A MFG approach for competing firms in the Emission Trading System

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Outline of the talk

- Economic Motivation
- Mathematical Model and MFG approximation
- Results for Exogenous price
- Results for Endogenous price
- Numerical Simulations
- Future work

Motivation

- The problem of excessive firm pollution has long been a part of economic theory, mainly because it imposes negative externalities on society;
- From an economic point of view, one possibility is to put a price on pollution; in this way, polluters will be more conscious about the social value of their private decisions.
- One of the most popular measures that help tackle this problem are the emission trading system and the carbon tax.

Motivation

The ETS works on a system of *allowances*.

Allowance = permit to emit a ton of Co2.

Each firm can obtain allowances in three ways: **free credits distributed by central authority**, **Carbon market**, **auctions**.

- A significant part of carbon emissions in the EU falls in the so-called EU Trading Systems of carbon allowances launched in 2005.
- If the total number of allowances under circulation falls under 400 milllions, the regulator adds allowances. If it reaches 800 million, allowances are withdrawn; allocation process is dynamic.

Motivations

Number of allowances available is capped and each year the cap is reduced.

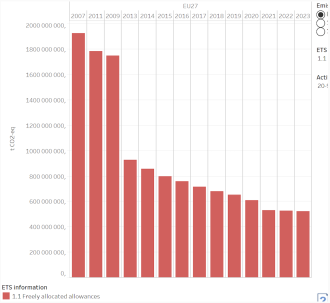


Figure 1: *European Environment Agency*

Main Goal

We develop a model based on N differential game and mean field games of competitive firms that

- produce similar goods according to a standard AK model
- generate pollution as a byproduct.

Firms are involved in the cap-and-trade pollution regulation. Under this regulation, firms

- can implement pollution abatement
- can reduce their output
- can participate in emission trading, while a regulator dynamically allocates emission allowances to each firm.

Our aim: Investigate the Impact of market structure on the ETS mechanism.

The model

The model

Consider $N \geq 1$, $N \in \mathbb{N}$ indistinguishable competing, profit-maximizing firms, whose carbon emissions are regulated in a cap-and-trade system.

The state of a firm i is characterized by the following two variables,

$$(K_t^i, \widetilde{X}_t^i)$$

where K_t^i is the level of capital at time t of firm i, for $i \in \{1, \ldots, N\}$, whereas $\widetilde{X}_t^i$ the value of emission allowances owned by the firm i, for $i \in \{1, \ldots, N\}$.

The evolution of the level of capital K_t^i

For the sake of simplicity, in this talk the technological level of firm i is set to one, the depreciation rate to zero, and the coefficients do not depend on i.

$K_t^i =$ is the capital stock of the firm i at time t.

$I_t^i =$ is the investment of the firm i at time t.

$$dK_t^i = I_t^i\, dt + \sigma K_t^i dW_t^{1,i}, \quad K^i(0) = \kappa_0^i.$$

Where

$$I_t^i = \kappa_f F^i(t) + \kappa_g G^i(t)$$

with κ_f is the productivity of the fossil energy and κ_g is the productivity of green energy, $F^i(t) =$ and $G^i(t)$ represent the amount of fossil-fuel and green energy based level of capital used by firm i for capital creation. Finally, $(W_t^{1,i})_{i=1,\dots,N}$ is a family of independent Brownian motions.

The value of emission allowances owned by firm i.

$\widetilde{X}^i_t$= value of emission allowances of firm i.
$E^{i,\alpha}_t$ = cumulative emissions of firm i.
A^i_t = the dynamic allocation of allowances

$$
\begin{cases}
d\widetilde{X}^i_t = \beta^i_t dt + dA^i_t - dE^{i,\alpha}(t) \\
dA^i_t = \tilde{a}^i dt + \tilde{\sigma}_2 dW^{0,2}_t, \\
dE^{i,\alpha}(t) = \kappa_1 F^i(t)\, dt - \alpha^i_t\, dt + \sigma_1\sqrt{1-\rho_i^2}\, dW^{2,i}(t) + \sigma_1\rho_i W^{0,1}(t), \\
E^i(0) = E^i_0, \quad \widetilde{X}^i(0) = x^i_0, \quad A^i(0) = a^i_0
\end{cases}
$$

where $\beta^i_t \in \mathbb{R}$ is the trading rate ($\beta^i_t > 0$ if credits are bought, $\beta^i_t < 0$ if credits are sold), where α^i_t is the abatement rate, $\tilde{\sigma}_2 > 0$, $\tilde{a}^i$ is the drift of dynamic allocation, $W^{0,1}_t, W^{0,2}_t$ are independent Brownian motions,$(W^{2,i}_t)_{i=1,\dots,N}$ is a family of independent Brownian motions.

The objective function

Each firm wants to minimize the objective function, over the controls $v^i = (F_t^i, G_t^i, \beta_t^i, \alpha_t^i)$,

$$J^N(v^i; v^{-i}) = \mathbb{E}\Bigg[\underbrace{\int_0^T c_\alpha(\alpha_t^i)\,dt}_{\text{cost of abatement}} + \underbrace{\int_0^T c_g(G_t^i)\,dt}_{\text{cost of green energy}} + \underbrace{\int_0^T c_f(F_t^i)\,dt}_{\text{cost of fossil energy}} + \underbrace{\int_0^T \frac{1}{2\nu}(\beta^i(t))^2\,dt}_{\text{cost of trading}} +$$

$$+ \underbrace{\int_0^T \beta^i(t)\overline{\omega}_t}_{\text{revenus from trading}} \underbrace{- \int_0^T p_t^i(\mu_t^N) K_t^i\,dt}_{\textit{revenues}} + \underbrace{g(\widetilde{X}_T)}_{\text{final penalization}}\Bigg]$$

with

- $\bar{\omega}$ is the price of allowances,
- $p_t^i(\mu_t^N)$ is the price of the good produced by the firm i,

$$p_t^i(\mu_t^N) = a - b(1-\gamma)\,K_t^i - \gamma b\left(\frac{1}{N}\sum_{j\neq i} K_t^j\right), \quad \gamma \in [0,1], a, b > 0,$$

firms compete *á la Cournot*.

The model - Objective function

- a realistic choice is $g(X_T^i) = \left(X_T^i - L_i\right)^+$
- a more treatable choice is $g(X_T^i) = \left(X_T^i\right)^2$
- We choose c_α, c_f, C_g, to be quadratic cost function and P as a linear production function.

$\epsilon-$ Nash equilibrium

We say that $\left(v^{*,1}, \cdots, v^{*,N}\right)$ is an $\epsilon-$ Nash equilibrium for $\left(J_i^N\right)_{i=1}^N$ if for all i and for all v, $J_i^N\left(v^{*,1}, \cdots, v^{*,N}\right) \leq J_i^N\left(\left(v^{*,j}\right)_{j\neq i}, v\right) + \epsilon$

How do we approach the problem?

Assuming that agents are symmetric and atomistic.

We pass to the limit $N \to \infty$ and study the Mean Field Game.

The MFG problem

Assuming $\rho_i \equiv \rho$, $\tilde{a}^i \equiv \tilde{a}$.
Agents are (conditionally) independent and for $N \to \infty$ they tends to behave as a SDE identically distributed, assuming $\rho_i \equiv \rho$, $k_0^i \equiv k_0$, $X_0^i \equiv X_0$, $\tilde{a}^i \equiv \tilde{a}$.

$$\begin{cases} dK^i(t) = (\kappa_f F^i(t) + \kappa_g G^i(t))\,dt + \sigma K^i(t)\,dW^{1,i}(t), \quad K^i(0) = \kappa_0^i, \\ dX^i(t) = (\beta^i(t) + \tilde{a} + \alpha^i(t) - \kappa_1 F^i(t))\,dt + \tilde{\sigma}_2\,dW^{0,2}(t) - \sigma_1 \rho_i\,dW^{0,1}(t) \\ \qquad\qquad -\sigma_1\sqrt{1-\rho_i^2}\,dW^{2,i}(t), \quad X^i(0) = X_0^i, \end{cases}$$

$$\Big\downarrow N \to \infty$$

$$\begin{cases} dK(t) = (\kappa_f F(t) + \kappa_g G(t))\,dt + \sigma K(t)\,dW^1(t), \quad K(0) \sim m_{k_0}, \\ dX(t) = (\beta(t) + \tilde{a} + \alpha(t) - \kappa_1 F(t))\,dt + \tilde{\sigma}_2\,dW^{0,2}(t) - \sigma_1 \rho\,dW^{0,1}(t) \\ \qquad\qquad -\sigma_1\sqrt{1-\rho^2}\,dW^2(t), \quad X(0) = X_0 \sim m_{X_0} \end{cases}$$

where $m_{k_0} = \lim_N \sum_i \delta_{k_0^i}$, $m_{X_0} = \lim_N \sum_i \delta_{X_0^i}$.

The MFG problem

The objective function for the representative player:

$$J_i^N(v^i; v^{-i}) = \mathbb{E}\Bigg[\int_0^T c_\alpha(\alpha_t^i)\,dt + \int_0^T c_g(G_t^i)\,dt + \int_0^T c_f(F_t^i)\,dt + \int_0^T \frac{1}{2\nu}(\beta^i(t))^2 dt +$$
$$+ \int_0^T \beta^i(t)\overline{\omega}_t - \int_0^T p_t^i(\mu_t^N)K_t^i\,dt + \lambda(X_T)^2\Bigg]$$

$$\Big\downarrow N \to \infty$$

$$J(v;\xi) = \mathbb{E}\Bigg[\int_0^T c_\alpha(\alpha_t)\,dt + \int_0^T c_g(G_t)\,dt + \int_0^T c_f(F_t)\,dt +$$
$$\int_0^T \frac{1}{2\nu}(\beta(t))^2 dt + + \int_0^T \beta(t)\overline{\omega}_t - \int_0^T p_t(\xi)K_t\,dt +$$
$$+\lambda(X_T)^2\Bigg]$$

The MFG problem

We rewrite the problem using the vectorial formulation,

$$J(v;\bar{\xi}) = \mathbb{E}\Big\{ \int_t^T \Big[\langle Q(s)X(s), X(s)\rangle + 2\langle \bar{Q}(s)\bar{\xi}(s), X(s)\rangle + \langle R(s)v(s), v(s)\rangle + 2\langle q(s), X(s)\rangle + 2\langle r(s), v(s)\rangle\Big] \, ds + \langle HX(T), X(T)\rangle \Big\}$$

under the state

$$\begin{cases} dX(s) = \{A_0(s) + B(s)v(s)\}\; ds + C_{0,2}dW^2(s) + C_{0,3}dW^3(s) + C_1X(s)dW^1(s) \\ \qquad\qquad + F_{0,1}dW^{0,1} + F_{0,2}dW^{0,2} \\ X(0) = X_0 \end{cases}$$

The MFG problem: matrices involved in the problem

$$A_0(s) = \begin{bmatrix} 0 \\ \tilde{a}(s) \end{bmatrix}, \ B(s) = \begin{bmatrix} \kappa_f & \kappa_g(\gamma) & 0 & 0 \\ -\kappa_1 & 0 & 1 & 1 \end{bmatrix},$$

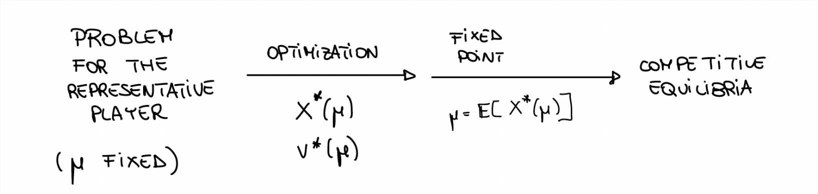
$$C_{0,2}(s) = \begin{bmatrix} 0 \\ -\sigma_1\sqrt{1-\rho^2} \end{bmatrix}, \ C_{0,3}(s) = \begin{bmatrix} 0 \\ -\sigma_2 \end{bmatrix}, \ C_1(s) = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_{0,1}(s) = \begin{bmatrix} 0 \\ -\sigma_1\rho \end{bmatrix}, \ F_{0,2}(s) = \begin{bmatrix} 0 \\ \tilde{\sigma}_2 \end{bmatrix},$$

$$Q(s) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{Q}(s) = \begin{bmatrix} \frac{b\gamma}{2} & 0 \\ 0 & 0 \end{bmatrix} \quad R(s) = \begin{bmatrix} c_{1,2} & 0 & 0 & 0 \\ 0 & c_{2,2} & 0 & 0 \\ 0 & 0 & \frac{1}{2\eta} & 0 \\ 0 & 0 & 0 & \frac{1}{2\nu} \end{bmatrix}$$

$$q(s) = \begin{bmatrix} -\frac{a}{2} \\ 0 \end{bmatrix} \quad r(s) = \begin{bmatrix} \frac{c_{1,1}}{2} \\ \frac{c_{2,1}}{2} \\ \frac{h}{2} \\ \frac{\bar{\omega}(s)}{2} \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

Mean Field Game



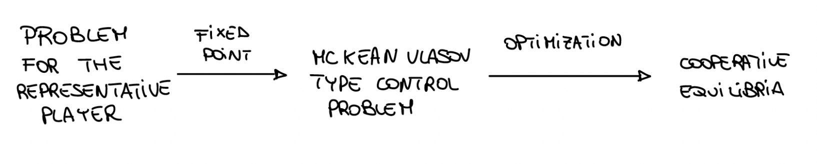
The goal is to find c^* such that:

(i) $J(v^*;\xi) = \inf_v J(v;\xi)$;

(ii) find a fixed point, i.e. ξ such that $\xi \overset{\mathcal{L}}{=} \hat{X}^*_s(\xi)$.

We call v^* the Mean Field Nash equilibria.

Mean Field Control



We rewrite the problem using the vectorial formulation,

$$J(v;\bar{\xi}) = \mathbb{E}\Big\{ \int_t^T \Big[\langle Q(s)X(s), X(s)\rangle + 2\langle \bar{Q}(s)\bar{\xi}(s), X(s)\rangle + \langle R(s)v(s), v(s)\rangle + 2\langle q(s), X(s)\rangle + 2\langle r(s), v(s)\rangle\Big]\, ds + \langle HX(T), X(T)\rangle \Big\}$$

under the state

$$\begin{cases} dX(s) = \{A_0(s) + B(s)v(s)\}\; ds + C_{0,2}dW^2(s) + C_{0,3}dW^3(s) + C_1X(s)dW^1(s) \\ \qquad + F_{0,1}dW^{0,1} + F_{0,2}dW^{0,2} \\ X(0) = X_0 \end{cases}$$

- $\bar{\omega}$ is a given $\overline{\mathcal{F}}^0$-progressively measurable process, where the latter is the completion of the filtration generated by the Brownian motion $(W^0(t))$.
- $\bar{\xi}$ is a given process adapted to the filtration $\overline{\mathcal{F}}^0$.

Reference: Graber, P. J.: Linear quadratic mean field type control and mean field games with common noise with application to production of an exhaustible resource: Applied Mathematics & Optimization 74: 459-486. (2016)

Some Notations

- $\mathbb{S}^2((\mathcal{F}_t);\mathbb{R}^n)$ be the set of $\mathbb{R}^n$-valued $(\mathcal{F}_t)$-adapted continuous processes $(U(t))$ such that
  $\|U\|_{\mathbb{S}^2} := \mathbb{E}\left[\sup_{t\in[0,T]} |U(t)|^2\right]^{\frac{1}{2}} < \infty$.
- $\mathbb{H}^2((\mathcal{F}_t);\mathbb{R}^n)$ be the set of $\mathbb{R}^n$-valued $(\mathcal{F}_t)$-progressively measurable processes $(U(t))$ such that
  $\|U\|_{\mathbb{H}^2} := \mathbb{E}\left[\int_0^T |U(t)|^2\, dt\right] < \infty$.
- $\bar{X} = \mathbb{E}\left[X|\mathcal{F}_s^0\right]$

Exogenous allowance price ($\bar{\omega}$)

Characterization of equilibrium strategy

Maximum Principle

The Mean Field Nash Equilibria (X^*, v^*) is characterized by $(X, v, Y, Z, Z_0) \in \mathbb{S}^2((\overline{\mathcal{F}}_t); \mathbb{R}^2) \times \mathbb{H}^2((\overline{\mathcal{F}}_t); \mathbb{R}^4) \times \mathbb{S}^2((\overline{\mathcal{F}}_t); \mathbb{R}^2) \times \mathbb{H}^2((\overline{\mathcal{F}}_t); \mathbb{R}^{2\times 3}) \times \mathbb{H}^2((\overline{\mathcal{F}}_t); \mathbb{R}^{2\times 2})$, solution of the FBSDE

$$\begin{cases} dY(s) = -\Big(C_1^T(s)Z_1(s) + Q(s)X(s) + \bar{Q}(s)\bar{X}(s) + q(s)\Big)\, ds \\ \qquad\qquad + \sum_{j=1}^3 Z_j(s)dW^j(s) + \sum_{j=1}^2 Z_{0,j}(s)dW_0^j(s), \\ Y(T) = HX(T) \\ dX(s) = \{A_0(s) + B(s)v(s)\}\, ds + C_{0,2}dW^2(s) + C_{0,3}dW^3(s) \\ \qquad\qquad + C_1X(s)dW^1(s) + F_{0,1}dW^{0,1} + F_{0,2}dW^{0,2} \\ X(0) = x_0 \\ Rv(s) + B^T(s)Y(s) + r(s) = 0, \end{cases}$$

Characterization of equilibrium strategy

Proof:

1. Observe that our MFG is potential, so we solve the MFC in its place.

$$\text{Nash equilibria of MFG with coefficients } f, g$$

$$\Big\updownarrow \quad f = \frac{\delta F}{\delta m}, g = \frac{\delta G}{\delta m}$$

$$\text{Solution of MFC with coefficients } F, G$$

2. Since the functional J is convex, we know that an optimal pair (X, v) for the Mean Field control problem exists.

3. By the optimality condition $\frac{d}{dh}J(v^h)|_{h=0} = 0$, we found a relation between the optimal control v and Y.

Characterization of equilibrium strategy

Let v_i^* be a mean field Nash equilibrium for the Mean Field Problem. with $X = X^i$ and $W^1 = W^{1,i}$,$W^2 = W^{2,i}$ and $W^3 = W^{3,i}$. Then for any $\epsilon > 0$ there exists N_ϵ large enough such that if $N \geq N_\epsilon$, then $\{v_i^*\}_{i=1}^N$ is an ϵ-Nash equilibrium for the N-player game.

Characterization of equilibrium strategy

Lemma (Price of Anarchy)

$$\mathcal{J}^{mfg}(v) - \mathcal{J}^{mfc}(v) = b\frac{\gamma}{2}\mathbb{E}\left[\int_0^T \left(\mathbb{E}\left[K(s) \mid \overline{\mathcal{F}}_s^0\right]\right)^2 ds\right]$$

Note that $\mathcal{J}^{mfg}, \mathcal{J}^{mfc}$ are not precisely the same. However v solves both a fixed point Nash equilibrium and a mean field type control problem.

If $\bar{\omega}$ exogenous:

- Maximum Principle
- Existence and uniqueness of the solution of the FBSDE
- Riccati Equations
- Price of Anarchy

Endogenous allowance price ($\bar{\omega}$)

The Equilibrium Price

The price of allowances $\bar{\omega}$ is in principle derived by some internal mechanism. By imposing a balance between the sales and the purchases, called market clearing condition.

Asymptotic Market Equilibrium

At the N player game, a market equilibrium is a vector of processes $(\hat{\beta})$ such that

- $\hat{\beta}^i$ is a component of the ϵ-Nash equilibrium for the N-player game.
- "$\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i_t \to 0$" (Market Clearing Condition)

Source: Fujii, M. and Akihiko T.: A mean field game approach to equilibrium pricing with market clearing condition. *SIAM Journal on Control and Optimization*, 60(1): 259-279 (2022).

The mean field Nash equilibrium is an ϵ-Nash equilibrium for the N -player game.

By the coupling condition, we have that the optimal trading rate $\hat{\beta}^i$ is

$$\hat{\beta}^i(s) = \Big(-2\nu Y^{(2),i}(s) - \nu\bar{\omega}(s)\Big)$$

We impose market clearing condition,

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i(s) = 0 \quad \Longrightarrow \bar{\omega}^N(s) = -2\frac{1}{N}\sum_{i=1}^{N} Y^{(2),i}(s)$$

However, we expect $\bar{\omega}$ to be $\overline{\mathcal{F}}^0$ progressively measurable process. Therefore,

$$\bar{\omega}^N(s) = -2\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} Y^{(2),i}(s)|\overline{\mathcal{F}}_t^0\right]$$

and in the large-N limit, we expect the market price of allowances to be given by:

$$\bar{\omega}(s) = -2\mathbb{E}\left[Y^{(2)}(s)\middle|\overline{\mathcal{F}}_t^0\right]$$

By plugging

$$\bar{\omega}(s) = -2\mathbb{E}\left[Y^{(2)}(s) \middle| \overline{\mathcal{F}}^0_t \right]$$

in the FBSDE, we get a new FBSDE

$$\begin{cases} dX(s) = (A_0(s) + Bv(s))\, ds \\ \qquad\qquad + C_{0,2}\, dW^2(s) + C_{0,3}\, dW^3(s) + C_1 X(s)\, dW^1(s) \\ \qquad\qquad + F_{0,1}\, dW^{0,1}(s) + F_{0,2}\, dW^{0,2}(s), \\ X(0) = x_0. \\ dY(s) = -(C_1^T Z_1(s) + QX(s) + \overline{Q}\,\overline{X}(s) + q)\, ds \\ \qquad\qquad + \sum_{j=1}^{3} Z_j(s) dW^j(s) + \sum_{j=1}^{2} Z_{0,j}(s) dW^{0,j}(s), \\ Y(T) = HX(T). \\ v(t) = -R^{-1}(B^T\, Y(t) + \tilde{r} + D\, \overline{Y}(s)), \end{cases} \tag{1}$$

where

$$\tilde{r} = \begin{bmatrix} \frac{c_{1,1}}{2} \\ \frac{c_{2,1}}{2} \\ \frac{h}{2} \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{2}. \end{bmatrix}$$

The idea of asymptotic in a scheme

MFG ⟵ N-player game

↕ ↕

Mean Field Nash Equilibria ⟵ ϵ-Nash equilibria

solution of the FBSDE

solution of the FBSDE
with $W^j \equiv W^{j,i}$, $j = 1, 2, 3.$
$+$
$\overline{\omega}_t = -2\mathbb{E}[Y^{(2)}(t)|\overline{\mathcal{F}}^0_t]$
with $Y^{(2)}(t)$ solution of (1)

$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \beta^i = 0 \Longleftarrow$

Existence of solution for the FBSDE

Theorem

Under the assumption that

$$\left(\frac{\kappa_f^2}{c_{1,2}} + \frac{\kappa_g^2}{c_{2,2}} - \frac{\kappa_f \kappa_e}{c_{1,2}}\right) > 0, \quad \left(2\eta + \nu + \frac{\kappa_e^2}{c_{1,2}} - \frac{\kappa_f \kappa_e}{c_{1,2}}\right) > 0,$$

there exists a unique strong solution $(X, Y, Z, Z_0) \in \mathbb{S}^2((\mathcal{F}_t); \mathbb{R}^2) \times \mathbb{S}^2((\mathcal{F}_t); \mathbb{R}^2) \times \mathbb{H}^2((\mathcal{F}_t); \mathbb{R}^{2\times 3}) \times \mathbb{H}^2((\mathcal{F}_t); \mathbb{R}^{2\times 2})$ to the FBSDE

Existence of solution for the FBSDE

Proof:

1. Verify monotonicity condition on the coefficients

$$u = \begin{bmatrix} x \\ y \\ z \\ z_0 \end{bmatrix}, \qquad A(s,u) = \begin{bmatrix} -f \\ b \\ \sigma \\ \sigma_0 \end{bmatrix}(s,u).$$

$$\mathbb{E}\left[\langle A(s,u) - A(s,u'), u - u' \rangle\right] \leq -\beta_1 \mathbb{E}[|\hat{y}|^2]$$

Existence of solution for the FBSDE

2. Introduce a family of FBSDE indexed by a parameter $\varrho \in [0,1]$,

$$\begin{aligned} dx_t^\varrho &= [-(1-\varrho)\beta_2(y_t^\varrho) + \varrho b(t, u_t^\varrho, \tilde{a}(t), \theta) + \varphi_t]\, dt \\ &+ [\varrho\sigma(t, u_t) + \psi_t] dW(t) + \sigma_0(\theta)\, dW^0(t) \\ dy_t^\varrho &= -[\varrho f(t, u_t^\varrho) + \gamma_t]\, dt + z_t^\varrho dW(t) + z_{0,t}^\varrho\, dW^0(t), \\ x_0^\varrho &= x_0, \quad y_T^\varrho = \varrho\Phi(x_T^\varrho) + (1-\varrho)x_T^\varrho + \xi, \end{aligned} \tag{2}$$

3. $\rho = 0$ its solution is straightforward, whereas for $\rho = 1$ it implies the existence of a unique strong solution to the FBSDE. In particular, the monotone conditions above allow to extend the existence from $\varrho = 0$ to $\varrho = 1$

Equilibrium Price

Proposition

Since the MFG is LQ, we can explicitly write down the solution of problem in terms of the following system of Riccati.

$$\begin{cases} \dot{P}(t) + C_1^T P(t) C_1 + Q - P(t) B R^{-1} B^T P(t) = 0; \\ P(T) = H. \end{cases}$$

$$\begin{cases} \dot{\Pi}(t) + C_1^T P(t) C_1 + (Q + \bar{Q}) - \Pi(t) B R^{-1} B^T \Pi(t) = 0; \\ \Pi(T) = H. \end{cases}$$

$$\begin{cases} \dot{\varphi}(t) - \Pi(t) B R^{-1} \tilde{r} + q + \Pi(t) A_0(t) - \Pi(t) B R^{-1} (B^T + D) \varphi(t) = 0 \\ \varphi(T) = 0. \end{cases}$$

$$\Rightarrow Y(t) = \mathcal{P}(t)(X(t) - \bar{X}(t)) + \Pi(t)\bar{X}(t) + \varphi(t)$$

Theorem

Let $T > 0$ and $\beta^{\star,i}(t) := -2\nu Y^{(2),i}(t) + 2\nu\mathbb{E}[Y^{(2)}(t)|\overline{\mathcal{F}}^0_t]$
Then

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\beta^{\star,i}(t)\right|^2\right] = 0.$$

Equilibrium Price

Proof: Since Y^i are $\mathcal{F}_0$ conditionally iid, we have

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i\right|^2\right] = \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\left(-2\nu Y^{2,i}(s) + 2\nu\mathbb{E}\left[Y^2(s)|\mathcal{F}_s^0\right]\right)\right|^2\right] \leq$$

$$\leq \frac{2\nu}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[\left|Y^{2,i}(s) - \mathbb{E}\left[Y^2(s)|\mathcal{F}_s^0\right]\right|^2\right]$$

$$\leq \frac{2\nu}{N^2}\sum_{i=1}^{N}\left(\mathbb{E}\left[\left|Y^{2,i}(s)\right| + \mathbb{E}\left[\left|\mathbb{E}\left[Y^2(s)|\mathcal{F}_s^0\right]\right|^2\right]\right]\right)$$

$$\leq \frac{C}{N}\mathbb{E}\left[\left|Y^{2,1}(s)\right|^2\right]$$

By proving a bound on $Y^{2,1}(s)^2$ we conclude that

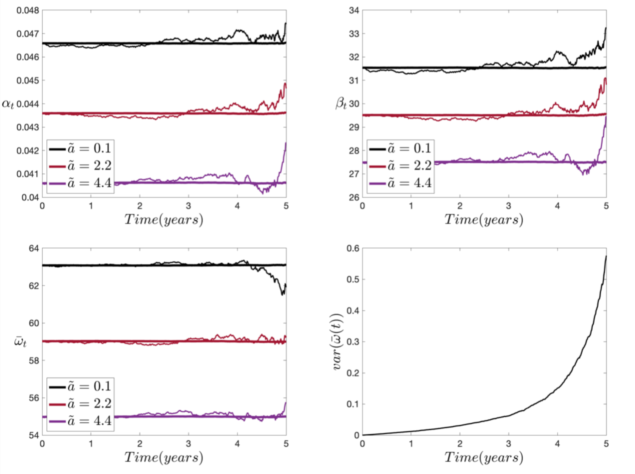
$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^i\right|^2\right] \leq \frac{C}{N}.$$

In particular, if $\bar{\omega}$ is endogenous:

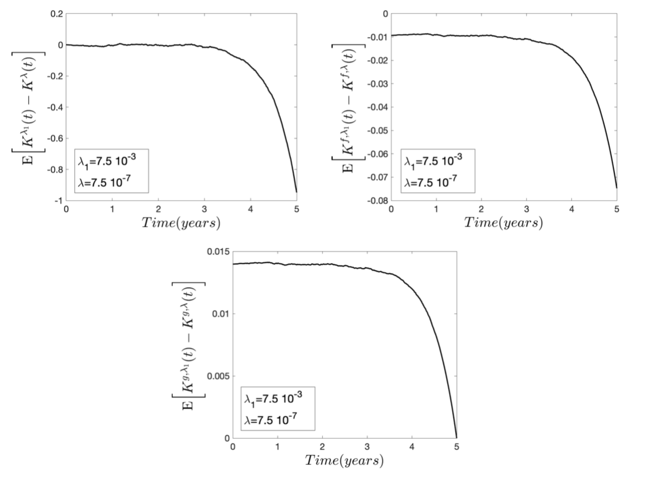
- Local existence and uniqueness of the FBSDE; Theorem 6.2 in our paper.
- Global existence and uniqueness of the FBSDE; Theorem 6.3 in our paper.
- Asymptotic Market clearing condition; Theorem 6.5 in our paper.
- Derivation of the Riccati Equation.

Some Numerics

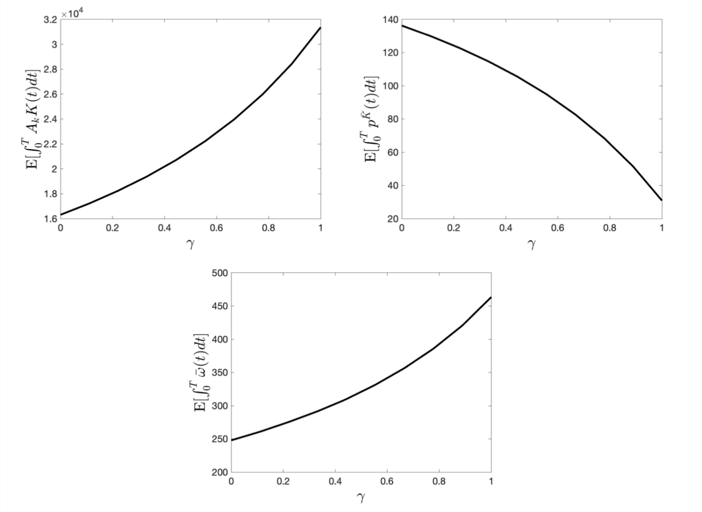
The role of the regulator



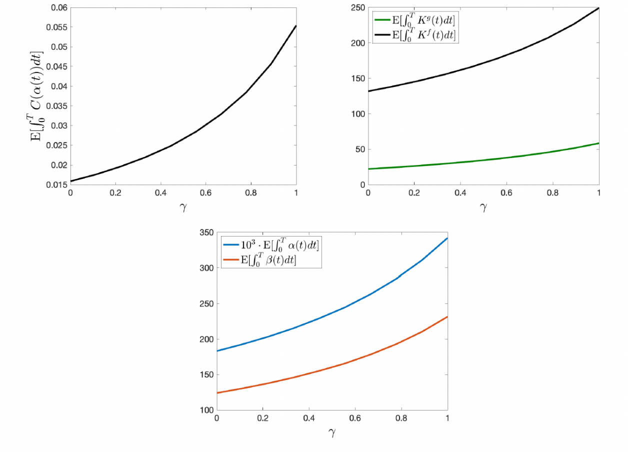
The role of the regulator



The economics of competition



The economics of competition



Future work

The model proposed in this work introduces several fundamental elements regarding pollution generation, abatement and costs, and regulation, which can serve as a basis for future research.

- More general, i.e., non-linear production function.
- Extension of the model to multiple population, where firms within each population share the same cost and coefficient functions, but differ across populations.
- Account for the way in which the regulator allocates allowances to individual firms, in particular analysing the case when the initial allocation is through auctions, as initially intended by the European Union, which switched back to grandfathering in the third phase (after 2012).

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