Optimal Execution under Liquidity Uncertainty

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Contents

Motivation

The Execution Problem:

- An agent must execute $\overline{X}$ number of shares of a financial asset by time $T < +\infty$.
- The agent is a trader with continuous observations of the Limit Order Book (LOB).
- Both continuous and discrete trading are allowed.

Liquidity Constraints

- Market liquidity is defined by the minimization of transaction costs/market impact.
- Price impact: immediate impact affecting the trader, transient impact decaying over time, and permanent impact reflecting the lasting effect on the price.

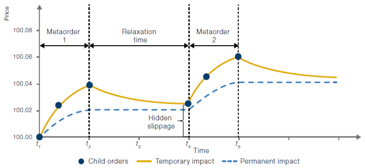

Figure 1: Stylized representation of average permanent vs. temporary price impact, Harvey et al. (2021).

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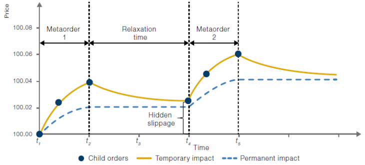

Figure 1: Stylized representation of average permanent vs. temporary price impact, Harvey et al. (2021).

Motivation

→ Stochastic liquidity and regime changes.

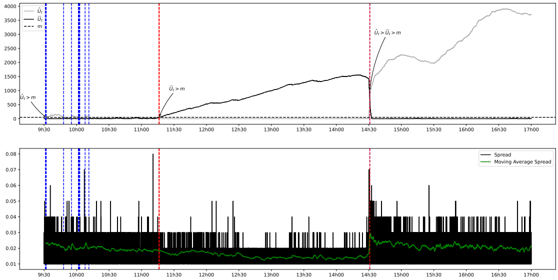

Figure 2: Changes in the spread observed in BNP stock data on 11/05/2022, Chevalier, Hafsi, and Ly Vath (2023).

Literature review

Optimal Execution with transcient price Impact:

- Obizhaeva and Wang (2013), Alfonsi, Fruth and Schied (2010), Predoiu, Shaikhet, and Shreve (2011), Gatheral, Schied and Slynko (2012), Fruth, Schöneborn and Urusov (2014), Alfonsi and Blanc (2016), Abi Jaber and Neuman (2025) ...

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Focus:

→ Singular control problem under (stochastic) liquidity constraints.

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Focus:

$\rightarrow$ Singular control problem under (stochastic) liquidity constraints.

Contents

Liquidity Dynamics: Market Impact

- Fix a liquidity regime i
- Impact function derived from the limit order book (LOB) shape F_i.
- Quantity at distance $x \geq 0$ from fundamental price A_t for a continuous LOB: $F_i(x) := \int_0^x f_i(p)\mathrm{d}p$.
- Price impact for order size y: $\psi_i(y) := \sup\{a \geq 0 : F_i(a) < y\}$, with $\psi_i(0) = 0$.
- Post-trade price: $A_t + \psi_i(y)$.

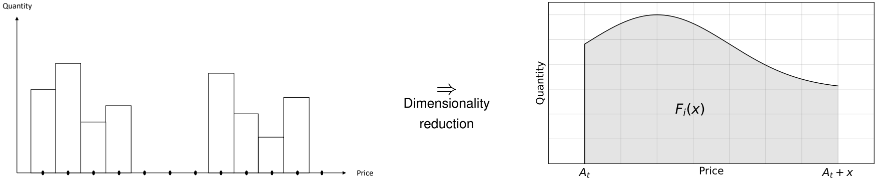

Figure 3: Illustration of a limit order book density and resulting price impact.

Liquidity Dynamics: Price Modeling and Volume Effect

- **Liquidity Regimes I:** A stable and conservative Markov chain $\{I_t\}_{t\geq 0}$ on finite state space $E = \{1, \ldots, d\}$.
- **Volume effect** Y: an $\mathscr{F}$-adapted nonnegative process, such that for all $t \geq 0$ and $u \in [t, T]$,

$$\begin{cases} \mathrm{d}Y_u = -h(Y_{u^-})\mathrm{d}u + \sigma(Y_{u^-})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u^-}, z)M(\mathrm{d}u, \mathrm{d}z), & \text{(comp. of } M(du,dz) = \lambda_u \nu(dz)du) \\ Y_{t^-} = y. \end{cases}$$

- **Price P:** $P_t := \underbrace{A_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{price deviation}}$, with $D_t := \psi_{I_t}(Y_t)$, for all $t \geq 0$.

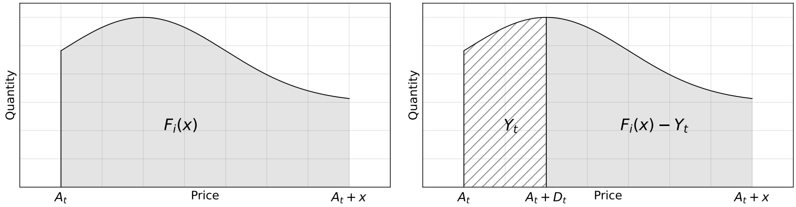

Figure 4: Representation of a limit order book at time t. The left-hand figure showing the state before a transaction and the right-hand figure showing it after. The dashed area Y_t denotes consumed shares.

Liquidity Dynamics: Price Modeling and Volume Effect

Assumptions 2.1

(A1) *There exists* $b > 0$, $\beta > 0$ *and* $a > 0$ *such that*

$$F_i(x) \geq bx^{\beta}, \quad \forall (x, i) \in [a, +\infty[\times \mathbb{I}_m.$$

(A2) *The measure* $\nu(\mathrm{d}z)$ *satisfies*

$$\int_{\mathbb{R}} (1 + z^2)\nu(\mathrm{d}z) < +\infty,$$

and $y \mapsto y + q(y, z)$ *is non-decreasing for every* z

(A3) *There exists a constant* $C > 0$ *such that, for all* $y \geq 0$,

$$|h(y)| + |\sigma(y)| + \int_{\mathbb{R}} |q(y, z)|\nu(\mathrm{d}z) \leq C(1 + |y|).$$

(A4) *There exists a constant* $L > 0$ *such that, for all* $y, y' \geq 0$,

$$|h(y) - h(y')| + |\sigma(y) - \sigma(y')| + \int_{\mathbb{R}} |q(y, z) - q(y', z)|\nu(\mathrm{d}z) \leq L|y - y'|.$$

Contents

Problem Formulation

Objective:

$\rightarrow$ Purchase a position $\bar{X}$ within a finite time horizon T, starting from $X_{t-} = x$ if $I_{t-} = i$ and $Y_{t-} = y$.

- Supposing $y = 0$, the purchase cost for a large trade of size Δy, in excess of A_t at time t is equal to

$$\Phi_i(\Delta y) := \int_0^{\psi_i(\Delta y)} \xi \, \mathrm{d}F_i(\xi) = \int_0^{\Delta y} \psi_i(\zeta)\, \mathrm{d}\zeta, \quad \forall (y,i) \in \mathbb{R}_+ \times \mathbb{I}_m.$$

- The total cost is

$$\pi_t^i(\Delta y) = \int_0^{\Delta y} \left[A_t + \psi_i(\zeta)\right] \mathrm{d}\zeta = \underbrace{A_t \Delta y}_{\text{Cost at the current price}} + \underbrace{\Phi_i(\Delta y)}_{\text{Impact cost}}.$$

- An admissible purchase strategy consists of a non-decreasing $\mathscr{F}$-adapted right-continuous process $X = (X_u)_{t \leq u \leq T}$ such that $X_{t^-} = x$ and $X_T = \bar{X}$.

Optimal Execution: Value Function

Controlled Dynamics:

$$\begin{cases} \mathrm{d}Y_u^{t,y,X} = \mathrm{d}X_u - h(Y_{u^-}^{t,y,X})\mathrm{d}u + \sigma(Y_{u^-}^{t,y,X})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u^-}^{t,y,X}, z)M(\mathrm{d}u, \mathrm{d}z), \\ Y_{t^-}^{t,y,X} = y, \end{cases}$$

with $u \in [t, T]$ and $y \geq 0$.

Value Function:

- Minimize the costs over admissible strategies $X \in \mathscr{A}_t(x)$:

$$v(i,t,x,y) = v_i(t,x,y) := \inf_{X \in \mathscr{A}_t(x)} \mathbb{E}\Big[\int_t^T \psi_{I_u}(\check{Y}_{u^-}^{t,y,X})\mathrm{d}X_u^c + \sum_{t \leq u \leq T} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_u}(\check{Y}_{u^-}^{t,y,X})\Big],$$

  for all $(t,x,y) \in \mathscr{S} := [0, T[\times[0,\bar{X}[\times\mathbb{R}_+^*$ and $i \in \mathbb{I}_m$.

$\rightarrow$ *Boundary conditions:* $v_i(T,x,y) = \Phi_i(y+\bar{X}-x) - \Phi_i(y)$ and $v_i(t,\bar{X},y) = 0$.

$\rightarrow$ *Growth condition:* $0 \leq v_i(t,x,y) \leq \Phi_i(y+\bar{X}-x) - \Phi_i(y)$.

Analytical Properties of the Value Function

Proposition 3.1

The value function v is finite.

Proposition 3.2 (Monotonicity)

For any $(i,t,x,y) \in \mathbb{I}_m \times \overline{\mathscr{S}}$, the following results hold:

1. $t_0 \mapsto v_i(t_0,x,y)$ *is non-decreasing on* $[0,T]$.
2. $x_0 \mapsto v_i(t,x_0,y)$ *is non-increasing on* $[0,\bar{X}]$.

Theorem 3.1

The value function v is a continuous function on $\overline{\mathscr{S}}$.

Analytical Properties of the Value Function

Sketch of proof: Compare the sample paths of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$. Design a strategy that ensures the paths of $u \mapsto Y_u^{t,y,X}$ and $u \mapsto Y_u^{t,y',\widetilde{X}}$ meet at a specific point in time τ, facilitating their comparison and enabling us to establish bounds for $v_i(t,x,y') - v_i(t,x,y)$.

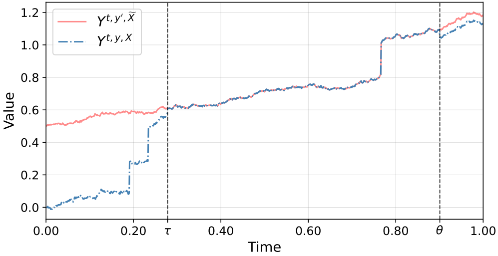

Figure 5: Illustration of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$ sample paths in the scenario where $y < y'$, $\tau < \theta < T$ and $\Delta X_\tau + Y_\tau^{t,y,X} - Y_\tau^{t,y',\widehat{X}} < \bar{X} - \widehat{X}_{\tau^-}$.

Optimal Execution: Dynamic Programming Principle

Value Function and HJBQVI:

- v solves the Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI):

$$\max\Big(-\frac{\partial v_i}{\partial t} - \mathscr{L} v_i - \sum_{j\neq i}(v_j - v_i)Q_{ij}, -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i\Big) = 0, \text{ on } \mathscr{S}.$$

where $\mathscr{S} := [0,T]\times[0,\bar{X}]\times\mathbb{R}_+$. $\rightarrow$ **viscosity solutions**

- The partial integro-differential operator $\mathscr{L}$ is given by

$$\mathscr{L}\varphi := \frac{1}{2}\sigma^2(y)\frac{\partial^2\varphi}{\partial y^2} - h(y)\frac{\partial\varphi}{\partial y} + \lambda_t\int_{\mathbb{R}}(\varphi(t,x,y+q(y,z)) - \varphi)\,\nu(\mathrm{d}z),$$

Lemma 1 (Dynamic Programming Principle)

For any stopping time τ in $[t,T]$, $(t,x,y)\in\mathscr{S}$ and $i\in\mathbb{I}_m$, we have

$$v_i(t,x,y) = \inf_{X\in\mathscr{A}_t(x)}\mathbb{E}\Big[\int_t^\tau \psi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})\mathrm{d}X_u^c + \sum_{t\le u\le\tau}\big(\Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_u}(\check{Y}_{u^-}^{t,y,X})\big) + v_{I_\tau}(\tau, X_\tau, Y_\tau^{t,y,X})\Big].$$

Viscosity Characterization

Definition 3.1 (Viscosity Solution)

We define a viscosity solution of HJBQVI as follows :

1. *v is a continuous viscosity supersolution (resp. subsolution) of HJBQVI on $\mathbb{I}_m \times \mathscr{S}$ if it satisfies the growth conditions, and if*

$$\max\Big(-\big(\frac{\partial\varphi}{\partial t} + \mathscr{L}\varphi + \sum_{j\neq i}(v_j - v_i)Q_{ij}\big)(t,x,y),\; -\big(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \psi_i\big)(t,x,y)\Big) \geq (\text{resp. } \leq) 0,$$

for any $(i,t,x,y) \in \mathbb{I}_m \times \mathscr{S}$ and any smooth test function $\varphi \in C^{1,2}(\mathscr{S})$ such that $(v_i - \varphi)$ attains a local minimum (resp. maximum) at (t,x,y) over the set $[t, t+\delta[\times[x, x+\delta[\times B_\delta(y) \subset \mathscr{S}$ for some $\delta > 0$, with $(v_i - \varphi)(t,x,y) = 0$.

2. *v is a continuous viscosity solution on $\mathbb{I}_m \times \mathscr{S}$ if it is both a viscosity supersolution and subsolution of the HJBQVI.*

Viscosity Characterization

Theorem 3.2

The value function v is a viscosity subsolution of the HJBQVI.

Theorem 3.3

The value function v is a viscosity supersolution of HJBQVI.

Theorem 3.4 (Strong Comparison Principle)

If v_i is a continuous viscosity subsolution and w_i is a continuous viscosity supersolution of the HJBQVI, such that

$$v_i(t,\bar{X},y) \leq w_i(t,\bar{X},y), \;\; \text{and} \;\; v_i(T,x,y) \leq w_i(T,x,y),$$

for all $(i,t,x,y) \in \mathbb{I}_m \times \overline{\mathscr{S}}$, then $v_i \leq w_i$ on $\mathscr{S}$.

Strong comparison principle

Sketch of proof: Let $\beta > 0$ such that

$$\lim_{y\to+\infty} \max_{i\in I_m} \frac{\Phi_i(y+\bar{X}-x)-\Phi_i(y)}{y^\beta} = 0, \quad \forall (x,y)\in[0,\bar{X}]\times\mathbb{R}_+.$$

Define $\varphi_i : \overline{\mathscr{S}} \to \mathbb{R}$ such that

$$\varphi_i(t,x,y) := -e^{-ct}\big((-a_1x+a_2)y^\beta - b_1x + b_2\big), \quad \forall (t,x,y)\in\overline{\mathscr{S}},$$

where a_1, a_2, b_1, b_2 and c are positive constants. Define $v_{i,m} : \overline{\mathscr{S}} \to \mathbb{R}$ such that

$$v_{i,m} := v_i + \frac{1}{m}\varphi_i.$$

$\Rightarrow$ $v_{i,m}$ is a strict subsolution. We get a contradiction using *Ishii's lemma*.

The Free Boundary Problem

Definition 3.2

We define the exercise region $\mathscr{E}_i := \overline{int(\mathscr{E}_i^{diff})}$ *as the closure of interior of the set* $\mathscr{E}_i^{diff}$, *on which* v_i *is differentiable, with*

$$\mathscr{E}_i^{diff} := \{(t,x,y) \in \mathscr{S} \backslash \mathscr{N}_i : -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i = 0\},$$

and we define the continuation region $\mathscr{C}_i := \mathscr{S} \backslash \mathscr{E}_i$ *as its complement.*

Proposition 3.3 (Connectedness)

Assume that the interiors of $\mathscr{C}_i$ *and* $\mathscr{E}_i$ *are non-empty, and that* $\sigma + \lambda > 0$. *Then, for each* $i \in \mathbb{I}_m$, *the free boundary* $\partial \mathscr{E}_i$ *is non-empty and path-connected. Moreover, if* $(t,x,y) \in \mathscr{E}_i$, *then*

$$(t,x,y') \in \mathscr{E}_i \quad \text{for all } 0 \leq y' \leq y, \quad \text{and} \quad (t,x',0) \in \mathscr{E}_i \quad \text{for all } 0 \leq x' \leq x.$$

The Free Boundary Problem

Sketch of proof: Assume that $\mathring{\mathscr{C}}_i$ and $\mathring{\mathscr{E}}_i$ are non-empty. Let $z_0 := (t_0, x_0, y_0) \in \mathring{\mathscr{E}}_i$ be an interior point of $\mathscr{E}_i$. Suppose there exists $\delta > 0$ one of the following options,

$$\mathscr{O}_c := \{(t_0, x_0, y) \in \overline{\mathscr{S}} : y + \delta < y_0\} \subset \mathscr{C}_i, \text{ and } \mathscr{O}_e := \{(t_0, x_0, y) \in \overline{\mathscr{S}} : y_0 < y + \delta\} \subset \mathscr{E}_i,$$

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where $\mathscr{O}_c$ is non-empty segment.

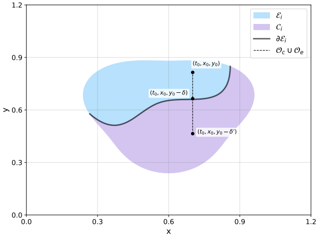


Figure 6: Illustration of $\mathscr{O}_e$, $\mathscr{O}_c$ and $\widetilde{\mathscr{O}}$.

The Free Boundary Problem

Sketch of proof: Construct a test function $\varphi_\varepsilon \in C^2(\mathscr{S})$ such that z_0 achieves a local maximum of $v_i - \varphi_\varepsilon$, $v_i(z_0) = \varphi_\varepsilon(z_0)$ and $\lim_{\varepsilon\to 0} -\mathscr{L}\varphi_\varepsilon(z_0) = +\infty$. Since v_i is a viscosity subsolution of the HJBQVI, we get

$$\max\Big(-\big(\frac{\partial \varphi_\varepsilon}{\partial t} + \mathscr{L}\varphi_\varepsilon + \sum_{j\neq i} Q_{ij}(v_j - \varphi_\varepsilon)\big)(z_0),\ -\big(\frac{\partial \varphi_\varepsilon}{\partial x} + \frac{\partial \varphi_\varepsilon}{\partial y} + \psi_i\big)(z_0)\Big) \leq 0.$$

The constructed funciton φ_ε satisfies $\lim_{\varepsilon\to 0} -\mathscr{L}\varphi_\varepsilon(z_0) = +\infty$, while

$$-\big(\mathscr{L}\varphi_\varepsilon + \sum_{j\neq i} Q_{ij}(v_j - v_i)\big)(z_0) \leq 0$$

which leads to a contradiction.

Contents

Numerical Examples and Parameterization

- Volume effect:

$$h(y) = cy, \quad \sigma(y) = dy, \quad \text{and} \quad q(y,z) = eyz, \quad \forall y, z \in \mathbb{R}_+,$$

- Order volumes follow an exponential distribution: $\nu(\mathrm{d}z) = \eta e^{-\eta z} \mathbb{1}_{\{z>0\}} \mathrm{d}z, \quad \forall z \in \mathbb{R}$.
- Limit order book shape and impact functions:

$$\psi_i(y) = \begin{cases} \frac{y}{\kappa}, & \text{if } \gamma_i = 0, \\ e^{\frac{y}{\kappa}} - 1, & \text{if } \gamma_i = 1, \\ \left[1 + (1-\gamma_i)\frac{y}{\kappa}\right]^{\frac{1}{1-\gamma_i}} - 1, & \text{else.} \end{cases}$$

- Default parameter values:

Parameter	c	d	e	η	κ	γ_0	λ	$\bar{X}$	$\bar{Y}$	T
Value	0.5	0.1	0.2	1.0	0.8	-1	0.5	5	5	4.0

Numerical Results: Single Regime Case

→ Impact of drift *h*, jump intensity λ, and volatility σ on the execution costs (i.e, value function).

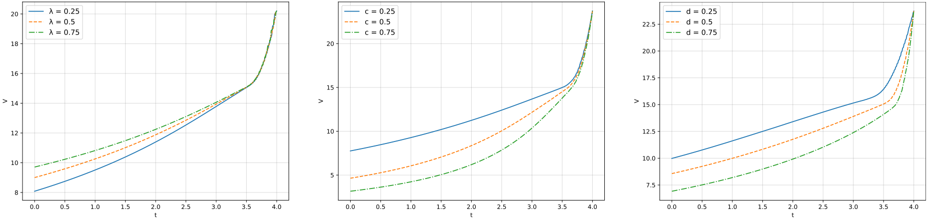


Figure 7: Variation of the value function over time under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

Numerical Results: Single Regime Case

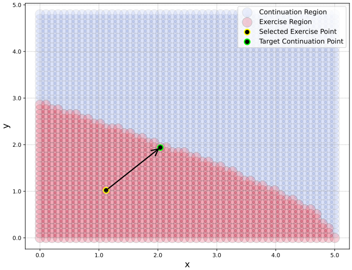


Figure 8: Illustration of continuation (blue) and exercise (red) regions, plotted against purchased quantity (x-axis) and volume effect (y-axis). The arrow shows an impulse shifting a state from exercise to continuation along $y = x$.

Numerical Results: Single Regime Case

→ Impact of drift h, jump intensity λ, and volatility σ on the exercise and continuation regions.

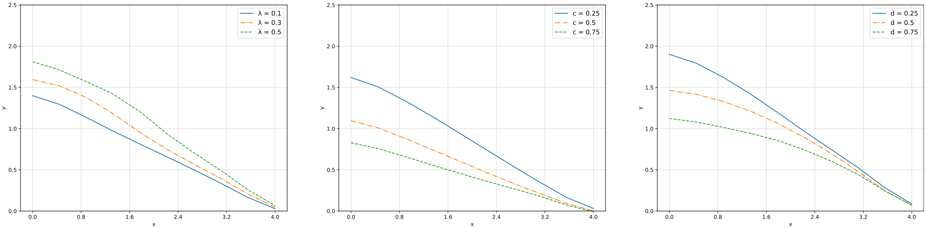


Figure 9: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

Numerical Results: Regime-Switching case

We assume a two-state homogeneous Markov chain, with the transition rate matrix $Q(t)$ at time $t \in [0, T]$ given by

$$Q(t) = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

We fix $q_1 = q_2 = 0.2$ and $\gamma_1 = 0$. With these parameters set, we then vary γ_2.

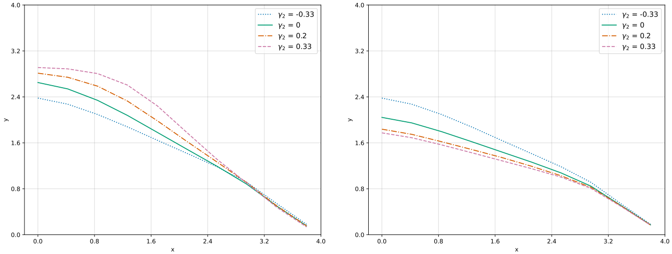
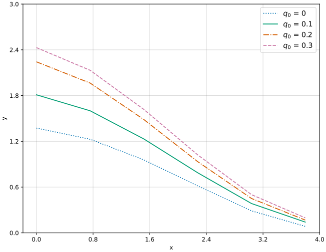


Figure 10: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different price impacts.

Numerical Results: Regime-Switching case

→ The price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.5$ in regime 2. We consider $q_1 = q_2 = q_0$ and vary the values of q_0. The probability of being in either regime is equal, but the frequency of transitions between regimes changes.



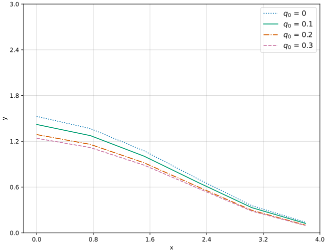


Figure 11: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different regime switching intensities.

Numerical Results: Regime-Switching case

$\rightarrow$ The price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.5$ in regime 2. We fix $q_1 = 0.2$ and vary q_2, and study how the asymmetry in switching probabilities affects execution.

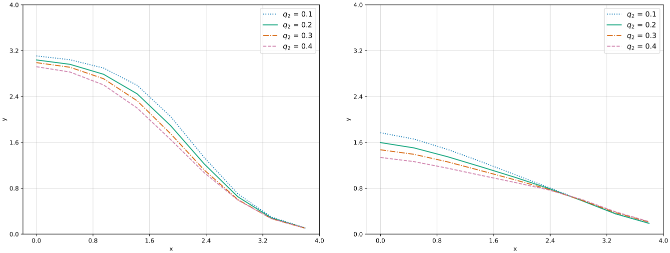


Figure 12: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different asymmetric regime switching intensities.

Conclusions and future work

- Developed a stochastic control framework for optimal execution under general price impact and stochastic liquidity, modeled via a jump diffusion volume effect and regime-switching Markov processes.
- Incorporated transient market impact using a general LOB execution cost function, allowing for arbitrary order book shapes and non-linear price responses.
- Modeled market resilience dynamically through a stochastic deviation process, and captured structural changes in market depth via liquidity regime shifts.
- Characterized the value function as the unique viscosity solution of a system of HJBQVIs. Proved regularity properties of the value function, including continuity, monotonicity, and connectedness of the free boundary.
- Approximated the solution to quantify the effect of liquidity uncertainty on optimal strategies.

Thank You !