Bank-El Karoui's representation of stochastic processes and its applications in the exit contract problems

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Based on joint works with

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Main references

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Outline

1. Exit Incentives for Carbon Emissive Firms
2. Exit Contract Problem and Representation of Stochastic Processes
3. A Mean-Field Extension
   - An Entry Time Game in Electricity Market
   - Mean-Field Version of Bank-El Karoui's Representation

A Carbon Intensive Production Problem

• A market with n producers of carbon intensive good (e.g. crude oil, coal, etc.).

• For producer $i \in \{1, 2, \cdots, n\}$, the production at time t is X_t^i. Assume that the unit profit is $\pi_i > 0$, then the optimal time to stop the production is $\tau_i = \infty$, i.e.

$$\sup_{\tau_i} \mathbb{E}\Big[\int_0^{\tau_i} e^{-\rho t}\pi_i X_t^i dt\Big] = \mathbb{E}\Big[\int_0^{\infty} e^{-\rho t}\pi_i X_t^i dt\Big].$$

• With the total production $X_t := X_t^1 + \cdots + X_t^n$, the corresponding social utility is

$$\mathbb{E}\Big[\int_0^{\infty} e^{-\rho t}\big(\bar{\pi} X_t - \ell X_t^2\big) dt\Big].$$

A Carbon Intensive Production Problem : $n = 1$

- Example : $n = 1$ and

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \ t \geq 0,$$

and assume that

$$\mu < \rho < 2\mu + \sigma^2.$$

Then reward value of the producer is

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\pi X_t dt\Big] = \frac{\pi}{\rho - \mu}X_0 < \infty,$$

and the corresponding social utility is

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\left(\bar{\pi}X_t - \ell X_t^2\right)dt\Big] = \frac{\pi}{\rho - \mu} - \ell\int_0^\infty e^{(-\rho+2\mu+\sigma^2)t}dt = -\infty.$$

Exit Incentives for Carbon Intensive Production : $n = 1$

• Exit Incentive : the social planner provides a compensation Y_τ if the producer stops the production at time τ, so that the producer's problem becomes an optimal stopping problem :

$$\sup_\tau \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\pi_i X_t dt + e^{-\rho\tau} Y_\tau\Big].$$

• The social planner's problem becomes :

$$\sup_Y \mathbb{E}\Big[\int_0^{\hat\tau(Y)} e^{-\rho t}\big(\bar\pi X_t - \ell X_t^2\big)dt - e^{-\rho\hat\tau(Y)} Y_{\hat\tau(Y)}\Big].$$

• Observation : to make the producer to stop at time τ, the corresponding compensation must satisfies $Y_\tau \geq \frac{\pi}{\rho-\mu} X_\tau$, so that the social planner's problem is equivalent to

$$\sup_\tau \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\big(\bar\pi X_t - \ell X_t^2\big)dt - e^{-\rho\tau}\frac{\pi}{\rho-\mu} X_\tau\Big] > -\infty.$$

Solving the Optimal Exit Incentives Problem : $n = 1$

- Let

$$u(x) := \sup_{\tau} \mathbb{E}\Big[\int_0^{\tau} e^{-\rho t}\big(\bar{\pi} X_t - \ell X_t^2\big)dt - e^{-\rho\tau}\frac{\pi}{\rho-\mu}X_\tau \Big| X_0 = x\Big],$$

then u satisfies the dynamic programming equation :

$$\min\Big\{\rho u(x) - \mu x u'(x) - \frac{1}{2}\sigma^2 x^2 u''(x) - (\bar{\pi}x - \ell x^2); u(x) - \frac{\pi}{\rho-\mu}x\Big\} = 0.$$

- By solving the above equation explicitly, one obtains that the optimal stopping time is a hitting time :

$\hat{\tau} := \inf\{t \geq 0 : X_t \geq x^*\}$, where x^* can be explicitly computed;

and an optimal compensation scheme can be given by

$$Y_t := \frac{\pi}{\rho-\mu}X_t \mathbf{1}_{\{X_t \geq x^*\}}, \quad t \geq 0.$$

Optimal Exit Incentives for Carbon Intensive Production

• A universal compensation scheme $(Y_t)_{t\geq 0}$ for all $n \geq 2$ producers, so that producer $i = 1, \cdots, n$ solves the optimal stopping problem :

$$\sup_{\tau_i} \mathbb{E}\Big[\int_0^{\tau_i} e^{-\rho t}\pi_i X_t^i dt + e^{-\rho\tau_i}\lambda_i Y_{\tau_i}\Big],$$

where λ_i is the market share of producer i, and the social planner's optimal compensation problem becomes :

$$\sup_{Y} \mathbb{E}\Big[\int_0^{\infty} e^{-\rho t}(\bar{\pi}X_t - \ell X_t^2)dt - \sum_{i=1}^{n} e^{-\rho\hat{\tau}_i}\lambda_i Y_{\hat{\tau}_i}\Big],$$

with

$$X_t := \sum_{i=1}^{n} X_t^i \mathbf{1}_{\{t<\hat{\tau}_i\}}.$$

• When $n \geq 2$, it is not trivial to manipulate the universal compensation scheme Y to stops different agents.

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A general exit contract problem with multiple agents

- A principal-agent problem with universal contract :
  - For each agent $i = 1, \cdots, n$:

$$\max_{\tau_i \in \mathcal{T}} \mathbb{E}\Big[\int_0^{\tau_i} f_i(t, X_{t\wedge\cdot})dt + Y_{\tau_i}\Big].$$

  - Principal's problem :

$$\max_{Y} \mathbb{E}\Big[\sum_{i=1}^{n}\int_0^{\hat{\tau}_i} g_i(t, X_{t\wedge\cdot})dt - Y_{\hat{\tau}_i}\Big], \text{ subject to } Y_T \geq \xi,$$

    for some random variable $\xi \in \mathcal{F}_T$.

- Principal-Agent problem in the optimal control setting : Sannikov (2008), Cvitanic-Possamaï-Touzi (2018), etc.

The exit contract problem : multiple agents

- More Motivations/Applications :
  - Layoff problem : because of the Labour Union or law constraint, the company can not choose the employees to fire, but needs to suggest a universal plan and ask the employees to take volunteer leave.
  - Retirement system design : people can choose to retire in some range of ages, and the retirement pension depends on the retirement age.

Bank-El Karoui's representation of stochastic processes

Theorem (Bank-El Karoui)

Let $\ell \mapsto f(t,x,\ell)$ is strictly increasing and $f(t,x,\pm\infty) = \pm\infty$. Then, for any optional process Y (u.s.c. in expectation), there exists an optional process L such that

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(s, X, \sup_{r\in[t,s)} L_r\Big)ds + \xi\Big|\mathcal{F}_t\Big], \text{ with } \xi := Y_T.$$

As one of the applications, for each $\ell \in [0,1]$, the hitting time

$$\tau_\ell := \inf\{t \geq 0 \ : L_t \geq \ell\}$$

is (the smallest) solution to the optimal stopping problem :

$$\sup_\tau \Big[\int_0^\tau f(t,X,\ell)dt + Y_\tau\Big].$$

Bank-El Karoui's representation of stochastic processes

• Other applications of the representation in optimal consumption problems (Bank-Riedel, 2001), singular control problem (Chiarolla-Ferrari, 2014), etc. see also the review paper of (Bank-Föllmer, 2003).

• Extensions to the nonlinear case by a variant of the reflected BSDE (Ma-Wang, 2009), the case where the measure μ has atom and Meyer-σ-field (Bank-Besslich, 2018), etc.

• Extensions to the mean-field setting (He-T.-Zou, 2023).

• Closely related to the Gittins index theory (see e.g. El Karoui and Karatzas, 1994).

The exit contract problem

- The case with multiple agents, who share the same exit contract :

For each agent $i = 1, \cdots, n$:

$$\max_{\tau_i \in \mathcal{T}} \mathbb{E}\Big[\int_0^{\tau_i} f_i(t, X_{t\wedge\cdot})dt + Y_{\tau_i}\Big].$$

- Principal's problem :

$$\max_{Y,\ Y_T \geq \xi} \mathbb{E}\Big[\sum_{i=1}^{n} \int_0^{\hat{\tau}_i} g_i(t, X_{t\wedge\cdot})dt - Y_{\hat{\tau}_i}\Big].$$

- Assumption :

$$f_1(\cdot) < f_2(\cdot) < \cdots < f_n(\cdot).$$

The exit contract problem

- By interpolation, setting $\ell \longrightarrow i$ in Bank-El Karoui's theorem, the process L provides
  - the principal's contract Y by

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(s, X, \sup_{r\in[t,s)} L_r\Big)ds + \xi\Big|\mathcal{F}_t\Big], \quad t \geq 0,$$

  - the agents' optimal stopping time $\hat{\tau}_i$ by

$$\hat{\tau}_i \ := \ \inf\{t \geq 0 \ : L_t \geq i\}.$$

- It follows that, at least formally,

$$\sup_{Y\in\mathcal{Y}} F\big(Y, \hat{\tau}_1(Y), \cdots, \hat{\tau}_n(Y)\big) = \sup_{L\in\mathcal{L}} F\big(Y(L), \hat{\tau}_1(L), \cdots, \hat{\tau}_n(L)\big).$$

One needs identify the corresponding admissible set $\mathcal{Y}$ and $\mathcal{L}$.

The exit contract problem

• Let $\mathcal{Y}$ denote the class of all optional processes in class (D), u.s.c. in expectation and s.t. $Y_T \geq \xi$. Then given $Y \in \mathcal{Y}$, there exists an optional process L such that

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(s, X, \sup_{r\in[t,s)} L_r\Big)ds + \xi\Big|\mathcal{F}_t\Big], \text{ a.s.}$$

• Given an optional process L, the corresponding process Y may not be u.s.c. in expectation. When L is in the class $\mathcal{L}^+$ of all increasing optional processes taking value in $[0, n]$ (or even in $\{0, 1, \cdots, n\}$), the process $Y \in \mathcal{Y}$.

At the same time, given $Y \in \mathcal{Y}$, there exits Y^+ such that the corresponding $L^+ \in \mathcal{L}^+$ and it induces the same optimal stopping times $\hat{\tau}_i$ for the agents and

$$Y_{\hat{\tau}_i} = Y^+_{\hat{\tau}_i}, \ i = 1, \cdots, n.$$

The exit contract problem

Theorem (He-T.-Zou, 2023)

One has

$$\max_{Y\in\mathcal{Y},\ Y_T\geq\xi} \mathbb{E}\Big[\sum_{i=1}^{n}\int_0^{\hat{\tau}_i(Y)} g_i(t,X_{t\wedge\cdot})dt - Y_{\hat{\tau}_i(Y)}\Big]$$

$$= \max_{L\in\mathcal{L}^+} \mathbb{E}\Big[\sum_{i=1}^{n}\int_0^{T}\Big(g_i(t,X)1_{\{L_t<i\}} - f_{L_t}(t,X)1_{\{L_t\geq i\}}\Big)dt - \xi\Big]$$

$$= \max_{\tau_1\leq\cdots\leq\tau_n} \mathbb{E}\Big[\sum_{i=1}^{n}\Big(\int_0^{\tau_i} g_i(t,X) - \sum_{j=i}^{n}\int_{\tau_j}^{\tau_{j+1}} f_j(t,X)dt - \xi\Big)\Big].$$

The exit contract problem, more results (He-T.-Zou, 2023)

• A discrete time version of the problem and its convergence to the continuous time one (numerical approximation);

• Existence of the optimal contract.

• When the principal can choose from the class of Markovian contract, and/or continuous contracts, i.e.

$$Y_t = h(t, X_t), \quad \text{for all } t \geq 0,$$

with measurable and/or continuous functions h, we also identify the corresponding space for L.

Back to the Carbon Emissive Firms problem

- Assume that $X_t^i = \lambda_i X_t$ and $\pi_1 < \cdots < \pi_n$, so that agent i solve

$$\sup_{\tau_i} \mathbb{E}\Big[\int_0^{\tau_i} e^{-\rho t}\pi_i X_t dt + e^{-\rho\tau_i} Y_{\tau_i}\Big],$$

and the principal's problem

$$\sup_{Y} \mathbb{E}\Big[\int_0^{\infty} e^{-\rho t}\Big(\sum_{i=1}^n \pi_i\lambda_i X_t \mathbb{1}_{\{t\le\hat{\tau}_i\}} - \ell\Big(\sum_{i=1}^n \lambda_i X_t \mathbb{1}_{\{t\le\hat{\tau}_i\}}\Big)^{\gamma}\Big)dt - \sum_{i=1}^n e^{-\rho\hat{\tau}_i}\lambda_i Y_{\hat{\tau}_i}\Big],$$

and by the above methodology, it can be reduced to

$$\sup_{\tau_1\le\cdots\le\tau_n} \mathbb{E}\Big[\sum_{i=1}^n\Big(\int_{\tau_{i-1}}^{\tau_i} e^{-\rho t}(\bar{\pi}_i X_t - \ell_i X_t^{\gamma})dt - e^{-\rho\tau_i}\bar{\lambda}_i X_{\tau_i}\Big)\Big].$$

Back to the Carbon Emissive Firms problem

• In the paper with R. Aïd and X. Pang, in the context where X_t follows a Geometric B.M. dynamic, we provide an explicit expression of the optimal stopping time $\hat{\tau}_1 < \cdots < \hat{\tau}_n$ as hitting time of consecutive barriers.

• More numerical simulations and comparison with other situations (e.g. with individual exit incentives, without exit incentives, etc.) are provided in the paper.

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An entry time game in electricity market (R. Aïd, R. Dumitrescu and P. Tankov 2021)

• We consider an electricity market with 1 conventional producer and many many small renewable (green) producers.

• Take a representative green producer $\ell \in [0,1]$, it has constant maximum capacity M_ℓ and the capacity factor $S_t^\ell \in [0,1]$ with Jacobi dynamic :

$$dS_t^\ell = \lambda(\theta - S_t^\ell)dt + \sigma\sqrt{S_t^\ell(1-S_t^\ell)}dW_t^\ell,$$

and it enters the market at time τ_ℓ. The green producers have the distribution $\mu(d\ell)$, the total production at time t of the renewable electricity is given by

$$R_t := \int_0^1 M_\ell \, \mathbb{E}\left[S_t^\ell \mathbf{1}_{\{t \geq \tau_\ell\}}\right] \mu(d\ell).$$

An entry time game in electricity market (R. Aïd, R. Dumitrescu and P. Tankov 2021)

• Electricity price P_t : Let D_t denote the total market demand of electricity. Assume that the green electricity has the priority to supply the market, and the conventional producer supplies the rest of the demand. Therefore, the electricity price P_t equals to the marginal cost of the conventional producer.

Concretely, let $F(p)$ be the production function of the conventional producer when the electricity price is p, then the market equilibrium electricity price is given by

$$P_t := \inf\left\{p : (D_t - R_t)_+ \leq F(p)\right\} \wedge \overline{P}.$$

• Remark : $P_t = P_t(m_t)$ with $m_t := \mathcal{L}((\tau_\ell \wedge t, S_t^\ell)_{\ell\in[0,1]})$, i.e. P_t depends on R_t, which depends on the law of the entry time τ_ℓ of the green producers.

An entry time game in electricity market (R. Aïd, R. Dumitrescu and P. Tankov 2021)

- The optimal entry time of individual green producer $\ell \in [0,1]$:

$$\sup_{\tau_\ell \in \mathcal{T}} \mathbb{E}\Big[\int_{\tau_\ell}^{T} e^{-\rho t}(P_t M_\ell S_t^\ell - \lambda)dt - Ke^{-\rho\tau_\ell} + Ke^{-\rho T-\gamma(T-\tau_\ell)}\Big]$$

$$\Leftrightarrow \sup_{\tau_\ell \in \mathcal{T}} \mathbb{E}\Big[\int_{0}^{\tau_\ell} e^{-\rho t} f_\ell(t, P_t, S^\ell)dt\Big],$$

with $f_\ell(t,p,s) := -pM_\ell s + \lambda + \rho K + \gamma K e^{-(\rho+\gamma)(T-t)}$.

- A Mean-Field Game (MFG), a fixed point problem :

$$(P_t)_{t\in[0,T]} \Longrightarrow (\hat{\tau}_\ell)_{\ell\in[0,1]} \Longrightarrow m := \mathcal{L}\big((\hat{\tau}_\ell \wedge t, S_t^\ell)_{\ell\in[0,1]}\big) \Longrightarrow (\widehat{P}_t)_{t\in[0,T]}.$$

Subsidy in electricity market (work in progress with R. Dumitrescu, X. Pang and P. Tankov)

- To encourage the entry of the green producers, the government provides a (universal) subsidy Y_τ at the each entry time τ of the green producer, so that the green producer's problem becomes :

$$\sup_{\tau_\ell \in \mathcal{T}} \mathbb{E}\Big[\int_0^{\tau_\ell} e^{-\rho t} f_\ell(t, P_t, S^\ell) dt + Y_\tau(S^\ell)\Big].$$

- Lemma : For every Y, there exists a unique MFG Nash equilibrium electricity price P_t.
- The optimal subsidy policy problem :

$$\min_Y \mathbb{E}\Big[\int_0^T F(P_t) Q_t dt + \int_0^1 Y_{\hat{\tau}_\ell} \mu(d\ell)\Big].$$

In above, $F(P_t)$ is the conventional production, Q_t is the social cost per unit of the conventional production.

A mean-field version of the representation

Theorem (He-T.-Zou 2023)

Under some technical conditions, given an optional processes Y, there exists an optional process L such that

$$Y_\tau = \mathbb{E}\Big[\int_\tau^{\hat{T}} f\Big(s, X, \mathcal{L}(L), \sup_{r\in[\tau,s]} L_r\Big) ds \Big| \mathcal{F}_\tau\Big], \text{ a.s. for all } \tau \leq \hat{T}.$$

- Remark : it is a fixed point problem :

$$\mathcal{L}(L) := m_1 \longmapsto \widetilde{L} \longmapsto m_2 := \mathcal{L}(\widetilde{L}).$$

A mean-field version of the representation

- Let

$$\hat{\tau}_\ell \ := \ \inf\{t \geq 0 \ : L_t \geq \ell\},$$

then

$$\mathcal{L}(L) \implies \mathcal{L}\big((\hat{\tau}_\ell)_{\ell\in[0,1]}\big).$$

In particular, the above representation theorem provides solution to the corresponding MFG of timing.

- It would provide a unified approach to study different MFG problems, e.g. MFG of timing, MFG with singular control, MFG of optimal consumption, etc.

Back to the electricity market subsidy problem (work in progress with R. Dumitrescu, X. Pang and P. Tankov)

By applying the mean-field version of the Bank-El Karoui's representation theorem and with similar arguments, one can show that the electricity market subsidy problem

$$\min_{Y} \mathbb{E}\Big[\int_0^T F(P_t)Q_t dt + \int_0^1 Y_{\hat{\tau}_\ell}\mu(d\ell)\Big]$$

is equivalent to

$$\min_{L} \mathbb{E}\Big[\int_0^T F(P_t)Q_t dt + \int_0^T \int_0^1 f_{L_t}(t, S_t, P_t)\mathbf{1}_{\{L_t \geq \ell\}}\mu(d\ell)dt\Big].$$

In above, from the process L, one can compute the optimal entry time $\hat{\tau}_\ell$ of the green producers, and then deduce the equilibrium electricity price $(P_t)_{t\in[0,T]}$.