

Volatility and order flow: a tale of three fractional Brownian motions

J. Mühle-Karbe¹, Y. Ouazzani-Chahdi², M. Rosenbaum³, G. Szymanski⁴

¹Imperial College, ²École Polytechnique, ³Université Paris Dauphine-PSL,
⁴Université du Luxembourg

October 2025

What we know

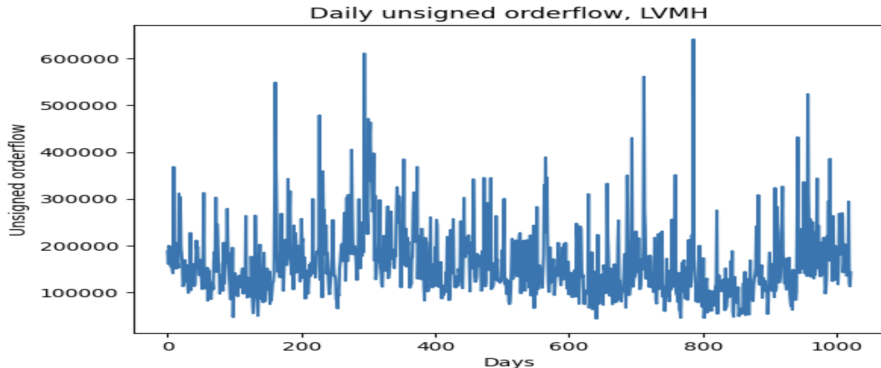
- There is obviously a huge literature about price modeling and volatility modeling.
- Prices are diffusive and volatility is rough.
- Much less is known about order flow and volume, although it is the main transmission channel of information towards prices (market impact).
- Volume/number of trades and volatility are closely related :

$$N_t \approx \int_0^t \sigma_s^2 ds.$$

- So time series of volume behave as time series of volatilities ?

Unsigned order flow

A typical unsigned order flow sample path



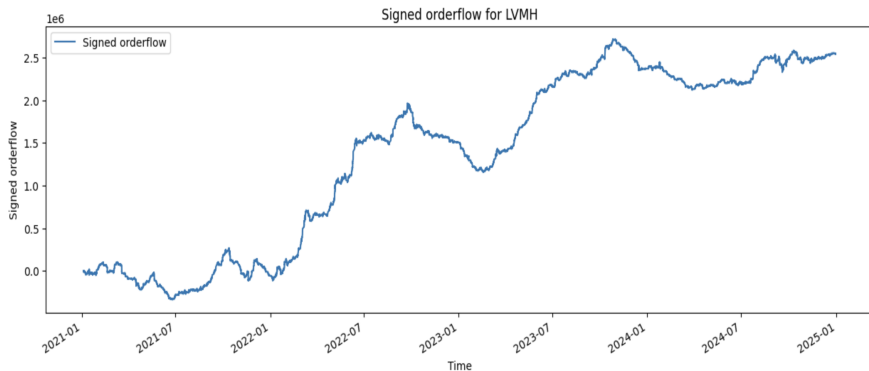
- Unsigned order flow behaves as an integrated rough process, with estimated value of H_1 ranging in $[0.1, 0.3]$.

The volume puzzle

- So it seems that Vol(atility) is rough and so Vol(ume) is rough.
- However there is a large (not very recent) econophysics and econometrics literature about long memory in trade signs and the use of fractional Brownian motion with $H > 1/2$ to model volume.
- Estimations in these works are based on classical estimators for the Hurst parameter.
- So let us look at the signed volume and empirical values for H_0 .

Signed order flow

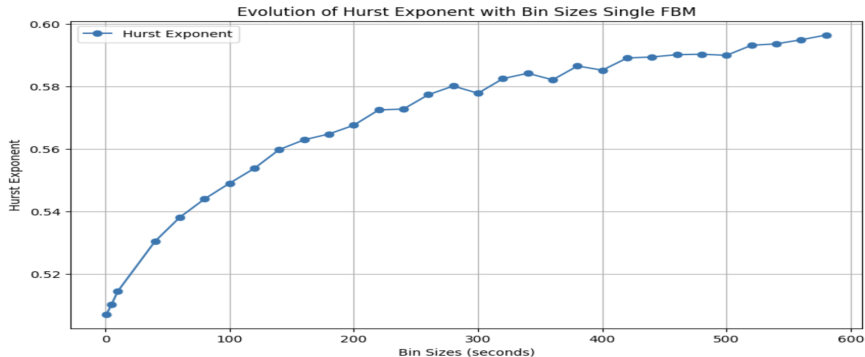
A typical signed order flow sample path



- At the visual level, rather smooth fBm like behavior.

Signed order flow

Estimation of the Hurst parameter



- H_0 seems around 0.6 but estimated values depends on frequency...

Two fractional Brownian motions

Rough and smooth order flow

- Unsigned order flow seems to behave as the integral of a rough process, with H_1 around 0.25.
- Signed order flow seems to behave as the smooth fBm process, with H_0 around 0.6.
- We want to explain and understand these two results and show that they are compatible.

Micro model for order flow

Fundamental orders

- In practice some participants have long term views, trade long term signals, follow long term drift, split metaorders... This type of fundamental trading surely generates a very autocorrelated flow.
- We consider a natural statistical model for this fundamental trading over $[0, T]$. We assume that $F^{\pm, T}$ are two i.i.d. Hawkes processes with baseline ν^T and self-exciting kernel φ_0^T . The unsigned and signed flows are respectively :

$$F_t^{+, T} + F_t^{-, T}, \quad F_t^{+, T} - F_t^{-, T}.$$

- Usual assumptions :

$$\varphi_0^T = a_0^T \varphi_0, \quad a_0^T \xrightarrow[T \rightarrow +\infty]{} 1, \quad \alpha_0 t^{\alpha_0} \int_t^\infty \varphi_0(s) ds \xrightarrow[t \rightarrow \infty]{} K_0.$$

$$T^{\alpha_0}(1 - a_0^T) \xrightarrow[T \rightarrow +\infty]{} \lambda_0 \delta_0, \quad T^{1-\alpha_0} \nu^T \xrightarrow[T \rightarrow +\infty]{} \mu_0^* \delta_0^{-1}, \quad \text{with } \delta_0 = K_0 \frac{\Gamma(1 - \alpha_0)}{\alpha_0}.$$

Micro model for order flow

Limit theorem for fundamental orders

Theorem

Let $\bar{F}_t^{\pm, T} = \frac{1-a_0^T}{T\nu^T} F_{tT}^{\pm, T}$ and f^{α_0, λ_0} be the Mittag-Leffler density. The process $(\bar{F}_t^{+, T}, \bar{F}_t^{-, T})$ is tight for the Skorohod topology. Furthermore, if (F_t^+, F_t^-) is a limit of $(\bar{F}_t^{+, T}, \bar{F}_t^{-, T})$, then (F_t^+, F_t^-) satisfies

$$F_t^{\pm} = \int_0^t s f^{\alpha_0, \lambda_0}(t-s) ds + \frac{1}{\sqrt{\mu_0^* \lambda_0}} \int_0^t f^{\alpha_0, \lambda_0}(t-s) B_{F_t^{\pm}}^{\pm} ds$$

where B^+ and B^- are two independent Brownian motions. This implies

$$\bar{F}_t^{+, T} + \bar{F}_t^{-, T} \rightarrow F_t^+ + F_t^-, \quad \bar{F}_t^{+, T} - \bar{F}_t^{-, T} \rightarrow F_t^+ - F_t^-$$

with the same scaling.

Micro model for order flow

Limit theorem for fundamental orders

Comments on the theorem

- $F_t = F_t^+ + F_t^-$ satisfies

$$F_t = 2 \int_0^t s f^{\alpha_0, \lambda_0}(t-s) ds + \frac{1}{\sqrt{\mu_0^* \lambda_0}} \int_0^t f^{\alpha_0, \lambda_0}(t-s) B_{F_t}^F ds.$$

- $U_t = F_t^+ - F_t^-$ satisfies

$$U_t = \frac{1}{\sqrt{\mu_0^* \lambda_0}} \int_0^t f^{\alpha_0, \lambda_0}(t-s) B_{F_t}^U ds.$$

- For $\alpha_0 < 1/2$, F_t and U_t have Hölder regularity $2\alpha_0 - \varepsilon$.
- So both F_t and U_t behave like functional of a fBm with Hurst parameter $H_0 = 2\alpha_0$.

Micro model for order flow

Reaction orders

- Some participants react to shorter term signals, extracting information from past orders.
- It is natural to model the arrival of these orders as Hawkes processes built upon the fundamental flow dynamics.
- So we consider two dimensional Hawkes process $N^{\pm, T}$ with self-exciting kernel

$$\phi^T = \begin{pmatrix} \varphi_1^T & \varphi_2^T \\ \varphi_2^T & \varphi_1^T \end{pmatrix} \quad (1)$$

with time dependent baseline intensity $\mu^T = (\mu^{+, T}, \mu^{-, T})$ given by

$$\phi^T * d\mathbf{F}^T, \quad \text{with } \mathbf{F}^T = \begin{pmatrix} F_t^{+, T} \\ F_t^{-, T} \end{pmatrix}.$$

- Usual assumptions :

$$\phi^T = a_1^T \phi, \quad a_1^T \xrightarrow{T \rightarrow +\infty} 1, \quad \varrho(\|\phi\|_{L^1}) = 1, \quad \alpha_1 x^{\alpha_1} \int_t^\infty \lambda_1(s) ds \xrightarrow{t \rightarrow +\infty} K_1.$$

Micro model for order flow

Reaction orders

Replacing the usual assumption on the Poisson part

- Usual asymptotic setting would be

$$T^{\alpha_1}(1 - a_1^T) \xrightarrow{T \rightarrow +\infty} \lambda_1^*, \quad T^{1-\alpha_1}\mu^T \xrightarrow{T \rightarrow +\infty} \mu_1.$$

- But there is no μ^T here. Usually we need the average number of jumps of the Poisson baseline on $[0, T]$ of order T^{α_1} . Here the number of jumps on this time interval is exactly $F_t^{+,T} + F_t^{-,T}$, which is of order $\frac{T\nu^T}{1-a_0^T}$. So we assume

$$\frac{T\nu^T}{1-a_0^T} \sim T^{\alpha_1}.$$

- This is equivalent to

$$T^{\alpha_1}(1 - a_1^T) \xrightarrow{T \rightarrow +\infty} \lambda_1^*, \quad 2\alpha_0 = \alpha_1.$$

Micro model for order flow

Limit theorem for the unsigned order flow

Theorem

Let $U_t^T = F_t^{T,+} + F_t^{T,-} + N_t^{+,T} + N_t^{-,T}$. We have

$$\frac{(1 - a_0^T)(1 - a_1^T)}{T\nu^T} U_{tT}^T \xrightarrow{T \rightarrow +\infty} 2X_t, \text{ with}$$

$$X_t = \frac{1}{2} \int_0^t f^{\alpha_1, \lambda_1}(t-s) F_s ds + \frac{c}{2} \int_0^t f^{\alpha_1, \lambda_1}(t-s) Z_s ds,$$

with $Z_t = Z_t^+ + Z_t^-$ and Z^+, Z^- are two martingales with quadratic variation X and zero quadratic covariation.

Comments on the theorem

- Due to scaling, the fundamental part almost vanishes in the limit.
- The unsigned flow behaves an integrated rough process with Hurst exponent $H_1 = \alpha_1 - 1/2 = H_0 - 1/2$.

Micro model for order flow

Limit theorem for the signed order flow

Theorem

Let $S_t^T = F_t^{T,+} - F_t^{T,-} + N_t^{+,T} - N_t^{-,T}$. We have

$$\left(\frac{(1 - a_0^T)(1 - a_1^T)}{T\nu^T} \right)^{1/2} S_{tT}^T \xrightarrow[T \rightarrow +\infty]{} 2U_t + Z_t^+ - Z_t^-.$$

Comments on the theorem

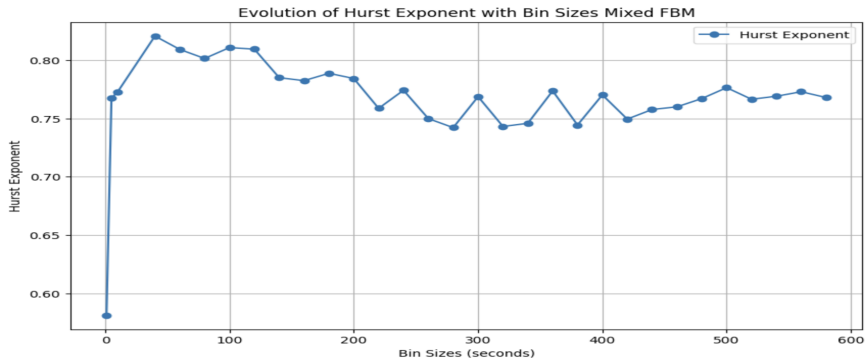
- The natural scaling is now different, both the fundamental part and the endogenous part play a role in the limit.
- The total signed flow behaves as :

rough process with Hurst exponent H_0 + martingale.

- This explains the empirical results for the estimation of the Hurst parameter of the signed flow !

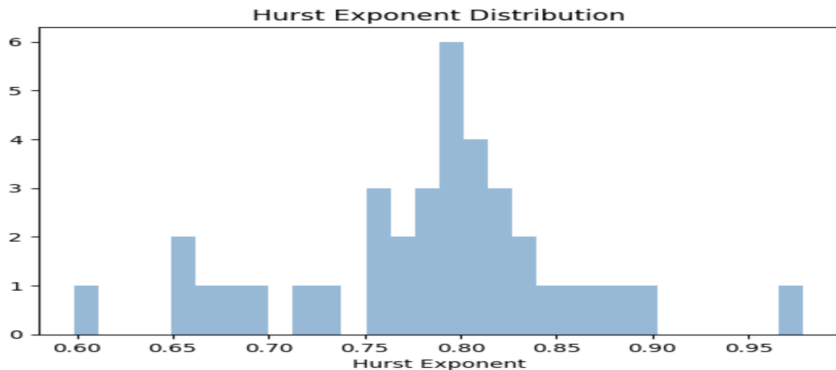
Estimating the Hurst parameter of the signed flow

Corrected estimator, averages of estimations over 40 stocks (20 Nasdaq, 20 CAC40)



Estimating the Hurst parameter of the signed flow

Corrected estimator, histogram over 40 stocks



From order flow to prices

- We now have a convincing model for order flow dynamics.
- Based on no-arbitrage argument, it is explained in previous works how to derive a propagator based martingale price dynamic from an order flow model :

$$P_t = \int_0^t G(t-s) dV_s.$$

- From this propagator price, we can deduce the power law shape of market impact and roughness exponent for the volatility.

Mixed fractional Brownian motion

- Here our order flow is essentially (at the intermediary scale) $W_t^H + W_t$.
- This is the so-called mixed fractional Brownian motion that has been notably studied by Patrick Cheredito.
- From Cheredito's work, we know that for $H > 3/4$, the mixed fractional Brownian is a semi-martingale.
- From the semi-martingale decomposition of the mixed fractional Brownian motion and the analysis of its finite variation part, we can deduce the shape of our propagator (conjecture).

Three fractional Brownian motions ?

- Let H_0 be the smoothness of the fundamental flow. We get the following preliminary results :
- Unsigned order flow as the integral of a rough process with exponent $H_1 = H_0 - 1/2$.
- Power law market impact with exponent $2 - 2H_0$.
- Rough volatility with exponent $H = 2H_0 - 3/2$.
- If $H = 0.1$, then $H_0 = 0.8$, $H_1 = 0.3$ and impact in 0.4.
- For square root law of impact, $H = 0$, $H_0 = 3/4$, $H_1 = 1/4$.