Dynkin Games with Partial and Asymmetric Information

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FDD-FiME Paris IHP
4 April 2025

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Research Education Outreach CCA

Outline

Introduction

Aims of the talk: illustrate general results and explicit solutions for zero-sum and nonzero-sum Dynkin games with asymmetric/incomplete information

The talk is based on

- De Angelis, Merkulov, Palczewski (2022). *On the value of non-Markovian Dynkin games with partial and asymmetric information.* **Ann. Appl. Probab.** 32 (3), 1774–1813.
- De Angelis, Ekström, Glover (2022). *Dynkin games with incomplete and asymmetric information.* **Math. Oper. Res.** 47 (1), pp. 560–586.
- De Angelis, Ekström (2020). *Playing with ghosts in a Dynkin game.* **Stoch. Process. Appl.** 130, pp. 6133–6156.

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Introduction I

Related topics/literature:

- Standard Dynkin games and Dynkin games with incomplete information: broad body of existing literature
- Stoch. diff. games with asymmetric information (Cardaliaguet and Rainer, 2009) – idea of randomised strategies
- Dynkin games with asymmetric information (Grün, 2013, Gensbittel and Grün, 2019) – viscosity solution of variational inequalities
- Radomised stopping times as increasing processes (Baxter and Chacon, 1977, Meyer, 1978)
- Min-max theorems (Sion, 1958, Touzi and Vieille, 2002)

Introduction II

Recent developmets:

- Private information: Kwon-Palczewski, MOR 2024, Gaitsgori-Groenewald, SICON 2025.
- General results, equilibria and mixed strategies: Decámp, Gensbittel, Mariotti, arXiv 2022 and 2024, Christensen-Schultz, arXiv 2024, Christensen-Lindensjö, arXiv 2024.
- Uncertain competition: Ekström-Lindensjö-Olofsson, SICON 2022, Ekström-Milazzo-Olofsson, SPA 2024.

General results:

value and optimal strategies in zero-sum games

Setting I: Processes

- Complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and $T \in (0, \infty]$
- $X \in \mathcal{L}$: real-valued càdlàg $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable processes with the norm

$$\|X\|_{\mathcal{L}} := \mathbf{E}\Big[\sup_{t \in [0,T]} |X_t|\Big] < \infty.$$

- Regular process:

$$\mathbf{E}[X_\eta - X_{\eta-} | \mathcal{F}_{\eta-}] = 0 \quad \mathbf{P}-a.s. \text{ for all predictable } (\mathcal{F}_t)\text{-stopping times } \eta.$$

(e.g., quasi left-cont., std. Markov processes, strong/weak solutions of SDEs, etc.)

Setting II: Game

- (P1) Player 1's information is contained in $(\mathcal{F}_t^1) \subseteq (\mathcal{F}_t)$
- (P2) Player 2's information is contained in $(\mathcal{F}_t^2) \subseteq (\mathcal{F}_t)$
- In general $(\mathcal{F}_t^1) \neq (\mathcal{F}_t^2)$ and $(\mathcal{F}_t^1), (\mathcal{F}_t^2) \subsetneq (\mathcal{F}_t)$
- P1 selects $\tau \in [0, T]$ and P2 selects $\sigma \in [0, T]$: the game ends at $\tau \wedge \sigma$ when P1 pays P2 the amount

$$\mathcal{P}(\tau, \sigma) = f_\tau 1_{\{\tau<\sigma\}} + g_\sigma 1_{\{\sigma<\tau\}} + h_\tau 1_{\{\tau=\sigma\}}. \tag{1}$$

So, P1 is the minimiser and P2 is the maximiser

Setting III: Payoffs

The *payoff processes* f, g and h satisfy the following conditions:

Assumptions

(A1) $f, g \in \mathcal{L}$,

(A2) f and g have the decomposition $f = \tilde{f} + \hat{f}$, $g = \tilde{g} + \hat{g}$ with

- (i) $\tilde{f}, \tilde{g} \in \mathcal{L}$,
- (ii) $\tilde{f}, \tilde{g}$ are $(\mathcal{F}_t)$-adapted regular processes,
- (iii) $\hat{f}, \hat{g}$ are $(\mathcal{F}_t)$-adapted (right-continuous) piecewise-constant processes of integrable variation with $\hat{f}_0 = \hat{g}_0 = 0$, $\Delta\hat{f}_T = \hat{f}_T - \hat{f}_{T-} = 0$ and $\Delta\hat{g}_T = \hat{g}_T - \hat{g}_{T-} = 0$,
- (iv) either $\hat{f}$ is non-increasing or $\hat{g}$ is non-decreasing.

(A3) $f_t \geq h_t \geq g_t$ for all $t \in [0, T]$, $\mathbf{P}$-a.s., (second-mover advantage)

(A4) h is an $(\mathcal{F}_t)$-adapted, measurable process.

Setting IV: randomised stopping times

Given a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$ let

$$\mathcal{A}(\mathcal{G}_t) := \{\rho : \rho \text{ is } (\mathcal{G}_t)\text{-adapted with } t \mapsto \rho_t(\omega) \text{ càdlàg}, \\ \text{non-decreasing}, \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) = 1 \text{ for all } \omega \in \Omega\}.$$

Definition (Randomised stopping times)

A $(\mathcal{G}_t)$-*randomised stopping time* is defined as

$$\eta = \eta(\rho, Z) := \inf\{t \in [0,T] : \rho_t > Z\}, \quad \mathbf{P} - a.s.$$

with $Z \sim U([0,1])$, independent of $\mathcal{F}_T$, and $\rho \in \mathcal{A}(\mathcal{G}_t)$.

The set of $(\mathcal{G}_t)$-randomised stopping times is denoted by $\mathcal{T}^R(\mathcal{G}_t)$. Randomisation devices of different stopping times are independent.

Terminology: We say that $\rho \in \mathcal{A}$ generates η.

Value of the game and optimal strategies

Definition (Value of the game)

Define

$$V_* := \sup_{\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)} \inf_{\tau \in \mathcal{T}^R(\mathcal{F}_t^1)} \mathsf{E}[\mathcal{P}(\tau,\sigma)] \quad (\text{lower value})$$

$$V^* := \inf_{\tau \in \mathcal{T}^R(\mathcal{F}_t^1)} \sup_{\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)} \mathsf{E}[\mathcal{P}(\tau,\sigma)] \quad (\text{upper value}).$$

The game has a *value in randomised strategies* if $V = V_* = V^*$.

Definition (Optimal strategies)

An admissible pair $(\tau_*,\sigma_*) \in \mathcal{T}^R(\mathcal{F}_t^1) \times \mathcal{T}^R(\mathcal{F}_t^2)$ is a *saddle point* (or a pair of *optimal strategies*) if

$$\mathsf{E}[\mathcal{P}(\tau_*,\sigma)] \le \mathsf{E}[\mathcal{P}(\tau_*,\sigma_*)] \le \mathsf{E}[\mathcal{P}(\tau,\sigma_*)],$$

for all other admissible pairs (τ,σ).

Main Result

Theorem (Value of the game and optimal strategies)

Under the assumptions stated above the game has a value in randomised strategies. Moreover, if $\hat{f}$ and $\hat{g}$ are non-increasing and non-decreasing, respectively, there exists a pair (τ_*, σ_*) of optimal strategies.

Remark. Our assumptions are minimal and we provide counterexamples in the paper.

Key ideas in the proof...

Game of singular controls

For $\tau \in \mathcal{T}^R(\mathcal{F}_t^1)$, $\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)$,

$$\mathbf{E}[\mathcal{P}(\tau,\sigma)] = \mathbf{E}\Big[\int_{[0,T)} f_t(1-\zeta_t)\mathrm{d}\xi_t + \int_{[0,T)} g_t(1-\xi_t)\mathrm{d}\zeta_t + \sum_{t\in[0,T]} h_t \Delta\xi_t \Delta\zeta_t\Big],$$

where (ξ_t) and (ζ_t) are the generating processes for τ and σ, respectively.

Interpretation:

$\xi_t = \mathbf{P}(\tau \le t | \mathcal{F}_t^1)$ Rightarrow $\mathrm{d}\xi_t$ is the pdf (conditional on $\mathcal{F}_t^1$) of the stopping distribution at time t

$\mathbf{P}(\tau > t | \mathcal{F}_t^1) = 1 - \xi_t$ and $\Delta\xi_t\Delta\zeta_t = \mathbf{P}(\tau = \sigma = t | \mathcal{F}_t^1 \vee \mathcal{F}_t^2)$

Notation: $N(\xi,\zeta) = \mathbf{E}[\mathcal{P}(\tau,\sigma)]$

Smoothing

$\mathcal{A}_{ac} \subset \mathcal{A}$ with absolutely continuous paths. We use an auxiliary game:

$$W_* = \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} \inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi, \zeta) \quad \text{and} \quad W^* = \inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} N(\xi, \zeta).$$

Clearly $W_* \geq V_*$ and $W^* \geq V^*$. Then we prove $W_* = V_*$ so that $W_* = V_* \leq V^* \leq W^*$.

The final step is to prove that $W_* = W^*$.

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Sion's min-max theorem

We consider

$$\mathcal{S} := L^2\Big([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F}, \lambda\times\mathbf{P}\Big),$$

equipped with its weak topology.

- $\mathcal{A}(\mathcal{F}^2_t)$ and $\mathcal{A}_{ac}(\mathcal{F}^1_t)$ are convex and $\mathcal{A}(\mathcal{F}^2_t)$ is weakly compact in $\mathcal{S}$
- $\mathcal{A}(\mathcal{F}^2_t) \ni \zeta \mapsto N(\xi,\zeta)$ is quasi-concave and upper semi-continuous (weakly in $\mathcal{S}$)
- $\mathcal{A}_{ac}(\mathcal{F}^1_t) \ni \xi \mapsto N(\xi,\zeta)$ is quasi-convex and lower semi-continuous (weakly in $\mathcal{S}$)

Sion's theorem (1958) gives us existence of a value and of a maximiser ζ^*.

Swapping the roles of $P1$ and $P2$ we also obtain existence of a minimiser ξ^* since the same arguments apply to the game with payoff $\mathcal{P}'(\tau,\sigma) := -\mathcal{P}(\tau,\sigma)$, where τ-player is a maximiser and the σ-player is a minimiser.

A zero-sum game:

partially observable dynamics

Setting I

On a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we have:

- Two random variables θ and $\mathcal{U}$ and a Brownian motion W
- The above are mutually independent
- $\mathbf{P}(\theta = 1) = \pi$ and $\mathbf{P}(\theta = 0) = 1 - \pi$ and $\mathcal{U} \sim U([0,1])$

The process underlying the game reads

$$dX_t = ((1-\theta)\mu_0 + \theta\mu_1)X_t \, dt + \sigma X_t \, dW_t \qquad (2)$$

with $\mu_0 < 0 < \mu_1$.

Setting II

Game's structure.

- Player 1 chooses a (random) time τ and Player 2 chooses a (random) time γ
- At $\tau \wedge \gamma$, Player 1 receives the amount

$$\mathcal{P}(\tau, \gamma) := X_\tau 1_{\{\tau < \gamma\}} + (1 + \epsilon) X_\gamma 1_{\{\tau \geq \gamma\}}$$

  from Player 2, with $\epsilon > 0$
- Player 1 is maximiser and Player 2 is minimiser (of the expected future payoff)

Setting III

Incomplete and asymmetric information.

- Both players observe X
- Player 2 knows the true value of θ
- Player 1 only knows the prior distribution of θ

We let the information available to Player 1 be given by the filtration

$$\mathcal{F}_t^X := \sigma(X_s, 0 \le s \le t) = \mathcal{F}_t^1,$$

whereas the information available to Player 2 is given by the filtration

$$\mathcal{F}_t^{X,\theta} := \sigma(\theta, X_s, 0 \le s \le t) = \mathcal{F}_t^2 \supsetneq \mathcal{F}_t^1.$$

(Both filtrations are augmented with $\mathbf{P}$-null sets.)

The set-up of the game is known to both players: they both know π, μ_0, μ_1 and σ and Player 1 is aware that Player 2 knows the true value of the drift

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Setting IV

Strategies.

- The peculiarity of this game is that an equilibrium exists in which P1 only uses a pure stopping time $\tau \in \mathcal{T}(\mathcal{F}_t^X)$
- P2 uses a randomised stopping time

$$\gamma_\theta = \gamma_0 1_{\{\theta=0\}} + \gamma_1 1_{\{\theta=1\}},$$

with γ_i generated by $\Gamma^i \in \mathcal{A}(\mathcal{F}_t^{X,\theta})$, $i = 0, 1$

Markovian formulation

Player 1's learns about the true drift through observations of the process X: for $t \geq 0$ we denote

$$\Pi_t := \mathbf{P}(\theta = 1 | \mathcal{F}_t^X). \tag{3}$$

By standard filtering theory we have

$$\mathrm{d}X_t = (\mu_0(1 - \Pi_t) + \mu_1 \Pi_t) X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \qquad X_0 = x$$

and

$$\mathrm{d}\Pi_t = \omega \Pi_t (1 - \Pi_t) \,\mathrm{d}B_t, \qquad \Pi_0 = \pi, \tag{4}$$

where $(B_t)_{t \geq 0}$ is a $(\mathbf{P}, \mathcal{F}^X)$-Brownian motion known as innovation process and $\omega := (\mu_1 - \mu_0)/\sigma$ is referred to as the signal-to-noise ratio.

We denote the likelihood ratio

$$\Phi_t := \frac{\Pi_t}{1 - \Pi_t}.$$

The process $(X_t, \Phi_t)_{t \geq 0}$ is Markovian and adapted to $\mathcal{F}^X$.

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The belief process

The adjusted posterior probability process Φ^*.

Assume $(\tau^*, \gamma^*_\theta)$ is an equilibrium. The uninformed player calculates the adjusted posterior probability, for $t \geq 0$

$$\Pi^*_t := \mathbf{P}(\theta = 1 \big| \mathcal{F}^X_t, \gamma^*_\theta > t) = \frac{\mathbf{P}(\theta = 1, \gamma^*_\theta > t \big| \mathcal{F}^X_t)}{\mathbf{P}(\gamma^*_\theta > t \big| \mathcal{F}^X_t)}$$

$$= \frac{\mathbf{P}(\gamma^*_1 > t \big| \mathcal{F}^X_t, \theta = 1)\mathbf{P}(\theta = 1 \big| \mathcal{F}^X_t)}{\mathbf{P}(\gamma^*_\theta > t \big| \mathcal{F}^X_t)} = \frac{(1 - \Gamma^{*,1}_t)\Pi_t}{\mathbf{P}(\gamma^*_\theta > t \big| \mathcal{F}^X_t)}$$

$$= \frac{(1 - \Gamma^{*,1}_t)\Phi_t}{1 - \Gamma^{*,0}_t + (1 - \Gamma^{*,1}_t)\Phi_t}.$$

Then the adjusted posterior probability satisfies

$$\Phi^*_t := \frac{\Pi^*_t}{1 - \Pi^*_t} = \Phi_t \frac{1 - \Gamma^{*,1}_t}{1 - \Gamma^{*,0}_t}, \qquad t \geq 0.$$

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Change of measure & dimension reduction

It is convenient to use the the measures $\mathbf{P}^0$ and $\mathbf{P}^1$ specified by

$$\mathbf{P}^i(A) := \mathbf{P}(A \,|\, \theta = i), \quad \text{for } A \in \mathcal{F}_\infty^X.$$

Using the explicit dynamic of X under $\mathbf{P}^i$ we also define

$$\left.\frac{\mathrm{d}\widetilde{\mathbf{P}}^i}{\mathrm{d}\mathbf{P}^i}\right|_{\mathcal{F}_t^X} := e^{-\mu_i t} X_t\,, \tag{5}$$

we obtain

Lemma. (Game payoff). Letting $\widehat{\mathcal{P}} := (1+\varphi)\mathcal{P}$, we have

$$\widehat{\mathcal{P}}_{x,\varphi}(\tau, \gamma_\theta) = \mathcal{P}^0_{x,\varphi}(\tau, \Gamma^0) + \varphi \mathcal{P}^1_{x,\varphi}(\tau, \Gamma^1),$$

where, for $i = 0, 1$,

$$\mathcal{P}^i_{x,\varphi}(\tau, \Gamma^i) = x \widetilde{\mathbf{E}}^i_\varphi \left[e^{\mu_i \tau}(1 - \Gamma^i_\tau) + (1+\epsilon) \int_0^\tau e^{\mu_i t} \mathrm{d}\Gamma^i_t \right]. \tag{6}$$

The only relevant dynamic is that of Φ...

Heuristics

$$R(\tau,\gamma) := X_\tau 1_{\{\tau<\gamma\}} + (1+\epsilon) X_\gamma 1_{\{\tau\geq\gamma\}}$$

- If $\mu = \mu_0 < 0$, Player 2 does not stop and $\gamma_0 = +\infty$;
- If Player 1 were certain that $\mu = \mu_1$, she would never stop;
- If $\mu = \mu_1$, Player 2 stops when Φ is too high (i.e., Player 1 has a strong belief that $\mu = \mu_1$); hence Player 2 stops at some upper threshold B (for Φ) according to some 'intensity' (randomised stopping);
- Randomisation generates an adjusted likelihood ratio Φ^* (i.e.,the belief of the uninformed player after manipulation by the informed one);
- In all cases, Player 1 stops when she has a sufficiently strong belief that $\mu = \mu_0$, i.e. at a lower threshold A for Φ^*.

We are after two thresholds $0 < A < B < +\infty$:

1. A reflecting threshold B, such that we can construct processes

$$\Gamma^B \in \mathcal{A}(\mathcal{F}_t^{X,\theta}) \quad \text{and} \quad \Phi_t^* := \Phi_t^B = \Phi_t(1 - \Gamma_t^B) \tag{7}$$

   so that Φ^* is downward reflected at B; then γ_1 is generated by Γ^B.

2. A stopping threshold A for Φ^*, i.e. $\tau_A := \inf\{t \geq 0 : \Phi_t^* \leq A\}$.

ODE for Player 2 if $\mu = \mu_0$.

We expect $\mathcal{P}^0_{x,\varphi}(\tau_A, 0) = x\widetilde{\mathsf{E}}^0_\varphi\left[e^{\mu_0 \tau_A}\right] =: xV_0(\varphi)$. Then V_0 solves

$$\begin{cases} \frac{\omega^2\varphi^2}{2} V_0''(\varphi) + \sigma\omega\varphi V_0'(\varphi) + \mu_0 V_0(\varphi) = 0, & \varphi \in (A, B) \\ V_0(\varphi) = 1, & \varphi \in (0, A] \\ V_0'(B-) = 0. \end{cases} \tag{8}$$

ODE for Player 2 if $\mu = \mu_1$.

We need to compute $\mathcal{P}^1_{x,\varphi}(\tau_A, \Gamma^B) =: xV_1(\varphi)$ and we expect smooth-fit at B. Then V_1 solves

$$\begin{cases} \frac{\omega^2\varphi^2}{2} V_1''(\varphi) + (\omega^2 + \sigma\omega)\varphi V_1'(\varphi) + \mu_1 V_1(\varphi) = 0, & \varphi \in (A, B) \\ V_1(\varphi) = 1, & \varphi \in (0, A] \\ V_1(\varphi) = 1 + \epsilon, & \varphi \in [B, \infty) \\ V_1'(B-) = 0. \end{cases} \tag{9}$$

ODE for Player 1 in all cases.

Recall that

$$\widehat{\mathcal{P}}_{x,\varphi}(\tau_A, 0, \Gamma^{1,B}) = \mathcal{P}^0_{x,\varphi}(\tau_A, 0) + \varphi \mathcal{P}^1_{x,\varphi}(\tau_A, \Gamma^B).$$

Then, for $\widehat{\mathcal{P}}_{x,\varphi}(\tau_A, 0, \Gamma^{1,B}) =: xV(\varphi)$ we have

$$V(\varphi) := (V_0(\varphi) + \varphi V_1(\varphi)).$$

$$\begin{cases} \frac{\omega^2\varphi^2}{2} V''(\varphi) + \sigma\omega\varphi V'(\varphi) + \mu_0 V(\varphi) = 0, & \varphi \in (A, B) \\ V(\varphi) = 1 + \varphi, & \varphi \in (0, A] \\ V'(B-) = 1 + \epsilon. \end{cases} \tag{10}$$

The crucial link across ODEs for both players is smooth-fit at A, i.e.

$$V'(A+) = 1$$

Theorem.

The system of ODEs yields a unique solution: a couple of points A and B and functions V_0, V_1 and V.

Moreover, such solution fulfils all conditions in our verification theorem. Hence $(\tau_A, 0, \Gamma^B)$ is a Nash equilibrium.

Now, some pictures...
$\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$

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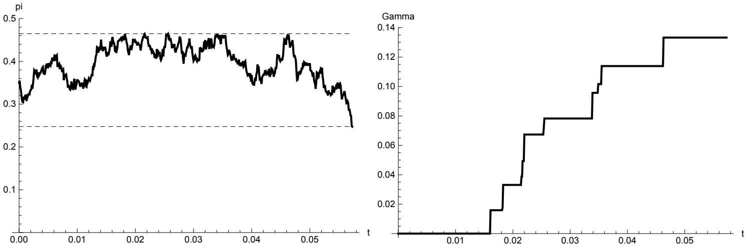


(a) (b)

Figure: A typical sample path of the Π^*-process (left) and its associated $\Gamma^{*,1}$-process (right) for our base-case parameters. Note that the dashed lines on the left represent the optimal boundaries $a = 0.248$ and $b = 0.465$ and that we have chosen $\pi = 0.35$.

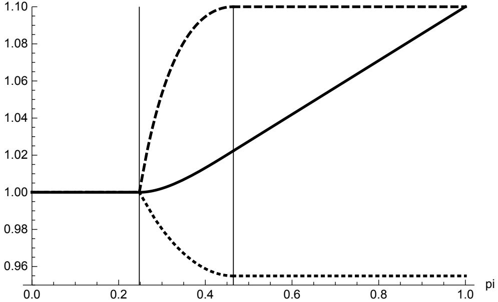


Figure: The value of the game to Player 1 (solid line; $(1-\pi)V_0 + \pi V_1$) along with the value of the game to Player 2 when $\mu = \mu_0$ (dotted line; V_0) and $\mu = \mu_1$ (dashed line; V_1). The shaded region corresponds to the interval $[a, b]$. The base-case parameters are $\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$; therefore $a = 0.248$ and $b = 0.465$.

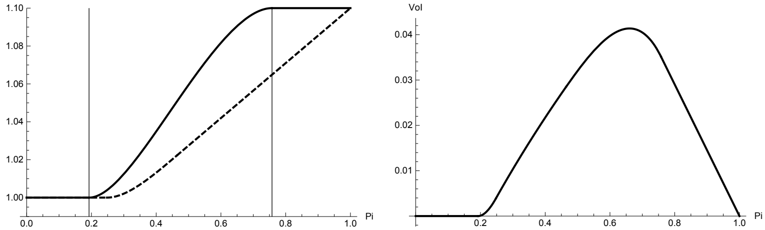


Figure: On the left: The common value function for both players in the symmetric incomplete information case (solid line) in comparison to the value function in the asymmetric case (dashed line). The shaded region corresponds to the interval $[a, b]$ (for the symmetric case). On the right: The difference between these values, which represents the value of information in our game. Base-case parameters: $\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$, which yields $a := A/(1+A) = 0.193$ and $b := B/(1+B) = 0.758$ (for the symmetric case).

Ghost Games:

Uncertain Competition

Motivating example

Sealed-bid auction with known competition:

- Two players bid for a good worth 1 EUR
- The bids are not public
- Both players know there are two bids (N bids)
- P1 bids $s \in [0,1]$ and P2 bids $t \in [0,1]$
- Payoffs:
$$\mathcal{J}_1(s,t) = (1-s)1_{\{s>t\}} \quad \text{and} \quad \mathcal{J}_2(s,t) = (1-t)1_{\{t>s\}}$$
- The only equilibrium is $(s_*, t_*) = (1,1)$ with $\mathcal{J}_1^* = \mathcal{J}_2^* = 0$

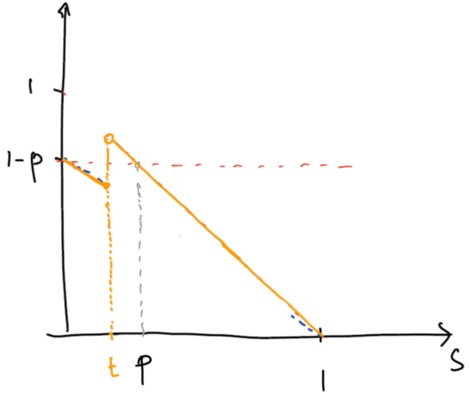
Sealed-bid auction with unknown competition (pure strategies):

- Same setup as above but now players are not sure whether there is another bidder
- For simplicity we take symmetric game
- P1 estimates that P2 is in the game (and viceversa) with probability $p \in (0,1)$
- Expected payoffs:

$$\mathcal{J}_1(s,t) = p(1-s)1_{\{s>t\}} + (1-p)(1-s) \quad \text{and} \quad \mathcal{J}_2(s,t) = p(1-t)1_{\{t>s\}} + (1-p)(1-t)$$

- There is no equilibrium in pure strategies:
  - If P2 bids $t < p$, then $P1$'s best response is $s = t + \varepsilon$ for $\varepsilon \downarrow 0$ (and viceversa)
  - If P2 bids $t > p$, then $P1$'s best response is $s = 0$ (and viceversa)
  - Players preempt each other for as long as they bid below p
  - The pair (p,p) is not an equilibrium

Figure: An illustration of Player 1's payoff when Player 2 picks $t \in [0, p)$.



Sealed-bid auction with unknown competition (mixed strategies):

- Players use mixed strategies, i.e., their bid is drawn from a cdf F supported on $[0,p]$ with $F(0)=0$
- If P2 bids according to F, then

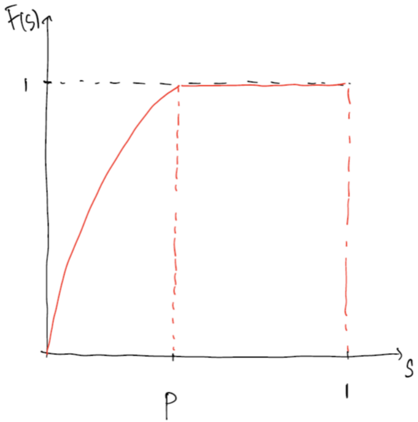
$$\mathcal{J}_1(s,F) = p(1-s)F(s) + (1-p)(1-s)$$

- In equilibrium P1 is indifferent across $s \in [0,p]$, i.e. $\mathcal{J}_1(s,F) = const.$
- In particular, $\mathcal{J}_1(s,F) = \mathcal{J}_1(0,F) = (1-p)$ for all $s \in [0,p]$. It follows:

$$F(s) = \left(\frac{1-p}{p}\right)\frac{s}{1-s},\ s \in [0,p]\ and\ F(s)=1,\ s \in (p,1].$$

- Notice that for $s \in (p,1]$, $\mathcal{J}_1(s,F) = (1-s) < 1-p \implies$ no bid above p
- Equilibrium in mixed strategies $(s,t) \sim (F,F)$ and $\mathcal{J}_1(F,F) = \mathcal{J}_2(F,F) = 1-p$

Figure: An illustration of the optimal mixed strategy.



Setting I

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space hosting the following:

(a) a continuous, $\mathbb{R}^d$-valued, strong Markov process X which is regular (it can reach any open set in finite time with positive probability, for any value of the initial point $X_0 = x$);

(b) two Bernoulli distributed random variables θ_i, $i = 1, 2$;

(c) two Uniform$(0, 1)$-distributed random variables U_i, $i = 1, 2$.

Furthermore, we assume that these processes and random variables are mutually independent, and that $\mathbf{P}(\theta_i = 1) = 1 - \mathbf{P}(\theta_i = 0) = p_i \in (0, 1]$.

Setting II

Here, the event $\{\theta_i = 1\}$ denotes the presence of active competition for the i-th player.

Game structure: a preemption game with uncertain competition.

- The payoff: $g : \mathbb{R}^d \to [0, \infty)$ is a continuous function such that $\sup_{x \in \mathbb{R}^d} g(x) > 0$.
- Player 1 chooses $\tau \in \mathcal{T}_1^R$ and Player 2 chooses $\gamma \in \mathcal{T}_2^R$
- The payoff for Player 1 at time τ is

$$R(\tau, \gamma) := \left(g(X_\tau) 1_{\{\tau < \hat{\gamma}\}} + \frac{1}{2} g(X_\tau) 1_{\{\tau = \hat{\gamma}\}} \right) 1_{\{\tau < \infty\}},$$

  where $\hat{\gamma} := \gamma 1_{\{\theta_1 = 1\}} + \infty 1_{\{\theta_1 = 0\}}$.
- For Player 2 at time γ the payoff is $R(\gamma, \tau)$ with $\hat{\tau} := \tau 1_{\{\theta_2 = 1\}} + \infty 1_{\{\theta_2 = 0\}}$.
- Both players are maximisers (of the expected future payoff)

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Setting III

Denote $\mathcal{J}_1(\tau,\gamma;p_1,x) := \mathsf{E}_x[R(\tau,\gamma)]$ and $\mathcal{J}_2(\tau,\gamma;p_2,x) := \mathsf{E}_x[R(\gamma,\tau)]$.

Definition (Nash equilibrium). Given $x \in \mathbb{R}^d$ and $p_i \in (0,1]$, $i = 1,2$, a pair $(\tau^*,\gamma^*) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$ is a Nash equilibrium if

$$\mathcal{J}_1(\tau,\gamma^*;p_1,x) \le \mathcal{J}_1(\tau^*,\gamma^*;p_1,x)$$

and

$$\mathcal{J}_2(\tau^*,\gamma;p_2,x) \le \mathcal{J}_2(\tau^*,\gamma^*;p_2,x)$$

for all pairs $(\tau,\gamma) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$. Given an equilibrium pair $(\tau^*,\gamma^*) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$ we define the equilibrium payoffs as

$$v_i(p_i,x) := \mathcal{J}_i(\tau^*,\gamma^*;p_i,x), \qquad \text{for } i = 1,2.$$

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The belief processes

If $\gamma \in \mathcal{T}_2^R$ is generated by $\Gamma^2 \in \mathcal{A}$, then Player 1 dynamically evaluates the conditional probability of Player 2 being active as

$$\Pi_t^1 := \mathbf{P}(\theta_1 = 1 | \mathcal{F}_t^X, \hat{\gamma} > t) = \frac{p_1(1 - \Gamma_t^2)}{1 - p_1 \Gamma_t^2}$$

provided $p_1 \in (0,1)$. Likewise, if $\tau \in \mathcal{T}_1^R$ is generated by $\Gamma^1 \in \mathcal{A}$, then

$$\Pi_t^2 := \mathbf{P}(\theta_2 = 1 | \mathcal{F}_t^X, \hat{\tau} > t) = \frac{p_2(1 - \Gamma_t^1)}{1 - p_2 \Gamma_t^1}$$

provided $p_2 \in (0,1)$.

The single agent problem and three sets

Let

$$V(x) := \sup_\tau \mathsf{E}_x \left[e^{-r\tau} g(X_\tau) 1_{\{\tau<\infty\}} \right]$$

and assume $V \in C(\mathbb{R}^d)$.

Equilibria are fully determined in terms of three sets

$$\overline{\mathcal{C}} := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p)V(x) \geq g(x)\}$$

$$\mathcal{C}' := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p/2)g(x) < (1-p)V(x) < g(x)\}$$

$$\mathcal{S} := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p)V(x) \leq (1-p/2)g(x)\}$$

and note that $\overline{\mathcal{C}} \cup \mathcal{C}' \cup \mathcal{S} = (0,1) \times \mathbb{R}^d$.

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The (explicit) boundaries

It is easy to see that

$$\overline{\mathcal{C}} = \{(p,x) \in (0,1) \times \mathbb{R}^d : p \leq b(x)\},$$
$$\mathcal{C}' = \{(p,x) \in (0,1) \times \mathbb{R}^d : b(x) < p < c(x),\}$$
$$\mathcal{S} = \{(p,x) \in (0,1) \times \mathbb{R}^d : c(x) \leq p\},$$

with continuous boundaries $b \leq c$ given by

$$b(x) = 1 - \frac{g(x)}{V(x)} \qquad \text{and} \qquad c(x) = \frac{V(x) - g(x)}{V(x) - g(x)/2}.$$

Construction of Nash equilibria I

From now on we assume $0 < p_1 \leq p_2 < 1$. It turns out that in this setting Player 2 is the most active.

Equilibrium (part 1). If $(p_1, x) \in \mathcal{S}$, an equilibrium is for both players to stop at once.

Equilibrium (part 2). If $(p_1, x) \in \overline{\mathcal{C}}$,

- Player 2 picks $\Gamma^{2,*} \in \mathcal{A}$ such that the process $(\Pi^1_t, X_t)_{t \geq 0}$ is kept in $\overline{\mathcal{C}}$ with minimal effort (recall $\Pi^1_t = p_1(1 - \Gamma^2_t)/(1 - p_1 \Gamma^2_t)$).
- Player 1 picks

$$\Gamma^{1,*}_t := \frac{p_1}{p_2} \Gamma^{2,*}_t 1_{\{t < \tau^*_V\}} + 1_{\{t \geq \tau^*_V\}}.$$

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Construction of Nash equilibria II

Equilibrium (part 3). If $(p_1, x) \in \mathcal{C}'$,

- Player 2 picks $\Gamma^{2,*} \in \mathcal{A}$ such that the process $(\Pi^1_t, X_t)_{t\geq 0}$ makes an immediate jump to a point (q_1, x) with $q_1 < b(x)$. Then $(\Pi^1_t, X_t)_{t\geq 0}$ is kept in $\overline{\mathcal{C}}$ with minimal effort. (Note: We have an explicit expression for q_1 depending on $p_1, V(x)$ and $g(x)$.)
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Remark. The jump of $\Gamma^{2,*}$ corresponds to saying that Player 2 ‘flicks a (biased) coin’ and stops immediately with probability $\Gamma^{2,*}_0$ (known explicitly) or continues with probability $1 - \Gamma^{2,*}_0$.

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An example

Competing for a real option.

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad \text{and} \quad g(x) = (x - K)^+$$

with $\mu < r$.

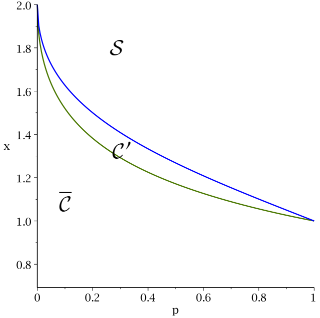

Figure: The figure displays the curves $p = b(x)$ (lower one) and $p = c(x)$ (top one). The parameter values are $K = 1$ and $\eta = 2$ so that $B = 2$.

Summary and conclusions

Summary and conclusions

In this talk:

- Existence of a value and of a saddle point in (non-Markovian) zero-sum Dynkin games with general information structure.
- Explicit solution of a zero-sum Dynkin game with partially observed dynamics and asymmetry of information.
- Explicit solution of a nonzero-sum Dynkin game with uncertain competition.

In future talks:

- Dynamic view of the games: martingale theory for the construction of saddle points in (non-Markovian) zero-sum Dynkin games with general information structure (PhD thesis of J. Smith (2024), paper in preparation with J. Palczewski)
- More explicit solutions to Dynkin games with asymmetric information (e.g., with D. Hobson and J. Palczewski)
- Extension of the theory to nonzero-sum Dynkin games.

Thank you