Séminaire FDD-FiME

# Principal–agent problem with Mckean–Vlasov dynamics and its connections

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Problem **Principal** and **Agent** may have objectives that are not align

Goal Design the right incentives for both parties

## Contract theory: Principal–Agent problems

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1–**Principal** suggests a contract to the **Agent**

2–From this contract, **Principal** estimates the best response from the **Agent**

3–**Principal** designs the best contract that aligns with her own interest given the best response from the **Agent**

$\longrightarrow$ Stackelberg type equilibrium

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Literature Holmström and Milgrom (1987), Schättler and Sung (1993, 1997), Sung (1995), Y. Sannikov (2008, 2013), N. Williams (2008), Cvitanić and Zhang (2012), Cvitanić, Possamai and Touzi (2015, 2017), $\cdots$

$\longrightarrow$ Classical Principal–Agent problem

Contract for the Agent $\mathfrak{C} = \xi$,

$$\xi : C([0,T];\mathbb{R}) \to \mathrm{E}.$$

The contracts are only depending of the production of the agent $X^{\alpha}$ (moral hazard)

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Agent's problem Given $\mathfrak{C} = \xi$ and control $\alpha : [0,T] \times C([0,T];\mathbb{R}) \to A$, let $X^\alpha$ be a process representing the production satisfying (in a weak sense):

$$\mathrm{d}X_t^\alpha = b\left(t, X_t^\alpha, \alpha(t, X^\alpha)\right) \mathrm{d}t + \mathrm{d}W_t.$$

The goal is to find $\alpha^\star \in \mathrm{argmax}_\alpha J_A^{\mathfrak{C}}(\alpha)$ where $J_A^{\mathfrak{C}}(\alpha) := \mathbb{E}\left[U_A\left(\xi(X^\alpha)\right) - \int_0^T \frac{1}{2}|\alpha(t, X^\alpha)|^2 \,\mathrm{d}t\right]$

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Principal's problem The goal is to find $\mathfrak{C}^\star \in \operatorname{argmax}_{\mathfrak{C}} J_P^{\alpha^\star}(\mathfrak{C})$ where $\alpha^\star \in \operatorname{argmax}_\alpha J_A^{\mathfrak{C}}(\alpha)$, $J_A^{\mathfrak{C}}(\alpha) \geq R_0$ and

$$J_P^{\alpha^\star}(\mathfrak{C}) := \mathbb{E}\left[X_T^{\alpha^\star} - \xi(X^{\alpha^\star})\right].$$

$\longrightarrow$ Principal–Agent problem with McKean–Vlasov dynamics

Contract $\mathfrak{C} = \xi$,

$$\xi : C([0,T];\mathbb{R}) \times \boxed{\mathcal{P}_2\left(C([0,T];\mathbb{R})\right)} \to \mathrm{E}$$

Agent's problem Given $\mathfrak{C}$ and control

$$\alpha : [0,T] \times C([0,T];\mathbb{R}) \to A.$$

Let $X^\alpha$ be the process representing the production satisfying

$$\mathrm{d}X_t^\alpha = b\left(t, X_t^\alpha, \boxed{\mu_t^\alpha}, \alpha(t, X^{\boldsymbol{\alpha}})\right)\mathrm{d}t + \mathrm{d}W_t, \quad \mu_t^\alpha = \mathcal{L}(X_t^\alpha).$$

The reward of the agent is given by $J_A^{\mathfrak{C}}(\alpha) := \mathbb{E}\left[U_A\left(\xi\left(X^\alpha, \boxed{\mathcal{L}(X^\alpha)}\right)\right) - \int_0^T \frac{1}{2}|\alpha(t,X^\alpha)|^2 \mathrm{d}t\right]$.

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$$J_P^\alpha(\mathfrak{C}) := \mathbb{E}\left[X_T^\alpha - \xi\left(X^\alpha, \boxed{\mathcal{L}(X^\alpha)}\right)\right].$$

Difference with the classical setting

- Measure dependence in the contract i.e. $\xi(X^\alpha) \longrightarrow \xi\left(X^\alpha, \boxed{\mathcal{L}(X^\alpha)}\right)$
- The coefficients may also depend on the measure

$\longrightarrow$ Resolution of the classical setting

Resolution of the agent's problem $\longrightarrow$ through BSDE (Cvitanić, Possamai and Touzi (2015, 2017))

Let $(t,x,e,z) \mapsto \widehat{\alpha}(t,x,z) \in A$ the unique maximizer of

$$a \mapsto b(t,x,a)z - \frac{1}{2}a^2 =: h(t,x,z,a) \text{ and } (\widehat{b},\widehat{h})(t,x,z) := (b,h)\left(t,x,z,\widehat{\alpha}(t,x,z)\right).$$

Given a contract $\xi$, the unique optimal control is given by $\widehat{\alpha}(t,X_t,Z_t)$ where:

$$\mathrm{d}X_t = \widehat{b}(t,X_t,Z_t)\mathrm{d}t + \mathrm{d}W_t \text{ and } Y_\cdot = U_A(\xi(X)) - \int_\cdot^T \frac{1}{2}\widehat{\alpha}(t,X_t,Z_t)^2\mathrm{d}t + \int_\cdot^T Z_t\mathrm{d}W_t$$

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Idea of the proof: By Girsanov's Theorem, $\mathbb{P}\circ(X^\alpha) = \mathbb{P}^\alpha \circ (X)^{-1}$ and $W^\alpha_\cdot = W_\cdot - \int_0^\cdot \psi_t \mathrm{d}t$ is a $\mathbb{P}^\alpha$–Brownian motion where $\mathrm{d}\mathbb{P}^\alpha := L_T \mathrm{d}\mathbb{P}$, with $\frac{\mathrm{d}L_t}{L_t} = \psi_t \mathrm{d}W_t$ and $\psi_t := \widehat{b}(t,X_t,Z_t) - b(t,X_t,\alpha(t,X))$,

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$$J_A^\xi(\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\left[U_A(\xi(X)) - \int_0^T \frac{1}{2}|\alpha(t,X)|^2\mathrm{d}t\right] = \mathbb{E}^{\mathbb{P}^\alpha}\left[Y_T - \int_0^T \frac{1}{2}|\alpha(t,X)|^2\mathrm{d}t\right]$$
$$= \mathbb{E}^{\mathbb{P}^\alpha}\left[Y_0 + \int_0^T h(t,X_t,Z_t,\alpha(t,X)) - h(t,X_t,Z_t,\widehat{\alpha}(t,X_t,Z_t))\mathrm{d}t\right]$$
$$\leq \mathbb{E}^{\mathbb{P}^\alpha}[Y_0] = \mathbb{E}^{\mathbb{P}}[Y_0] = \mathbb{E}^{\mathbb{P}}\left[U_A(\xi(X)) - \int_0^T \frac{1}{2}|\widehat{\alpha}(t,X_t,Z_t)|^2\mathrm{d}t\right] = J_A^\xi(\widehat{\alpha}).$$

⟶ Resolution of the classical setting

Principal's problem ⟶ through stochastic control theory with two states variables: production and continuation utility

For any $Z_t = \gamma(t, X)$ where $X$ satisfies

$$\mathrm{d}X_t = \widehat{b}(t, X_t, Z_t)\mathrm{d}t + \mathrm{d}W_t \tag{1}$$

with the right integrability conditions, $U_A^{-1}(Y_T)$ is a contract where

$$Y_\cdot = Y_0 + \int_0^\cdot \widehat{h}\,(t, X_t, Z_t)\,\mathrm{d}t - \int_0^\cdot Z_t \mathrm{d}X_t \tag{2}$$

and $\widehat{\alpha}(t, X_t, Z_t)$ is an optimal control for the agent's problem for the contract $U_A^{-1}(Y_T)$, then

$$\mathbb{E}\left[X_T - U_A^{-1}(Y_T)\right] \leq \sup_{\mathfrak{C},\alpha^\star} J_P^{\alpha^\star}(\mathfrak{C}).$$

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Conversely, we have seen that, for any contract $\xi$, there exists $(X, Y, Z)$ verifying (1) and (2) with $Y_T = U_A(\xi(X))$ and $Z_t \,\widehat{\in}\, \sigma(X_s : s \leq t)$ and the unique optimal control is $\alpha^\star = \widehat{\alpha}$, then

$$\sup_{\mathfrak{C},\alpha^\star} J_P^{\alpha^\star}(\mathfrak{C}) \leq \sup_Z \mathbb{E}\left[X_T - U_A^{-1}(Y_T)\right].$$

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$$\sup_{\mathfrak{C}, \alpha^\star} J_P^{\alpha^\star}(\mathfrak{C}) \leq \sup_Z \mathbb{E}\left[X_T - U_A^{-1}(Y_T)\right].$$

$\longrightarrow$ $\sup_{\mathfrak{C}, \alpha^\star} J_P^{\alpha^\star}(\mathfrak{C}) = \sup_Z \mathbb{E}\left[X_T - U_A^{-1}(Y_T)\right]$

$\longrightarrow$ Stochastic control problem with control of the diffusion coefficient through $\int_0^\cdot Z_t \mathrm{d}W_t$ !

Main problems

- The approach by DPP of (Cvitanić, Possamai and Touzi (2015, 2017)) failed i.e. we cannot apply Girsanov Theorem

$$\mathbb{P} \circ (X^{\alpha}) \neq \mathbb{P}^{\alpha} \circ (X)^{-1}.$$

- As usual, we only want $\xi$ to be measurable in space i.e. for each $m$, $x \mapsto \xi(x,m)$ is just measurable

$\longrightarrow$ The Principal–Agent problem with McKean–Vlasov dynamics is actually the limit, when $n$ goes to infinity, of a problem with Principal with $n$ Agent.

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## Problem

- Dependencies and interactions between Agents

$\longrightarrow$ Decisions of one Agent may affect the incentives or outcomes of the other Agents leading to coordination problem

- What best response from the $n$ Agents means • The situation is s.t $n$ is very large

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Typical examples: **Government** (principal) and **Population** (agents), **Tech company** like Google, Facebook (Principal) and **content creators** (agents), **fast food company** like McDonald's (Principal) and **franchisees** (agents).

## Contract theory: Principal with $n$ agents

### Problem

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### Mechanism

1. Principal offers contracts
2. Estimates the best response from the $n$ Agents
3. Choosing the best possible contract

$$\boxed{\text{Principal}} \Longleftrightarrow \boxed{\begin{cases} \text{Agent } 1 \\ . \\ . \qquad\qquad\qquad \| \text{Interactions and Coordinations} \\ . \\ \text{Agent } n \end{cases}}$$

Contract for $n$ agents $\mathfrak{C}^n = \xi^n$,

$$\xi^n : C([0,T];\mathbb{R})^n \to \mathrm{E}$$

Agents' problem Given $\mathfrak{C}^n$ and $n$ controls $\boldsymbol{\alpha} = (\alpha^{1,n}, \cdots, \alpha^{n,n})$,

$$\alpha^{i,n} : [0,T] \times C([0,T];\mathbb{R})^n \to A.$$

Let $\mathbf{X}^{\boldsymbol{\alpha}} = (X^{\boldsymbol{\alpha},1}, \cdots, X^{\boldsymbol{\alpha},n})$ be the processes representing the productions satisfying

$$\mathrm{d}X_t^{\boldsymbol{\alpha},i} = b\left(t, X_t^{\boldsymbol{\alpha},i}, \mu_t^n, \alpha^{i,n}(t, \mathbf{X}^{\boldsymbol{\alpha}})\right)\mathrm{d}t + \mathrm{d}W_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{\boldsymbol{\alpha},i}}.$$

Reward of agent $i$, $J_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}) := \mathbb{E}\left[\mathrm{R}_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}^n)\right]$, $\mathrm{R}_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}^n) := U_A\left(\xi^n(\mathbf{X}^{\boldsymbol{\alpha}})\right) - \int_0^T \frac{1}{2}|\alpha^{i,n}(t, \mathbf{X}^{\boldsymbol{\alpha}})|^2 \mathrm{d}t$

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Reward of agent $i$, $J_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}) := \mathbb{E}\left[\mathrm{R}_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}^n)\right]$, $\mathrm{R}_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}^n) := U_A\left(\xi^n(\mathbf{X}^{\boldsymbol{\alpha}})\right) - \int_0^T \frac{1}{2}|\alpha^{i,n}(t,\mathbf{X}^{\boldsymbol{\alpha}})|^2 \mathrm{d}t$

$\longrightarrow$ Meaning of best response

- Pareto Equilibrium Given $\mathfrak{C}^n = \xi^n$, $\boldsymbol{\alpha} = (\alpha^1, \cdots, \alpha^n)$ is an Pareto equilibrium

$$\frac{1}{n}\sum_{i=1}^{n} J_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}) = \sup_{(\beta^1,\cdots,\beta^n)} \frac{1}{n}\sum_{i=1}^{n} J_{n,i}^{\mathfrak{C}^n}(\beta^1, \cdots, \beta^n).$$

$\longrightarrow$ Cooperative equilibrium, PE$[\mathfrak{C}^n] := \{$ All Pareto equilibria $\boldsymbol{\alpha}$ associated to $\mathfrak{C}^n\}$

- Nash equilibrium Given $\mathfrak{C}^n = \xi^n$, $\boldsymbol{\alpha} = (\alpha^1, \cdots, \alpha^n)$ is a Nash equilibrium

$$J_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}) \geq \sup_{\beta} J_{n,i}^{\mathfrak{C}^n}(\alpha^1, \cdots, \alpha^{i-1}, \beta, \alpha^{i+1}, \cdots, \alpha^n) \text{ for all } 1 \leq i \leq n.$$

$\longrightarrow$ Competitive equilibrium, NE$[\mathfrak{C}^n] := \{$ All Nash equilibria $\boldsymbol{\alpha}$ associated to $\mathfrak{C}^n\}$

Principal's problem Given $\boldsymbol{\alpha} \in \mathrm{PE}[\mathfrak{C}^n]$ or $\mathrm{NE}[\mathfrak{C}^n]$ s.t. $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \mathrm{R}_{n,i}^{\mathfrak{C}^n}(\boldsymbol{\alpha}^n)\middle|\mathbf{X}_0\right] \geq R_0$, Principal's rewrad

$$J_n^{\boldsymbol{\alpha}}(\mathfrak{C}^n) := \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \left(X_T^{\boldsymbol{\alpha},i} - \xi^n(\mathbf{X}^{\boldsymbol{\alpha}})\right)\right] \text{ and } V_n := \sup_{\mathfrak{C}^n} \sup_{\boldsymbol{\alpha}^n} J_n^{\boldsymbol{\alpha}^n}(\mathfrak{C}^n).$$

We use $V_n^{\mathrm{NE}}$ for Nash equilibrium and $V_n$ for Pareto equilibrium.

$\longrightarrow$ Competitive agents

Literature only [solves the limit problem $\Longrightarrow$ optimal contract $\mathfrak{C}^{\star,\infty}$] for competitive agents (Nash equilibrium)

Carmona and Wang (2018), Élie, Mastrolia, and Possamai (2019), Élie, Hubert, Mastrolia, and Possamai (2021), $\cdots$

General characterization in D. "Stackelberg Mean Field Games: convergence and existence results to the problem of Principal with multiple Agents in competition" (2023), https://arxiv.org/abs/2309.00640.

$\longrightarrow$ Competitive agents

The limit problem is

$$V_P^{\text{NE}} := \sup_{\mathfrak{C}} \sup_{\alpha \in \text{MFG}[\mathfrak{C}],\ J_A^{\mathfrak{C}}(\alpha) \geq R_0} J_P^{\mathfrak{C}}(\alpha) \text{ with } J_P^{\mathfrak{C}}(\alpha) := \mathbb{E}\left[X_T^{\alpha} - \xi(\mu)\right] \tag{3}$$

where $\alpha$ belongs to $\text{MFG}[\mathfrak{C}]$ if (representative agent problem)

$$J_A^{\mathfrak{C}}(\alpha) \geq J_A^{\mathfrak{C}}(\beta), \quad \text{for any } \beta \text{ with } J_A^{\mathfrak{C}}(\beta) := \mathbb{E}\left[U_A(\xi(\mu)) - \int_0^T \frac{1}{2}\beta(t, X^{\beta})^2 \mathrm{d}t\right]$$

$\mu$, $W$ and $X_0$ are independent, and

$$\mathrm{d}X_t^{\beta} = b\left(t, X_t^{\beta}, \mu_t, \beta(t, X_t^{\beta}, \mu)\right)\mathrm{d}t + \mathrm{d}W_t \text{ with } \mu_t = \mathcal{L}(X_t^{\alpha}|\mu).$$

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Theorem (D. (2023)) Under some conditions, we have $\lim_{n\to\infty} V_n^{\mathrm{NE}} = V_P^{\mathrm{NE}}$.

## Contract theory: Principal with $n$ agents

$\longrightarrow$ The cooperative case creates completely different difficulties !

- "Advantage" of competitive case A Nash equilibrium effect $\longrightarrow$ when one player deviates, this has a "negligible" effect on the contract
- In cooperative case, we have no longer one deviation but potentially many !

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We introduce, for $\mathbf{Z} = (Z^{1,n}, \dots, Z^{n,n})$,

$$\widehat{J}_n(\mathbf{Z}) := \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_T^{i,\mathbf{Z}} - U_A^{-1}\left(Y_T^{n,\mathbf{Z}}\right)\right]$$

with for each $1 \le i \le n$, $X^{i,\mathbf{Z}} := X^{i,n}$, $Y^{n,\mathbf{Z}} := Y^n$, $Y_0^n \ge R$,

$$\mathrm{d}Y_t^n = \frac{1}{n}\sum_{i=1}^{n}\left|\widehat{\alpha}\left(t, X_t^{i,n}, nZ_t^{i,n}\right)\right|^2 \,\mathrm{d}t + \sum_{i=1}^{n} Z_t^{i,n}\mathrm{d}W_t^i$$

and $\mathrm{d}X_t^{i,n} = \widehat{b}\left(t, X_t^{i,n}, nZ_t^{i,n}\right)\,\mathrm{d}t + \mathrm{d}W_t^i$. and

$$\widehat{V}_n := \sup_{\mathbf{Z}} \widehat{J}_n(\mathbf{Z})$$

By using similar arguments as in the one dimensional case, we have

Theorem With some integrability conditions and assumptions over $b$ and $U_A$,

$$V_n = \widehat{V}_n$$

.

Using the coercivity of the criterium, we can let $n \to \infty$ and we find that: any $\mathbf{Z}$ close to the optimal of the problem $\widehat{V}_n$ satisfies:

$$\sup_{n \geq 1} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\int_0^T |nZ_t^{i,n}|^2 \mathrm{d}t\right] < \infty \text{ and } \lim_{n\to\infty}\sum_{i=1}^{n}\int_0^T Z_t^{i,n}\mathrm{d}W_t^i = 0 \text{ in } \mathbb{L}^2.$$

$\longrightarrow$ Example: $-U_A^{-1}(y) \leq -y$ and $b(t,x,a) = a$. We can then characterize the limit

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$\Longrightarrow$ The limit problem is then a McKean–Vlasov control problem with the law of the control.

$\Longrightarrow$ No control in the diffusion coefficient

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Contract $\mathfrak{C} = \xi$, with $\overline{\xi}(m) := U_A^{-1}\left(\int_{C([0,T];\mathbb{R})} \xi(x,m)m(\mathrm{d}x)\right)$

$$\xi : C([0,T];\mathbb{R}) \times \mathcal{P}_2\left(C([0,T];\mathbb{R})\right) \to \mathrm{E}$$

Agent's problem Given $\mathfrak{C}$ and control

$$\alpha : [0,T] \times C([0,T];\mathbb{R}) \to A.$$

Let $X^\alpha$ be the process representing the production satisfying

$$\mathrm{d}X_t^\alpha = b\left(t, X_t^\alpha, \mu_t^\alpha, \alpha(t, X^{\boldsymbol{\alpha}})\right)\mathrm{d}t + \mathrm{d}W_t,\ \ J_A^{\mathfrak{C}}(\alpha) := \mathbb{E}\left[U_A\left(\overline{\xi}\left(\mathcal{L}(X^\alpha)\right)\right) - \int_0^T \frac{1}{2}|\alpha(t,X^\alpha)|^2\mathrm{d}t\right].$$

Theorem (D. (2024)) Given $\mathfrak{C}$, for any $\alpha$, there exists $\Gamma$ verifying

$$J_A^{\mathfrak{C}}(\alpha) \leq \sup_{\alpha'} J_A^{\mathfrak{C}}(\alpha') + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} h\left(t, X_t^\alpha, \mu_t^\alpha, z, \alpha(t,X^\alpha)\right) - \widehat{h}\left(t, X_t^\alpha, \mu_t^\alpha, z\right)\Gamma(t, X^\alpha)(\mathrm{d}z)\mathrm{d}t\right].$$

$\longrightarrow$ If $\alpha$ is optimal, there exists $\gamma$ s.t. $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e.

$$\alpha(t, X^\alpha) = \widehat{\alpha}(t, X_t^\alpha, \mu_t^\alpha, \gamma(t, X^\alpha))$$

Principal's problem We recall that, given $\alpha \in \mathrm{PE}[\mathfrak{C}]$ s.t. $J_A^{\mathfrak{C}}(\alpha) \geq R_0$, Principal's rewrad

$$J^\alpha(\mathfrak{C}) := \mathbb{E}\left[X_T^\alpha - \overline{\xi}(\mathcal{L}(X^\alpha))\right] \text{ and } V := \sup_{\mathfrak{C}} \sup_{\alpha \in \mathrm{PE}[\mathfrak{C}]} J^\alpha(\mathfrak{C}).$$

given $\alpha \in \mathrm{PE}[\mathfrak{C}]$ i.e. optimal, there exists $\gamma$ s.t.

$$\alpha(t, X^\alpha) = \widehat{\alpha}(t, X_t^\alpha, \mu_t^\alpha, \gamma(t, X^\alpha)) \text{ for some } \gamma$$

Theorem (D. (2024)) Under some conditions over the coefficients $(b, U_A)$,

$$U_P(V) = \widehat{V} = \lim_{n \to \infty} V_n.$$

Recall that

$$\widehat{V} := \sup_\gamma \widehat{J}(\gamma) \text{ with } \widehat{J}(\gamma) := \mathbb{E}\left[X_T^\gamma - U_A^{-1}\left(Y_T^\gamma\right)\right].$$

where $\mathrm{d}X_t = \widehat{b}(t, X_t, \gamma(t, X_t))\ \mathrm{d}t + \mathrm{d}W_t$ and $Y_\cdot^\gamma := Y_0 + \mathbb{E}\left[\int_0^\cdot \frac{1}{2}|\widehat{\alpha}(t, X_t, \gamma(t, X_t))|^2\ \mathrm{d}t\right].$

$\longrightarrow$ Construction of optimal contract from $\widehat{V}$

Let $\gamma$ and $Y_0(\geq R)$ be optimal for $\widehat{V}$ verifying: with $U_\cdot = X_0 + W_\cdot$,

$$\mathbb{E}\left[\exp\left(\int_0^T a|\gamma(t,U_t)|^2 \mathrm{d}t\right)\right] < \infty, \quad \text{for each } a \geq 0. \tag{4}$$

The optimal contract is given by

$$U_A\left(\overline{\xi}(\overline{\mu})\right) := \mathbb{E}\left[\underbrace{Y_0 - \int_0^T \widehat{h}\left(t, X_t, \mu_t, \gamma(t,X_t)\right)\,\mathrm{d}t + \int_0^T \gamma(t,X_t)\mathrm{d}X_t}_{\xi(X,\overline{\mu})}\right]$$

$\forall\ X$ s.t. $\mathcal{L}(X) = \overline{\mu}$, $\mu = (\mathcal{L}(X_t))_{t\in[0,T]}$, $X$ is a semi–martingale s.t. $\int_0^T \gamma(t,X_t)^2 \mathrm{d}\langle X\rangle_t < \infty$ a.e.

The optimal control of the problem of the agent is given by $(t,x) \mapsto \widehat{\alpha}(t, x_t, \mu_t^{\widehat{\alpha}}, \gamma(t,x_t))$

## A toy model

The agent is facing the problem: with $\mathrm{d}X_t^\alpha = \Big(\alpha(t, X^\alpha) + \overline{\kappa}\mathbb{E}\left[b(X_t^\alpha)\right]\Big)\,\mathrm{d}t + \mathrm{d}W_t$,

$$\sup_\alpha \mathbb{E}\left[\xi\left(\mathcal{L}(X^\alpha)\right) - \frac{1}{2}\int_0^T |\alpha(t, X^\alpha)|^2 \mathrm{d}t\right]$$

where $b$ is given by $b(x) = -\overline{b} \vee (\overline{b} \wedge x)$ with $\overline{b} > 0$ very large. The Principal is then trying to solve

$$V_P = \sup_\xi \sup_{\boldsymbol{\alpha} \in \mathrm{PE}[\mathfrak{C}]} \mathbb{E}\left[X_T^\alpha - \xi\left(\mathcal{L}(X^\alpha)\right)\right].$$

Let us define $\widehat{\gamma}(t,x) := e^{\overline{k}(T-t)}$, $\alpha^\star(t,x) = e^{\overline{k}(T-t)}$, and the contract $\mathfrak{C} := \xi$, for any $\alpha$

$$\xi(\mathcal{L}(\mathbf{X}^\alpha)) := \mathbb{E}\left[R - \int_0^T \frac{1}{2}|\widehat{\gamma}_t|^2 + \overline{\kappa}\mathbb{E}\left[b(X_t^\alpha)\right]\mathrm{d}t + \int_0^T \widehat{\gamma}_t \mathrm{d}X_t^\alpha\right].$$

Theorem (D. (2024)) There exists a positive constant $C$ independent $\overline{b}$ s.t.

$$V_P - J_P^{\hat{\alpha}}(\mathfrak{C}) \le C\overline{b}^{-1}$$

and $J_n^{\hat{\boldsymbol{\alpha}}^n}(\mathfrak{C}^n) = \mathbb{E}\left[-R + e^{\overline{k}T}\mathbb{E}\left[X_0\right] + \frac{1}{2}\int_0^T |\widehat{\gamma}_t|^2 \mathrm{d}t\right].$

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## Conclusion

- Convergence results for Principal–agent problems
- We can use the McKean–Vlasov problem to construct a contract that solves the problem of the Principal–Agent problem with McKean–Vlasov dynamics
- While the classical Principal–Agent problem is with control in the diffusion coefficient, the PA with McKean–Vlasov dynamics is not !

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Open questions / current research

- What happens when there is a common noise impacting all the agents .

  $\longrightarrow$ McKean–Vlasov control problem with control of volatility. Indeed, we end up with

$$\mathrm{d}Y_t^n = \frac{1}{n}\sum_{i=1}^n \left|\widehat{\alpha}\left(t, X_t^{i,n}, nZ_t^{i,n}\right)\right|^2 \mathrm{d}t + \sum_{i=1}^n Z_t^{i,n}\mathrm{d}W_t^i + \underbrace{\sum_{i=1}^n Z_t^{i,n}\mathrm{d}W_t^0}_{\text{common noise part}}$$

- What happens if the Principal does not know completely if the agents are in competition or in cooperation i.e. unknown parameter representing the belief of the Principal of the state of cooperation or not of the agents
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- Many more $\cdots$

THANK YOU FOR YOUR ATTENTION