his best response $\nu \in \mathcal{U}^\circ(\Sigma)$, with

$$\mathcal{U}^\circ(\Sigma) := \big\{ \nu \in \mathcal{U} \ : \ \nu_t \in \mathrm{U}^\circ_t(X_t, \Sigma_t), \ t \geq 0 \big\},$$
$$\mathrm{U}^\circ_t(x, S) := \big\{ u \in U \ : \ [\sigma\sigma^\top](t,x,u) = S \big\}.$$

- Define

$$\mathcal{S} := \{\Sigma\ \mathbb{F}\text{–progressively measurable process}, \Sigma_t \in \mathbb{S}_t(X_t)\},$$
$$\mathbb{S}_t(x) := \{S \in \mathbb{R}_+ \ :\ S = [\sigma\sigma^\top](t,x,u) \ \text{ for some } u \in U\}.$$

- Given $\xi \in \Xi$ and $\Sigma \in \mathcal{S}$, the agent solves:

$$V_A^\circ(\xi, \Sigma) := \sup_{\nu \in \mathcal{U}^\circ(\Sigma)} J_A(\xi, \nu).$$

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- Alternative (still non-standard) stochastic control problem for the principal:

Problem 2 ('First-best' reformulation of the original problem)

$$V_P^\circ := \sup_{(\xi,\Sigma)\in\Xi\times\mathcal{S}} \ \sup_{\nu\in\mathcal{U}^{\circ,\star}(\xi,\Sigma)} \mathbb{E}^{\mathbb{P}^\nu}\big[U_P(X_T - \xi)\big], \qquad (1)$$

where $\mathcal{U}^{\circ,\star}(\xi, \Sigma) := \{\nu \in \mathcal{U}^\circ(\Sigma) \ : \ V_A^\circ(\xi, \Sigma) = J_A(\xi, \nu)\} \neq \varnothing$.

- **Lemma 1.** $V_P^\circ \geq V_P$.

▶ Step 2. For fixed $\Sigma \in \mathcal{S}$, the agent's continuation value can be represented by the following BSDE, with terminal value $Y_T = U_A(\xi)$:

$$Y_t^Z = y_0 - \int_0^t \mathcal{H}_A^\circ(s, X_s, Z_s, \Sigma_s) ds + \int_0^t Z_s \cdot dX_s, \qquad (2)$$

$$\text{with} \quad \mathcal{H}_A^\circ(t, x, z, S) = \sup_{u \in U_t^\circ(x,S)} \Big\{ [\sigma\lambda](t, x, u) z - c(t, x, u) \Big\}.$$

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Theorem 2

Let $\Sigma \in \mathcal{S}$ and $\xi \in \Xi$. Then there exists $Z \in \mathcal{V}^\circ(\Sigma)$ and $y_0 \geq U_A^{-1}(R_A)$ such that $\xi = U_A^{-1}(Y_T^Z)$, where the process Y^Z is defined by (2). Moreover:

(i) The agent's optimal response ν° is a maximiser of the Hamiltonian $\mathcal{H}_A^\circ$;

(ii) $V_A^\circ(\xi, \Sigma) = y_0 \geq R_A$;

(iii) $V_P^\circ = \widetilde{V}_P^\circ$ (defined on the next slide).

Problem 3 (Restriction to contracts of the form (2))

$$\widetilde{V}_P^\circ := \sup_{y_0 \geq U_A^{-1}(R_A)} \sup_{\Sigma \in \mathcal{S}} \sup_{Z \in \mathcal{V}^\circ(\Sigma)} \sup_{\nu^\circ \in \mathcal{U}^{\circ,\star}(Y_T,\Sigma)} \mathbb{E}^{\mathbb{P}^{\nu^\circ}} \big[U_P(X_T - \xi) \big].$$

where the state variables X and Y are solution to the following system of SDEs:

$$dX_t = \sigma\big(t, X, \nu_t^\circ\big) \Big(\lambda\big(t, X, \nu_t^\circ\big) dt + dW_t \Big), \quad t \in [0,T], \tag{3a}$$

$$dY_t = c\big(t, X, \nu_t^\circ\big) dt + Z_t \, \sigma\big(t, X, \nu_t^\circ\big) dW_t, \quad t \in [0,T], \tag{3b}$$

coupled through the agent's optimal effort $\nu_t^\circ := u^\circ(t, X, Z_t, \Sigma_t)$, where the function u° is defined as a maximiser of the (constraint) Hamiltonian $\mathcal{H}_A^\circ$.

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coupled through the agent's optimal effort $\nu_t^{\circ} := u^{\circ}(t, X, Z_t, \Sigma_t)$, where the function u° is defined as a maximiser of the (constraint) Hamiltonian $\mathcal{H}_A^{\circ}$.

- One can write the HJB equation associated to this problem and solve it (explicitly or numerically).

Step 3. 'Forcing' contracts

- For now, we have $V_P \leq V_P^{\circ} = \widetilde{V}_P^{\circ}$.
- Step 3. Show that $V_P \geq V_P^{\circ}$, by introducing 'forcing' contracts.

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with $\mathcal{H}_A(t,x,z,\gamma) := \sup_{u\in U}\Big\{[\sigma\lambda](t,x,u)z + \frac{1}{2}\big[\sigma\sigma^\top\big](t,x,u)\gamma - c(t,x,u)\Big\}$,

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- By [4], we know this form of contracts is 'optimal' (by 2BSDEs). Here, we just want to prove that this form allow to achieve the 'first-best' utility V_P° (which will imply optimality of the contract form).

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- By [4], we know this form of contracts is 'optimal' (by 2BSDEs). Here, we just want to prove that this form allow to achieve the 'first-best' utility V_P° (which will imply optimality of the contract form).

- Given a contract of the form (4), the agent's optimal response is given by the maximiser of his hamiltonian: $\nu_t^\star := u^\star(t, X_t, Z_t, \Gamma_t)$ with

$$u^\star(t,x,z,\gamma) \in \arg\max_{u \in U} \Big\{[\sigma\lambda](t,x,u)z + \frac{1}{2}\big[\sigma\sigma^\top\big](t,x,u)\gamma - c(t,x,u)\Big\}.$$

- **Lemma 3.** $V_P \geq \widetilde{V}_P$.

Problem 4 (Restriction to contracts of the form (4))

$$\widetilde{V}_P := \sup_{y_0 \geq R_A} \sup_{(Z,\Gamma)\in\mathcal{V}} \sup_{\nu^\star \in \mathcal{U}^\star(Y_T^{Z,\Gamma})} \mathbb{E}^{\mathbb{P}^{\nu^\star}} \left[U_P\left(X_T - U_A^{-1}\left(Y_T^{Z,\Gamma}\right)\right)\right],$$

where the state variable (X, Y) is solution to the following system of SDEs:

$$dX_t = \sigma\left(t, X_t, \nu_t^\star\right)\left(\lambda\left(t, X_t, \nu_t^\star\right)dt + dW_t\right), \quad t \in [0, T], \quad X_0 = x_0 \tag{5a}$$

$$dY_t = c\left(t, X_t, \nu_t^\star\right)dt + Z_t\, \sigma\left(t, X_t, \nu_t^\star\right)dW_t, \quad t \in [0, T], \quad Y_0 = y_0, \tag{5b}$$

coupled through the agent's optimal effort $\nu_t^\star := u^\star(t, X, Z_t, \Gamma_t)$, where the function $u^\star$ is defined as a maximiser of the Hamiltonian $\mathcal{H}_A$.

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coupled through the agent's optimal effort $\nu_t^\star := u^\star(t, X, Z_t, \Gamma_t)$, where the function $u^\star$ is defined as a maximiser of the Hamiltonian $\mathcal{H}_A$.

- To prove that $\widetilde{V}_P = V_P$, one need to rely on *2BSDEs* (because of volatility control)...

- In terms of value for the principal:

$$\mathrm{V_P} \geq \widetilde{\mathrm{V}}_\mathrm{P} \quad \text{and} \quad \widetilde{\mathrm{V}}^\circ_\mathrm{P} = \mathrm{V}^\circ_\mathrm{P} \geq \mathrm{V_P}.$$

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  - In Problem 3, the principal controls (Z, Σ) instead of (Z, Γ) in Problem 4;
  - The agent's optimal response might be different:

$$\nu_t^\circ := u^\circ(t, X_t, Z_t, \Sigma_t) \overset{?}{\Longleftrightarrow} \nu_t^\star := u^\star(t, X_t, Z_t, \Gamma_t)$$

- In terms of value for the principal:

$$V_{\mathrm{P}} \geq \widetilde{V}_{\mathrm{P}} \quad \text{and} \quad \widetilde{V}^{\circ}_{\mathrm{P}} = V^{\circ}_{\mathrm{P}} \geq V_{\mathrm{P}}.$$

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- **Lemma 4.** For any fixed y_0, there exists a one-to-one correspondence between $(\Sigma, Z) \in \mathcal{S} \times \mathcal{V}^{\circ}(\Sigma)$ in Problem 3 and $(Z, \Gamma) \in \mathcal{V}$ in Problem 4, so that $\widetilde{V}_{\mathrm{P}} = \widetilde{V}^{\circ}_{\mathrm{P}}$.

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Theorem 5 (Final result)

Solving Problem 2 is equivalent to solving Problem 1, as

$$V_P \geq \widetilde{V}_P = \widetilde{V}_P^\circ = V_P^\circ \geq V_P \quad \Rightarrow \quad V_P = V_P^\circ = \widetilde{V}_P^\circ.$$

In particular, we deduce $V_P = \widetilde{V}_P$ as in [4] (but without using 2BSDEs).

'Proof' of Lemma 4

- Use the following link between the hamiltonians $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}^{\circ}_{\mathrm{A}}$:

$$\begin{aligned}
\mathcal{H}_{\mathrm{A}}(t,x,z,\gamma) &= \sup_{u\in U}\Big\{[\sigma\lambda](t,x,u)z + \frac{1}{2}\big[\sigma\sigma^{\top}\big](t,x,u)\gamma - c(t,x,u)\Big\} \\
&= \sup_{S\in\mathbb{S}_t(x)}\left\{\sup_{u\in U^{\circ}_t(x,S)}\Big\{[\sigma\lambda](t,x,u)z + \frac{1}{2}\big[\sigma\sigma^{\top}\big](t,x,u)\gamma - c(t,x,u)\Big\}\right\} \\
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- Let $\gamma^{\star}\in\mathbb{R}$, the corresponding $S^{\star}$ is:

$$S^{\star}\in\arg\max_{S\in\mathbb{S}_t(x)}\Big\{\mathcal{H}^{\circ}_{\mathrm{A}}(t,x,z,S) + \frac{1}{2}S\gamma^{\star}\Big\}.$$

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$$S^\star\in\operatorname*{arg\,max}_{S\in\mathbb{S}_t(x)}\Big\{\mathcal{H}^{\circ}_{\mathrm{A}}(t,x,z,S)+\frac{1}{2}S\gamma^\star\Big\}.$$

- Similarly, let $S^\star\in\mathbb{S}_t(x)$, the corresponding $\gamma^\star$ is such that

$$\begin{aligned}S^\star &\in\operatorname*{arg\,max}_{S\in\mathbb{S}_t(x)}\Big\{\mathcal{H}^{\circ}_{\mathrm{A}}(t,x,z,S)+\frac{1}{2}S\gamma^\star\Big\}\\ \Leftrightarrow\gamma^\star &\in\operatorname*{arg\,min}_{\gamma\in\mathbb{R}}\Big\{\mathcal{H}_{\mathrm{A}}(t,x,z,\gamma)-\frac{1}{2}S^\star\gamma\Big\}.\end{aligned}$$

Conclusion & work in progress

- Study an alternative 'first-best' problem (Problem 2), which can be solved relying on BSDEs (only), and show that the value corresponding to this 'first-best' problem can be achieved using contracts of the form (4) in the original problem (Problem 1).

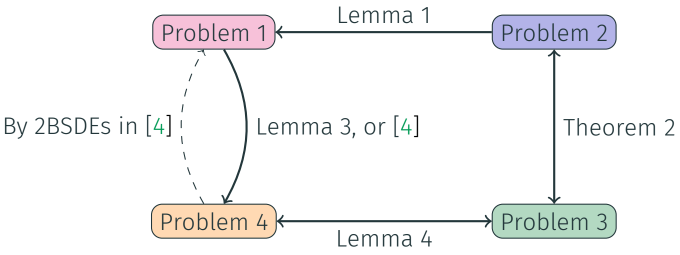

Figure 1: The circle is complete!

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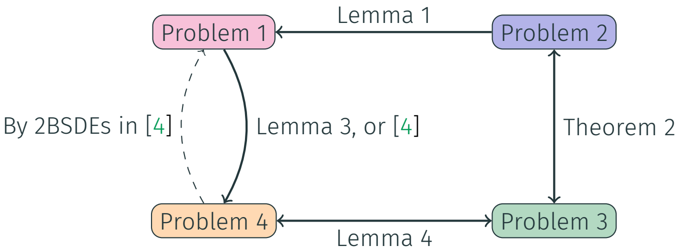

Figure 1: The circle is complete!

- No need to use 2BSDE to show that contracts of the form (4) are optimal.
- Should help to tackle extensions to multi/mean-field agents frameworks, non-continuous output processes...

A principal contracting with two agents

- Consider now that the output process is controlled by agent A through α and by agent B through β:

$$dX_t = (\alpha_t + \beta_t)(\lambda dt + dW_t), \; t \in [0, T].$$

- The principal observes X and $\langle X \rangle$, but cannot deduce individual efforts.

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- The principal observes X and $\langle X \rangle$, but cannot deduce individual efforts.
- Given $\xi := (\xi^A, \xi^B)$ and $\nu := (\alpha, \beta)$, consider

$$J_A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\big[-\exp(-\gamma_A \xi^A)\big], \ \gamma_A > 0$$
$$J_B(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\big[-\exp(-\gamma_B \xi^B)\big], \ \gamma_B > 0$$
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- Two ‘possible’ approaches:

(i) Extend the results from [4], as done in [7] $\Rightarrow$ Requires ‘multidimensional’ 2BSDEs...

(ii) New approach: look at a ‘first-best’ reformulation of the problem, in which the principal controls $\langle X \rangle$.

- **Step 1.** Formulate the 'first-best' associated problem:

(i) The principal chooses $\xi := (\xi^A, \xi^B)$ as well as the quadratic variation of X through its density $\boldsymbol{\Sigma}$.

(ii) The agents play a Nash $\nu^\star := (\alpha^\star, \beta^\star)$ such that $(\alpha^\star_t + \beta^\star_t)^2 = \boldsymbol{\Sigma}_t$, $t \in [0, T]$.

New approach (step 1)

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- The (usual) Hamiltonian for agent A can be decomposed as follows:

$$\mathcal{H}_A(t, x, z, \gamma, b) = \sup_{a \in \mathbb{R}} \left\{ (a+b)\lambda z + \frac{1}{2}\gamma(a+b)^2 \right\}$$

$$= \sup_{S \in \mathbb{R}_+} \left\{ \sup_{a \in \mathbb{R} : (a+b)^2 = S} \{(a+b)\lambda z\} + \frac{1}{2}\gamma S \right\}$$

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(ii) The agents play a Nash $\nu^\star := (\alpha^\star, \beta^\star)$ such that $(\alpha_t^\star + \beta_t^\star)^2 = \Sigma_t$, $t \in [0, T]$.

- The (usual) Hamiltonian for agent A can be decomposed as follows:

$$\mathcal{H}_A(t, x, z, \gamma, b) = \sup_{a \in \mathbb{R}} \left\{ (a+b)\lambda z + \frac{1}{2}\gamma(a+b)^2 \right\}$$
$$= \sup_{S \in \mathbb{R}_+} \left\{ \sup_{a \in \mathbb{R}:(a+b)^2 = S} \{(a+b)\lambda z\} + \frac{1}{2}\gamma S \right\}$$

- For fixed Σ, we shall consider the 'constrained' Hamiltonian:

$$\mathcal{H}_A^\circ(t, x, z, S, b) = \sup_{a \in \mathbb{R}:(a+b)^2 = S} \{(a+b)\lambda z\} \;\Rightarrow\; a^\circ(z, S, b) := \mathrm{sgn}(z)\sqrt{S} - b.$$

- **Step 1.** Formulate the 'first-best' associated problem:

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- For fixed Σ and $Z := (Z^A, Z^B)$ s.t. $\mathrm{sgn}(Z^A) = \mathrm{sgn}(Z^B)$, the set of Nash is:

$$\mathcal{N}(\Sigma, Z) := \left\{ \nu^\circ := (\alpha^\circ, \beta^\circ) \text{ s.t. } (\alpha^\circ_t + \beta^\circ_t)^2 = \Sigma_t,\ t \in [0, T] \right\}.$$

- **Step 2.** Using BSDEs, one can show that the optimal form of contracts is:

$$\xi^A = \xi_0^A - \int_0^T \widetilde{\mathcal{H}}^\circ(Z_t^A, \Sigma_t)dt + \int_0^T Z_t^A dX_t + \frac{\gamma_A}{2}\int_0^T |Z_t^A|^2 \Sigma_t dt,$$
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$$V_P \le V_P^\circ = \sup_{Z^A, Z^B, \Sigma} \mathbb{E}\big[-\exp\big(-\gamma_P(X_T - \xi^A - \xi^B)\big)\big]$$

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NEW APPROACH (STEP 3)

- **Step 3.** Inspired by [4], we consider the following contracts:

$$\xi^A = \xi_0^A - \int_0^T \widetilde{\mathcal{H}}(Z_t^A, \Gamma_t^A)\mathrm{d}t + \int_0^T Z_t^A \mathrm{d}X_t + \frac{1}{2}\int_0^T \big(\Gamma_t^A + \gamma_A |Z_t^A|^2\big)\mathrm{d}\langle X\rangle_t,$$

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- Here, we have:

$$\mathcal{H}_A(t, x, z, \gamma, b) = \sup_{a \in \mathbb{R}} \left\{ (a+b)\lambda z + \frac{1}{2}\gamma(a+b)^2 \right\} \Rightarrow \alpha_t = -\lambda \frac{Z_t^A}{\Gamma_t^A} - \beta_t.$$

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- To ensure existence of a Nash, we may restrict to contracts (ξ^A, ξ^B) s.t.

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- The Hamiltonian at any Nash is

$$\widetilde{\mathcal{H}}(z, \gamma) = -\frac{1}{2}\lambda^2 \frac{z^2}{\gamma}.$$

New approach: conclusion

- 'First-best' solution:

$$V_P \leq V_P^\circ = \sup_{Z^A, Z^B, \Sigma} \mathbb{E}\big[-\exp\big(-\gamma_P (X_T - \xi^A - \xi^B)\big)\big]$$

$$\text{with } \; dX_t = \operatorname{sgn}(Z_t^A)\sqrt{\Sigma_t}(\lambda dt + dW_t),$$

$$d\xi_t^A = \frac{1}{2}\gamma_A |Z_t^A|^2 \Sigma_t dt + Z_t^A \operatorname{sgn}(Z_t^A)\sqrt{\Sigma_t} dW_t,$$

$$\text{and } \; d\xi_t^B = \frac{1}{2}\gamma_B |Z_t^B|^2 \Sigma_t dt + Z_t^B \operatorname{sgn}(Z_t^B)\sqrt{\Sigma_t} dW_t.$$

- Using forcing contracts:

$$V_P \geq \sup_{Z^A, Z^B, \widetilde{Z}} \mathbb{E}\big[-\exp\big(-\gamma_P (X_T - \xi^A - \xi^B)\big)\big]$$

$$\text{with } \; dX_t = \widetilde{Z}_t(\lambda dt + dW_t),$$

$$d\xi_t^A = \frac{1}{2}\gamma_A |Z_t^A|^2 \widetilde{Z}_t^2 dt + Z_t^A \widetilde{Z}_t dW_t,$$

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- The two problems are the same!

- Multidimensional and non-Markovian output process X:

$$X_t = X_0 + \int_0^t \sigma(s, X_{\cdot\wedge s}, \nu_s)\big(\lambda(s, X_{\cdot\wedge s}, \nu_s)ds + dW_s^{\nu}\big),\ t \in [0,T],\ \mathbb{P}^{\nu}\text{-a.s.},$$

controlled by N agents: $\nu := (\nu^1, \dots, \nu^N)$.

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controlled by N agents: $\nu := (\nu^1, \dots, \nu^N)$.

- Given $\xi := (\xi^1, \dots, \xi^N)$, general reward function for each agent:

$$J_i(\xi, \nu^{-i}, \nu^i) := \mathbb{E}^{\mathbb{P}^{\nu}}\left[\mathcal{K}_i^{\nu}(T)U_i(X, \xi^i) - \int_0^T \mathcal{K}_i^{\nu}(t)c_i(t, X_{\cdot\wedge t}, \nu_t)dt\right],$$

$$\text{with } \mathcal{K}_i^{\nu}(t) := \exp\left(-\int_0^t k_i(s, X_{\cdot\wedge s}, \nu_s)\right),\ t \in [0,T].$$

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- General reward function for the principal:

$$J_P(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu}\left[\mathcal{K}_P(T)U_P(X_T, \xi)\right],$$

$$\text{with } \mathcal{K}_P(T) := \exp\left(-\int_0^T k_P(s, X_{\cdot\wedge s})ds\right).$$

- Step 1. Formulate the 'first-best' alternative problem:

(i) The principal chooses the contracts ξ and Σ;

(ii) The agent play a Nash $\nu^\star$ s.t. $[\sigma\sigma^\top](t, X, \nu_t^\star) = \Sigma_t$ for all $t \in [0, T]$.

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- Step 2. Use BSDE theory to characterise the optimal form of contracts: $\xi^i = U_i^{-1}(X, Y_T^i)$ where

$$Y_T^i = y_0^i - \int_0^T \widetilde{\mathcal{H}}_i^\circ(Z_t, \boldsymbol{\Sigma}_t)\mathrm{d}t + \int_0^T Z_t^i \mathrm{d}X_t,$$

where $\widetilde{\mathcal{H}}_i^\circ$ is the 'constrained' Hamiltonian at a Nash.

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- Step 3. Use 'forcing' contracts of the form $\xi^i = U_i^{-1}(X, Y^i_T)$ where

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where $\widetilde{\mathcal{H}}_i$ is the 'usual' Hamiltonian at a Nash.

- To be continued...

Thank you all for your attention!

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