When Lasry-Lions meet Krugman

A mean-field game theory of spatial dynamics

M. Bahlali*, R. Boucekkine*, Q. Petit†

FDD-FiME-MIRTE seminar – January 2026

*Aix-Marseille University, CNRS, AMSE
†EDF R&D & FiME Lab

Table of contents

Introduction and Motivation

The big picture: Spatial economics

- Economic activity is not uniformly distributed. Why do cities, industrial clusters, and economic disparities emerge?
- Seminal work by [Krugman, 1991] (core-periphery model): trade-off between economies of scale (agglomeration) and competition (dispersion).
- In these models, migration decisions are key. Workers move to locations offering higher utility (real wages).

The big picture: Spatial economics

- Economic activity is not uniformly distributed. Why do cities, industrial clusters, and economic disparities emerge?
- Seminal work by [Krugman, 1991] (core-periphery model): trade-off between economies of scale (agglomeration) and competition (dispersion).
- In these models, migration decisions are key. Workers move to locations offering higher utility (real wages).

A crucial assumption: Myopic vs. Forward-looking agents

- **Myopic models**: Agents only react to current wages. Simple, but unrealistic for long-term decisions like migration.
- **Forward-looking models**: Agents anticipate future changes in wages, prices, and population distribution. This is more realistic but mathematically much harder.

The structure of modern spatial models

Recent forward-looking spatial models share a **two-layer structure**:

1. Static Equilibrium (at each time t)

Given the current distribution of labor $\mu(t)$, a static trade model (e.g., Krugman, Eaton-Kortum) determines:

- Local wages $w(t, x)$
- Local price indices $P(t, x)$
- Instantaneous utility (real wage)

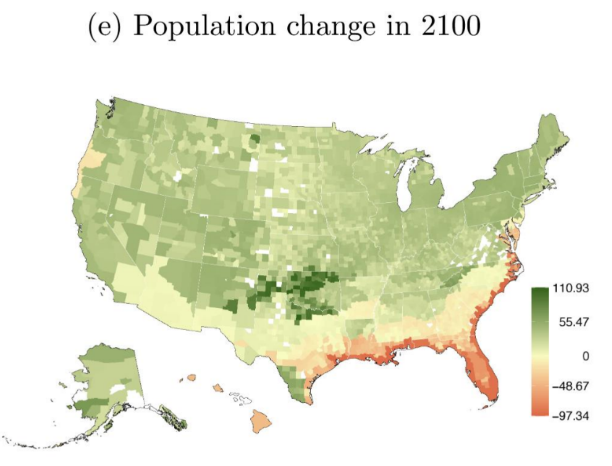
$$V(t, x) = w(t, x)/P(t, x).$$

2. Dynamic Migration (over time)

Forward-looking agents make migration decisions based on the future path of utilities

- This is an optimal control problem for each agent.
- The evolution of the labor distribution $\mu(t)$ results from these individual decisions.

Bilal and Rossi-Hansberg (2023)



Impact of 3°C additional warming by 2100 in US population distribution

The research gap: A lack of theoretical foundations

Many recent quantitative papers build complex forward-looking models ([Caliendo et al., 2019], [Kleinman et al., 2023], [Bilal & Rossi-Hansberg, 2023]).

They often rely on numerical solutions or linearization around a steady state.

But fundamental theoretical questions remain open:

- Does a dynamic equilibrium even **exist**? If it exists, is it **unique**?

These questions have not yet been solved, either in discrete or in continuous settings.

The research gap: A lack of theoretical foundations

Many recent quantitative papers build complex forward-looking models ([Caliendo et al., 2019], [Kleinman et al., 2023], [Bilal & Rossi-Hansberg, 2023]).

They often rely on numerical solutions or linearization around a steady state.

But fundamental theoretical questions remain open:

- Does a dynamic equilibrium even **exist**? If it exists, is it **unique**?

These questions have not yet been solved, either in discrete or in continuous settings.

Why is it not simple?

The problem involves a complex coupling between:

- A backward Bellman equation (individual optimization).
- A forward law of motion for the population (aggregation).
- A non-linear integral equation for wages at each point in time.

Our contributions

We provide the theoretical foundations for this class of models using **Mean-Field Game (MFG) theory** [Lasry & Lions, 2007].

1. **Static Equilibrium:** We prove the existence and uniqueness of the static trade equilibrium in **continuous space** for a broad class of models (Krugman, Armington, etc.).

Our contributions

We provide the theoretical foundations for this class of models using **Mean-Field Game (MFG) theory** [Lasry & Lions, 2007].

1. **Static Equilibrium:** We prove the existence and uniqueness of the static trade equilibrium in **continuous space** for a broad class of models (Krugman, Armington, etc.).
2. **Dynamic Equilibrium:** We use mean-field game tools to prove the existence of a dynamic equilibrium, formally establishing that these forward-looking spatial models are well-posed.

Our contributions

We provide the theoretical foundations for this class of models using **Mean-Field Game (MFG) theory** [Lasry & Lions, 2007].

1. **Static Equilibrium:** We prove the existence and uniqueness of the static trade equilibrium in **continuous space** for a broad class of models (Krugman, Armington, etc.).
2. **Dynamic Equilibrium:** We use mean-field game tools to prove the existence of a dynamic equilibrium, formally establishing that these forward-looking spatial models are well-posed.
3. **Analytical Insights:** In a simplified circular economy, we obtain closed-form solutions and use the MFG structure of the model to decompose the dynamics of agglomeration into four distinct forces:
   - Idiosyncratic shocks
   - Myopic adjustment
   - Uncertainty management
   - Forward-looking expectations

The Static Layer: Trade Equilibrium

The Krugman model in continuous space

To fix ideas, we use the [Krugman, 1996] model on a torus $\mathbb{T}^2$.

- A continuum of locations $x \in \mathbb{T}^2$.
- A distribution of workers (population) μ.
- Dixit-Stiglitz preferences, monopolistic competition.
- Trade between locations x and y is subject to iceberg costs $\tau(x, y) \geq 1$.

The Krugman model in continuous space

To fix ideas, we use the [Krugman, 1996] model on a torus $\mathbb{T}^2$.

- A continuum of locations $x \in \mathbb{T}^2$.
- A distribution of workers (population) μ.
- Dixit-Stiglitz preferences, monopolistic competition.
- Trade between locations x and y is subject to iceberg costs $\tau(x, y) \geq 1$.

Static Equilibrium Wage

Given a population distribution μ, a wage profile $w : \mathbb{T}^2 \to \mathbb{R}_+$ is a static equilibrium if it solves a non-linear integral equation and a normalization condition.

$$w(x)^\sigma = \int_{\mathbb{T}^2} \frac{\tau(y,x)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (\tau(y,z) w(z))^{1-\sigma} d\mu(z)} d\mu(y) \tag{1}$$

$$\int_{\mathbb{T}^2} w(x) d\mu(x) = 1 \tag{2}$$

Intuition for (1): The wage at x is high if it can sell to locations y with high

Deriving the Static Wage (1/3): The Consumer's Problem

Let's focus on a single worker at a location x. This worker earns a wage $w(x)$ and consumes a continuum of differentiated goods.

Maximisation problem: Love of Variety (Dixit-Stiglitz)

Let σ be greater than 1.

The worker's utility is derived from consuming quantities $q(y, x, i)$ of each variety i produced at every location y:

$$V(x) = \max \left(\int_{\mathbb{T}^2} \int_0^{n(y)} q(y, x, i)^{\frac{\sigma-1}{\sigma}} \, di \, dy \right)^{\frac{\sigma}{\sigma-1}}$$

under the budget constaint

$$\int_{\mathbb{T}^2} \int_0^{n(y)} p(y, x, i) \, q(y, x, i) \, di \, dy = w(x)$$

where $p(y, x, i)$ is the price at x of variety i produced at y.

Deriving the Static Wage (2/3): The Price Index

At the optimum we get:

1. **The Demand Function:** The demand for each variety is isoelastic:

$$q(y, x, i) = \frac{p(y, x, i)^{-\sigma}}{P(x)^{1-\sigma}} w(x).$$

Deriving the Static Wage (2/3): The Price Index

At the optimum we get:

1. **The Demand Function:** The demand for each variety is isoelastic:
$$q(y, x, i) = \frac{p(y, x, i)^{-\sigma}}{P(x)^{1-\sigma}} w(x).$$
2. **The Indirect Utility (Real Wage):** The maximum utility the worker can achieve is
$$V(x) = \frac{w(x)}{P(x)}.$$

Deriving the Static Wage (2/3): The Price Index

At the optimum we get:

1. **The Demand Function:** The demand for each variety is isoelastic:

$$q(y,x,i) = \frac{p(y,x,i)^{-\sigma}}{P(x)^{1-\sigma}} w(x).$$

2. **The Indirect Utility (Real Wage):** The maximum utility the worker can achieve is

$$V(x) = \frac{w(x)}{P(x)}.$$

The Price Index $P(x)$

The term $P(x)$ emerges naturally from the optimization as the *cost of living* at location x. It is defined as

$$P(x) = \left(\int_{\mathbb{T}^2} \int_0^{n(y)} p(y,x,i)^{1-\sigma} \, di \, dy \right)^{\frac{1}{1-\sigma}}.$$

Deriving the Static Wage (3/3): Market Clearing

The final steps involve the producer side and market clearing:

Producer Behavior

Firms set prices as a markup over marginal costs:

$$p(y, x, i) \propto w(y)\, \tau(y, x).$$

Deriving the Static Wage (3/3): Market Clearing

The final steps involve the producer side and market clearing:

Producer Behavior

Firms set prices as a markup over marginal costs:

$$p(y, x, i) \propto w(y)\,\tau(y, x).$$

By substituting these producer-side results into the price index $P(x)$ and then imposing the **market clearing condition**
(Total Supply = Total Demand for each variety), we arrive at the final wage equation:

The Equilibrium Wage Equation

$$w(x)^{\sigma} = \int_{\mathbb{T}^2} \frac{\tau(y, x)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (\tau(y, z) w(z))^{1-\sigma} d\mu(z)} d\mu(y)$$

This is a fixed-point equation where the wage at each location x depends on the wages and population distribution everywhere else.

Main Result 1: Static Equilibrium

While existence/uniqueness is known for discrete regions, the continuous case was an open problem.

Theorem 1 (Existence, Uniqueness, and Regularity)

For any population distribution μ with full support on $\mathbb{T}^2$ and regular trade costs τ:

1. There **exists a unique** static equilibrium wage profile w.
2. The wage profile w is Lipschitz continuous and bounded away from 0 and ∞.
3. **(Stability)** The map $W : \mathcal{P}_2(\mathbb{T}^2) \to (C^0(\mathbb{T}^2), \|\|_\infty)$ that from a distribution μ associates a profile wages $W(\mu)$ is Lipschitz.

Main Result 1: Static Equilibrium

While existence/uniqueness is known for discrete regions, the continuous case was an open problem.

Theorem 1 (Existence, Uniqueness, and Regularity)

For any population distribution μ with full support on $\mathbb{T}^2$ and regular trade costs τ:

1. There **exists a unique** static equilibrium wage profile w.
2. The wage profile w is Lipschitz continuous and bounded away from 0 and ∞.
3. **(Stability)** The map $W : \mathcal{P}_2(\mathbb{T}^2) \to (C^0(\mathbb{T}^2), \|\|_\infty)$ that from a distribution μ associates a profile wages $W(\mu)$ is Lipschitz.

Why is this important?

This result is the bedrock for the dynamic analysis. It ensures that the "static layer" of the model is well-behaved, which is essential for proving the existence of a dynamic equilibrium.

Sketch of proof: Exitence of an equilibrium

Let us define I and Λ as follows:

$$I[w](x) = \left(\int_{\mathbb{T}^2} \frac{\tau(y,x)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (\tau(y,z) w(z))^{1-\sigma} d\mu(z)} d\mu(y) \right)^{\frac{1}{\sigma}},$$

and

$$\Lambda[w](x) = \frac{I[w](x)}{\int I[w](y) d\mu(y)}.$$

From the definition of Λ (and with some computations), one can establish that $\Lambda[w]$

1. is uniformly bounded away from 0 and $+\infty$.
2. admits a Lispchitz constant that only depends on the regularity of τ.

Using a fixed-point strategy, one can prove the existence of a solution.

Sketch of proof: Uniqueness of equilibria (1/3)

Let us define for any $w \in C^0(\mathbb{T}^2, \mathbb{R}_+)$

$$G[w] = w - \Lambda[w] \quad \text{recalling} \quad \Lambda[w] = \frac{I[w]}{\int I[w] d\mu}.$$

The map $G : (C^0(\mathbb{T}^2, \mathbb{R}_+), \|\|_\infty) \to (C^0(\mathbb{T}^2, \mathbb{R}_+), \|\|_\infty)$ is C^1.

Moreover if w_0 is a static equilibrium and $\int h d\mu = 0$, then

$$DG[w_0](h) = h - DI[w_0](h).$$

Showing that $DG[w_0]$ is *invertible* will lead us to local uniqueness.

- Since I is 1-homotopic, then 1 is an egeinvalue of $DI[w_0]$ and $DI[w_0](w_0) = 0$.
- $DI[w_0]$ is linear, compact and strongly positive.

Krein-Rutman theorem

We use a strong version of the Krein-Rutman theorem to conclude that:

If $h \in C^0$ satisfies $h - DI[w_0](h) = 0$, then $h = tw_0$ for some t.

Sketch of proof: Uniqueness of equilibria (2/3)

Let us now work on a linear subspace of $(C^0(\mathbb{T}^2), \|\|_\infty)$:

$$\Theta^0 = \left\{ h \in C^0(\mathbb{T}^2) \ : \ \int h d\mu = 0 \right\}.$$

We can verify that

- $DI[w_0]_{|\Theta^0} : \Theta^0 \to \Theta^0$
- $\mathrm{Id} - DI[w_0]_{|\Theta^0}$ is a Fredholm operator on Θ^0.
- From the previous slide: $\mathrm{Ker}\, \mathrm{Id} - DI[w_0]_{|\Theta^0} = \{0\}$.

The Fredholm alternative ensures $\mathrm{Id} - DI[w_0]_{|\Theta^0}$ is an isomorphism of Θ^0.

The Local Inversion Theorem gives the **local uniqueness** of static equilibria.

Sketch of proof: Uniqueness of equilibria (3/3)

Global Uniqueness

1. **Start with a simple case:** If there are no trade costs ($\tau \equiv 1$), it is easy to show that the unique equilibrium is a constant wage profile, $w \equiv 1$.

Sketch of proof: Uniqueness of equilibria (3/3)

Global Uniqueness

1. **Start with a simple case:** If there are no trade costs ($\tau \equiv 1$), it is easy to show that the unique equilibrium is a constant wage profile, $w \equiv 1$.
2. **Create a continuous path:** We introduce a parameter $s \in [0, 1]$ and define a path of trade costs that continuously deforms the "no cost" case into our general case:

$$\tau_s(x, y) = (1 - s) \cdot 1 + s \cdot \tau(x, y).$$

Sketch of proof: Uniqueness of equilibria (3/3)

Global Uniqueness

1. **Start with a simple case:** If there are no trade costs ($\tau \equiv 1$), it is easy to show that the unique equilibrium is a constant wage profile, $w \equiv 1$.
2. **Create a continuous path:** We introduce a parameter $s \in [0, 1]$ and define a path of trade costs that continuously deforms the "no cost" case into our general case:

$$\tau_s(x, y) = (1 - s) \cdot 1 + s \cdot \tau(x, y).$$

3. **Track the number of solutions:** Using the **Leray-Schauder degree theory** and the results on DG, we show that the number of solutions remains constant along the path from $s = 0$ to $s = 1$.

Sketch of proof: Stability

Let $W : \mathcal{P}_2(\mathbb{T}^2) \to (C^0(\mathbb{T}^2), \|\|\|_\infty)$ be the map that from a distribution μ associates the static wage equilibirum $W(\mu)$.

- The uniqueness result combine with a compactness argument lead to the continuity of U.

Using the previous notations with $G = G[\mu, w]$, the definition of U leads to

$$G(\mu, W(\mu)) = 0,$$

where we already established that $D_w G(\mu, W(\mu))$ is invertible.

Lifting $\mathcal{P}_2(\mathbb{T}^2)$ into L^2, we can use the **Implicit Function Theorem**. Coming back to $\mathcal{P}_2(\mathbb{T}^2)$, we deduce that U is C^1 in the intrinsic sense and satisfies

$$D_\mu W(\mu) = -D_w G(\mu, W(\mu))^{-1}_{|\Theta^0} D_\mu G(\mu, W(\mu)).$$

By working a little bit more from this formula it is possible to bound the norm of $D_\mu W(\mu)$ and deduce that W is Lipschitz.

The Dynamic Layer: A Mean-Field Game

The agent's problem and the MFG interaction

Agents are forward-looking and choose their migration path to maximize lifetime utility.

- An agent's location X_t evolves according to a controlled stochastic process:

  $$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$$

  where α_t is the chosen velocity (control) and B_t is an idiosyncratic shock.
- The agent maximizes expected discounted utility:

  $$\max_{(\alpha_t)} \mathbb{E}\left[\int_0^T e^{-\rho t}\left(V(X_t, W(\mu(t)), \mu(t)) - \frac{c_0}{2}|\alpha_t|^2\right) dt\right]$$

  where V is the real wage (instantaneous utility) and $\frac{c_0}{2}|\alpha_t|^2$ is a migration cost.

The agent's problem and the MFG interaction

Agents are forward-looking and choose their migration path to maximize lifetime utility.

- An agent's location X_t evolves according to a controlled stochastic process:

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$$

  where α_t is the chosen velocity (control) and B_t is an idiosyncratic shock.
- The agent maximizes expected discounted utility:

$$\max_{(\alpha_t)} \mathbb{E}\left[\int_0^T e^{-\rho t}\left(V(X_t, W(\mu(t)), \mu(t)) - \frac{c_0}{2}|\alpha_t|^2\right) dt\right]$$

  where V is the real wage (instantaneous utility) and $\frac{c_0}{2}|\alpha_t|^2$ is a migration cost.

The Mean-Field Game Interaction

- Each agent is negligible, but their collective actions determine the population distribution $\mu(t)$.

The Mean-Field Game System

The equilibrium is characterized by a system of coupled partial differential equations (PDEs):

$$\begin{cases} -\partial_t u - \nu \Delta u - \frac{|\nabla u|^2}{2c_0} + \rho u &= V(x, w(t), \mu(t)) \quad \text{(HJB)} \\ \partial_t \mu - \nu \Delta \mu + \frac{1}{c_0} \operatorname{div}(\mu \nabla u) &= 0 \quad \text{(Fokker-Planck)} \\ \int_{\mathbb{T}^2} \frac{\tau(y,x)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (\tau(y,z) w(z))^{1-\sigma} d\mu(z)} d\mu(y) &= w(t,x)^\sigma \quad \text{(Static Wage Eq.)} \end{cases}$$

- The **Hamilton-Jacobi-Bellman (HJB)** equation describes the value function u for an individual agent. It is solved *backward* in time.
- The **Fokker-Planck (FP)** equation describes the evolution of the population distribution μ. It is solved *forward* in time.
- The **Static Wage Equation** links the wage profile w to the distribution μ at each instant.

Main Result 2: Dynamic Equilibrium

Theorem 2 (Existence of Dynamic Equilibrium)

Under standard regularity assumptions on the model primitives (trade costs, migration costs, etc.), there exists at least one mean-field game equilibrium (u, μ, w) solving the coupled PDE system.

Main Result 2: Dynamic Equilibrium

Theorem 2 (Existence of Dynamic Equilibrium)

Under standard regularity assumptions on the model primitives (trade costs, migration costs, etc.), there exists at least one mean-field game equilibrium (u, μ, w) solving the coupled PDE system.

Proof Idea: Schauder's Fixed-Point Theorem

We construct a map Ψ that takes a path of population distributions to another:

$$\Psi : \quad \mu \quad \xrightarrow{\text{Thm 1}} \quad w \quad \xrightarrow{\text{HJB}} \quad u \quad \xrightarrow{\text{FP}} \quad \hat{\mu}$$

A fixed point $\mu = \hat{\mu}$ is a dynamic equilibrium.

- We show this map Ψ is continuous and maps a compact, convex set into itself.
- The regularity results from our static analysis (Theorem 1) are crucial to ensure the map is well-behaved and that we can apply the fixed-point theorem.

A word on uniqueness

Uniqueness of dynamic equilibria is a major challenge in economic geography and MFG theory.

- **Agglomeration forces can lead to multiplicity.** If the model favors concentration (like our pure Krugman model), there can be multiple steady states (e.g., the economy can agglomerate in any location x_0). The path can depend on self-fulfilling prophecies.

A word on uniqueness

Uniqueness of dynamic equilibria is a major challenge in economic geography and MFG theory.

- **Agglomeration forces can lead to multiplicity.** If the model favors concentration (like our pure Krugman model), there can be multiple steady states (e.g., the economy can agglomerate in any location x_0). The path can depend on self-fulfilling prophecies.
- **Dispersion forces can lead to uniqueness.** If the model is dominated by dispersion forces (e.g., strong congestion effects), the equilibrium is often unique.

A word on uniqueness

Uniqueness of dynamic equilibria is a major challenge in economic geography and MFG theory.

- **Agglomeration forces can lead to multiplicity.** If the model favors concentration (like our pure Krugman model), there can be multiple steady states (e.g., the economy can agglomerate in any location x_0). The path can depend on self-fulfilling prophecies.
- **Dispersion forces can lead to uniqueness.** If the model is dominated by dispersion forces (e.g., strong congestion effects), the equilibrium is often unique.
- This is related to the **Lasry-Lions monotonicity condition** in MFG theory. The condition roughly states that utility should decrease as local density increases.

$$\int_{\mathbb{T}^2} (V(x,\mu_1) - V(x,\mu_2))\, d(\mu_1 - \mu_2)(x) \leq 0$$

  This is typically true for models with strong congestion but not for pure agglomeration models.

Extensions Studied

Our theoretical results are robust and extend beyond the specific Krugman model to a **broad class of trade models**.

- Other **monopolistic competition** models (e.g., with local amenities or productivity spillovers).
- **Perfect competition** models (e.g., Armington, Eaton-Kortum).

Extensions Studied

Our theoretical results are robust and extend beyond the specific Krugman model to a **broad class of trade models**.

- Other **monopolistic competition** models (e.g., with local amenities or productivity spillovers).
- **Perfect competition** models (e.g., Armington, Eaton-Kortum).

The dynamic analysis is also extended to different **time frameworks**.

- The **stationary case**, to characterize long-run steady-state equilibria.
- The **infinite time horizon** problem, which is a standard setting in macro-dynamic models.

Extensions Studied

Our theoretical results are robust and extend beyond the specific Krugman model to a **broad class of trade models**.

- Other **monopolistic competition** models (e.g., with local amenities or productivity spillovers).
- **Perfect competition** models (e.g., Armington, Eaton-Kortum).

The dynamic analysis is also extended to different **time frameworks**.

- The **stationary case**, to characterize long-run steady-state equilibria.
- The **infinite time horizon** problem, which is a standard setting in macro-dynamic models.

These extensions demonstrate the robustness and generality of the MFG framework for analyzing spatial economic dynamics.

The Racetrack Economy: Analytical Insights

The Racetrack Economy

To gain analytical insights, we study the model on a circle of radius R $\mathbb{T}^1_R$.

- This setting has a simple, spatially uniform steady state: $\bar{\mu} = 1, \bar{w} = 1$.
- We study the stability of this equilibrium by introducing a small sinusoidal perturbation to the initial population:

$$\mu_0(x) = 1 + \delta_\mu \cos(kx)$$

- We then linearize the MFG system around the steady state to study the evolution of this perturbation.

The linearized MFG system

The linearized MFG system is:

$$\begin{cases} -\partial_t \tilde{u} - \nu \Delta \tilde{u} + \rho \tilde{u} &= \tilde{V}(\tilde{\mu}) \quad \text{(HJB)} \\ \partial_t \tilde{\mu} - \nu \Delta \tilde{\mu} + \frac{1}{c_0} \Delta \tilde{u} &= 0 \quad \text{(Fokker-Planck)} \\ \tilde{V}(\tilde{\mu}) &= \frac{\delta_V}{\delta_\mu} \tilde{\mu} \quad \text{(Static Wage Eq.)} \end{cases}$$

completed with $\tilde{\mu}(0, x) = \delta_\mu \cos(kx)$ and $\tilde{u}(T, x) = 0$ for all $x \in \mathbb{T}^1_R$

Quantifying agglomeration: The HHI

We use the Herfindahl–Hirschman Index (HHI) to measure spatial concentration:

$$H[\mu](t) = \int_{\mathbb{T}^1_R} \mu(t,x)^2 dx$$

- $H[\mu]$ is minimized for a uniform distribution.
- $H[\mu] \to \infty$ as the distribution concentrates into a Dirac mass.

Quantifying agglomeration: The HHI

We use the Herfindahl–Hirschman Index (HHI) to measure spatial concentration:

$$H[\mu](t) = \int_{\mathbb{T}^1_R} \mu(t,x)^2 dx$$

- $H[\mu]$ is minimized for a uniform distribution.
- $H[\mu] \to \infty$ as the distribution concentrates into a Dirac mass.

The evolution of agglomeration is given by the time derivative $H'[\mu](t)$.

- $H'[\mu](t) > 0 \implies$ Increasing concentration (agglomeration).
- $H'[\mu](t) < 0 \implies$ Decreasing concentration (dispersion).

Decomposition of agglomeration forces

The core insight comes from decomposing $H'[\mu](t)$ using the linearized MFG equations.

Proposition (Decomposition of HHI evolution)

The change in spatial concentration can be decomposed into four forces:

$$\frac{1}{2}H[\mu]'(t) = \underbrace{\nu \int \Delta\tilde{\mu} \cdot \tilde{\mu}\, dx}_{\text{Idiosyncratic shocks}} \underbrace{- \frac{1}{\rho c_0} \int \tilde{V}[\tilde{\mu}] \cdot \Delta\tilde{\mu}\, dx}_{\text{Myopic adjustment}}$$
$$\underbrace{- \frac{\nu}{\rho c_0} \int \Delta\tilde{u} \cdot \Delta\tilde{\mu}\, dx}_{\text{Uncertainty}} \underbrace{- \frac{1}{\rho c_0} \int \partial_t \tilde{u} \cdot \Delta\tilde{\mu}\, dx}_{\text{Forward-looking expectations}} .$$

This allows us to analyze the sign and magnitude of each component's contribution.

The Four Forces of Spatial Dynamics

For Krugman trade model:

Dispersion Forces (-)

- **Idiosyncratic Shocks:** Random shocks ($\nu > 0$) always smooth out the distribution, acting as a powerful dispersion force. (Entropic effect)
- **Uncertainty Management:** Agents anticipate that noise will make less crowded areas more attractive in the future. They move there preemptively, reinforcing dispersion.

Agglomeration Forces (+)

- **Myopic Adjustment:** In the Krugman model, utility is higher in denser areas. Agents move towards these areas, reinforcing concentration.
- **Forward-looking Expectations:** Agents expect others to be drawn to dense areas, raising future utility there. This creates a self-fulfilling prophecy, reinforcing agglomeration.

What would happen with dispersive trade models?

For Armington trade model:

Dispersion Forces (-)	Agglomeration Forces (+)
• **Idiosyncratic Shocks**	• **Uncertainty Management**
• **Myopic Adjustment**	• **Forward-looking expectations**

Whatever the static trade model, forward-looking always act as an agglomeration force.

However, it cannot reverse the dominant pattern that would arise under myopic behavior.

Conclusion

Conclusion

- We provided the first rigorous theoretical foundations for a broad class of dynamic, forward-looking spatial equilibrium models in continuous space.

Conclusion

- We provided the first rigorous theoretical foundations for a broad class of dynamic, forward-looking spatial equilibrium models in continuous space.
- **Methodology:** Mean-Field Game theory is the natural framework to handle the interaction between individual optimization and aggregate dynamics.

Conclusion

- We provided the first rigorous theoretical foundations for a broad class of dynamic, forward-looking spatial equilibrium models in continuous space.
- **Methodology:** Mean-Field Game theory is the natural framework to handle the interaction between individual optimization and aggregate dynamics.
- **Key Results:**
  1. Proved existence and uniqueness of the static trade equilibrium in continuous space using a novel homotopy argument.
  2. Proved the existence of a dynamic MFG equilibrium, ensuring these models are well-posed.
  3. Decomposed the dynamics of agglomeration, explicitly quantifying the roles of uncertainty and expectations.

Future Research

The MFG framework opens up many avenues for future work:

- **Policy analysis:** Introduce a major player (e.g., a government setting taxes or subsidies) in a Mean-Field Game with a major player.
- **Richer dynamics:** Incorporate common noise (e.g., aggregate productivity shocks, climate shocks) or non-local migration (jumps).
- **Quantitative applications:** Apply numerical methods for solving MFG systems to solve realistic versions of these models.

Thank you! Questions?

References i

- Krugman, P. (1991).
  Increasing returns and economic geography.
  Journal of Political Economy, 99(3), 483-499.
- Krugman, P. (1996).
  The self-organizing economy.
  Blackwell Publishers.
- Caliendo, L., Opromolla, L. D., Parro, F., & Sforza, A. (2019).
  Goods and factor mobility.
  NBER Working Paper.
- Kleinman, B., Liu, E. M., & Redding, S. J. (2023).
  Dynamic spatial general equilibrium.
  NBER Working Paper.

References ii

Bilal, A., & Rossi-Hansberg, E. (2023).
The location of economic activity: First versus second nature.
NBER Working Paper.

Lasry, J. M., & Lions, P. L. (2007).
Mean field games.
Japanese Journal of Mathematics, 2(1), 229-260.