Dynamic Efficiency With Private Information

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Motivation

- Contribute to Macro models with frictions (output diversion) and *many* heterogeneous agents (different participation constraint).
- Rely on continuous-time contract theory (Sannikov 2008, Cvitanic-Possamai-Touzi 2018) and mean-field techniques to study with many agents.
- Mathematically speaking, we are dealing with a mean-field control problem with Value-Dependent Constraints.
- *Road Map:*
  - Focus on the *continuum limit* ($N \to \infty$) for tractability.
  - Characterize optimal allocations via a functional HJB equation and a *fixed point argument*.
  - Effect of the Common noise

Endogenous growth model: AK model

Agents: N infinitely lived agents with discount rate ρ and same utility u.

- Maximize continuation utility: $\mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^i)\, dt\right]$
- Participation constraint: $\mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^i)\, dt\right] \geq \omega^i$, ω^i exogenous.

Output of agent i $dY_t^i = \frac{k_t^i}{N}\left[\mu\, dt + \sigma\, dZ_t^i + \sigma_A\, dZ_t^A\right]$

Controls for the Social planner

1. c_t^i consumption for agent i.
2. k_t^i amount of capital managed by agent i.
3. μ, σ and σ_A exogenous.

Capital and the Resource Constraint

- **Average Capital:** $K_t := \frac{1}{N}\sum_{i=1}^{N} k_t^i$
- **Evolution of Capital:**

$$dK_t = (\mu K_t - \bar{c}_t)\,dt + \sigma_A K_t dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^i dZ_t^i, \quad \bar{c}_t = \frac{1}{N}\sum_{i=1}^{N} c_t^i$$

- If we have $\mathbb{E}\left(e^{-\mu t} K_t\right) \to 0$ when $t \to \infty$, then

$$K = \mathbb{E}\left[\int_0^\infty e^{-\mu t}\bar{c}_t\,dt\right].$$

The Promise-Keeping Condition

The planner tracks the agent's expected continuation utility

$$\omega_t^i = \mathbb{E}_t \left[\int_t^\infty \rho e^{-\rho t} u(c_t^i)\, dt \right]$$

By the representation theorem of Brownian martingales, the "promise keeping" condition

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i) \right] dt + \frac{\sigma}{N} \sum_{j=1}^{N} \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A$$

Observations: At this stage, all β are not independent control variables but depend on the consumption stream chosen by the social planner.

The Principal-Agent Problem as a Stackelberg game

- Let us assume $\sigma_A = 0$.
- **The Social Planner (Principal):** Signs bilateral contracts with firms (c_t^i, k_t^i) and does not consume.
- **Each individual Firm (Agent):** Reports output $d\hat{Y}_t = dY_t - \delta_t dt$ (output diversion).

$$\begin{aligned} d\hat{Y}_t^i &= dY_t^i - \delta_t\, dt \\ &= \mu k_t^i\, dt + \sigma k_t^i (dZ_t^i - \frac{\delta_t}{\sigma k_t^i}\, dt) \\ &= \mu k_t^i\, dt + \sigma k_t^i d\hat{Z}_t^i, \end{aligned}$$

- **Firm's objective:**

$$\omega_0^i = \sup_{\delta} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s} u(c_s^i + \delta_s) ds\right]$$

Martingale Optimality Principle

Step 1: Construct a process R_t^δ to apply the martingale optimality principle:

$$R_t^\delta = e^{-\rho t}\omega_t^\beta + \int_0^t e^{-\rho s} u(c_s + \delta_s) ds$$

$$d\omega_t^\beta = \rho(\omega_t^\beta - u(c_t^i))\, dt + \beta_t d\hat{Z}_t^i, \text{ with } \beta_t \geq \rho\sigma k_t^i u'(c_t^i).$$

Step 2: The drift of dR_t^δ is proportional to:

$$\rho(u(c_t + \delta_t) - u(c_t)) - \frac{\beta_t \delta_t}{\sigma k_t}$$

Step 3: Using the concavity of u:

$$u(c_t^i + \delta_t) - u(c_t^i) - \frac{\beta_t \delta_t}{\rho\sigma k_t^i} \leq \delta \left(u'(c_t^i) - \frac{\beta_t}{\rho\sigma k_t^i} \right) \leq 0$$

$(R_t^\delta)_t$ is a **supermartingale** for $\delta > 0$ and a **martingale** for $\delta = 0$.

Incentive Compatibility and Participation Constraints

To sum up

The social planner has to select capital and consumption flow such that

- $\beta_t^{ii} \geq \rho\sigma k_t^i u'(c_t^i)$.
- $\omega_0^i \geq \omega^i$.

A simple benchmark: N=1

With one agent, how much capital is needed to obtain utility ω?

- Inverse of Merton problem

$$V(k) = \sup_c \mathbb{E}[\int_0^\infty \rho e^{-\rho t} u(c_t) dt]$$

under

$$dk_t = (\mu k_t - c_t)dt + \sigma k_t dZ_t$$

- $$V(k) = \omega \iff k = k(\omega)$$

- When $u(c) = \log c$ we have a closed form solution:

$$k(\omega) = exp(\omega - a_1), c(\omega) = \rho k(\omega)$$

where

$$a_1 = \log \rho + \frac{\mu - \rho - \frac{\sigma^2}{2}}{\rho}.$$

Characterizing Optimal Allocations when $N \to \infty$

- Agents are indexed by their participation constraint ω, where $\mathbb{P}$ represents the initial distribution across the population.
- We assume Markov controls, i.e. $c(\mathbb{P}, \omega)$ and $k(\mathbb{P}, \omega)$.
- Control approach (Sannikov 2008,CPT 2018), starting from a random variable ω_0 with distribution $\mathbb{P}$, the social planner controls the continuation utility process

$$d\omega_t = \rho(\omega_t - u(c(\mathbb{P}_t, \omega_t)))\, dt + \sigma \beta_t dZ_t.$$

- β_t is now a second control that must satisfy the IC condition $\beta_t \geq \rho k u'(c)$.

Social Planner Problem

$$K(\mathbb{P}) = \min_{\{k,c,\beta\}} \int_0^\infty e^{-\mu t} \left[\int_{\mathbb{R}} c(\mathbb{P}_t, \omega)\, d\mathbb{P}_t(\omega) \right] dt \qquad (1)$$

subject to the following constraints for all t:

$$d\omega_t = \rho\,[\omega_t - u(c_t)]\,dt + \sigma\beta_t dZ_t \qquad \text{(Promise Keeping)}$$

$$\beta_t \geq \rho k_t u'(c_t) \qquad \text{(Incentive Compatibility)}$$

$$K(\mathbb{P}_t) = \int_{\mathbb{R}} k(\mathbb{P}_t, \omega)\, d\mathbb{P}_t(\omega) \qquad \text{(Capital Allocation)}$$

$$\mathbb{P}_0 = \mathbb{P} \qquad \text{(Initial Condition).}$$

Differential Calculus in $\mathcal{P}_2(\mathbb{R})$ and Chain Rule

Definition: First Variation and L-Differentiability

First variation exists at $\mathbb{P}$ if there is a continuous $\nabla F[\mathbb{P}] : \mathbb{R} \to \mathbb{R}$ such that:

$$\lim_{\varepsilon \to 0} \frac{F((1-\varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}) - F(\mathbb{P})}{\varepsilon} = \int_{\mathbb{R}} \nabla F[\mathbb{P}](\omega)\, d(\mathbb{Q} - \mathbb{P})(\omega)$$

- F is L-differentiable if the first variation $\nabla F[\mathbb{P}](\cdot)$ is twice differentiable in $\mathbb{R}$. Notation: $\partial_\omega \nabla F[\mathbb{P}]$ and $\partial_{\omega\omega} \nabla F[\mathbb{P}]$.

Itô's Lemma (Carmona & Delarue, Vol I,II)

$$F(\mathbb{P}_t) = F(\mathbb{P}_0) + \int_0^t \mathbb{E}\left[\partial_\omega \nabla F(\mathbb{P}_s)(\omega_s)\rho\big(\omega_s - u(c(\mathbb{P}_s, \omega_s))\big)\right] ds$$
$$+ \frac{1}{2}\int_0^t \mathbb{E}\left[\partial^2_{\omega\omega} \nabla F(\mathbb{P}_s)(\omega_s)\sigma^2\beta^2(\mathbb{P}_s, \omega_s)\right] ds$$

Functional Hamilton-Jacobi-Bellman (HJB)

Constrained Functional HJB

Taking $\lambda(\mathbb{P})$ as the Lagrange multiplier for $\int k(\mathbb{P},\omega)\, d\mathbb{P} = K(\mathbb{P})$:

$$(\mu - \lambda(\mathbb{P}))K(\mathbb{P}) = \inf_{c,k} \int_{\mathbb{R}} \Big[c + \rho(\omega - u(c))\partial_\omega \nabla K + \frac{(\rho\sigma k u'(c))^2}{2} \partial^2_{\omega\omega} \nabla K \\ -\lambda(\mathbb{P})k \Big]\, d\mathbb{P}$$

- **Binding IC:** Assuming $\partial^2_{\omega\omega}\nabla K > 0$ (convexity), the volatility constraint binds: $\beta = \rho k u'(c)$.
- **Structure:** This is a second-order PDE in the space of measures $\mathcal{P}_2(\mathbb{R})$. Existence, Uniqueness

Verification Theorem

Existence and Uniqueness

Let $\lambda(.)$ be a given Lagrange multiplier and let $K^{\lambda}(\mathbb{P})$ be a L-differentiable function such that:

1. **HJB & Transversality:** K^{λ} solves the constrained HJB equation and satisfies:
$$\lim_{t\to\infty} e^{-\mu t} K^{\lambda}(\mathbb{P}_t) = 0$$
2. **Admissibility:** There exists an optimal control pair $(c^*_{\lambda}, k^*_{\lambda})$ attaining the infimum.
3. **Consistency:** There is some λ^* such that:
$$\int k^*_{\lambda^*}(\omega)\, d\mathbb{P}_t(\omega) = K^{\lambda^*}(\mathbb{P}_t), \quad \forall t \geq 0$$

Then, $K = K^{\lambda^*}$

Explicit Solution: The CRRA Utility Case

Assume a Constant Relative Risk Aversion (CRRA) utility function:

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \quad \alpha > 1$$

Main result

Under CRRA utility, the optimal controls (k^*, c^*) are **independent** of the probability measure $\mathbb{P}$. They are parameterized by two constants (a^*, y^*):

- **Capital Allocation:** $k(\omega) = a^* u^{-1}(\omega)$
- **Consumption:** $c(\omega) = \frac{\rho k(\omega)}{y^*}$

Fixed Point Equation for a^*

The constant a^* is the unique solution to the scalar optimization:

$$\frac{\mu}{\rho} = \inf_{y} \left[\frac{1}{y} + \frac{\left(\frac{\rho a^*}{y}\right)^{1-\alpha} - 1}{\alpha - 1} + \frac{\alpha\sigma^2}{2\rho}(\rho a^*)^{2-2\alpha} y^{2\alpha} \right]$$

Sketch of the Proof I

A standard Guess and Verify approach and a fixed point argument.

- Ansatz: $K(\mathbb{P}) = \int g(\omega) d\mathbb{P}(\omega)$ with $g(\omega) = a^* u^{-1}(\omega)$.
- The first variation ∇K simply coincides with $g(\cdot)$.
- The functional equation must hold for *all* probability measures $\mathbb{P}$
- **Pointwise Identity:** Since the integrand is independent of $\mathbb{P}$, the following relation must hold for every state ω.

Individual HJB Equation

For any given λ, we seek for a function $g_\lambda(\omega)$ that satisfies:

$$(\mu - \lambda) g(\omega) = \inf_{c,k} \left[c + \rho(\omega - u(c)) g'(\omega) + \frac{(\rho \sigma k u'(c))^2}{2} g''(\omega) - \lambda k \right]$$

First-Order Conditions

We minimize over c and k the expression:

$$c + \rho(\omega - u(c))g'(\omega) + \frac{(\rho\sigma k u'(c))^2}{2} g''(\omega) - \lambda k$$

FOC with respect to c:

$$1 - \rho\, u'(c)\, g''(\omega) \left[(\omega - u(c)) - \rho\sigma^2 k^2 u''(c)\right] = 0$$

FOC with respect to k:

$$k^* = \frac{\lambda}{(\rho\sigma)^2\, (u'(c))^2\, g''(\omega)}$$

Guess and Change of Variables

With $u(c) = \dfrac{c^{1-\alpha}}{1-\alpha}$, so that $u^{-1}(\omega) = \big((1-\alpha)\omega\big)^{\frac{1}{1-\alpha}}$.

Guess: $g(\omega) = a^* \, u^{-1}(\omega) \equiv a^* \, \phi(\omega)$, with derivatives:

$$\phi'(\omega) = \frac{\phi(\omega)}{(1-\alpha)\,\omega}, \qquad \phi''(\omega) = \frac{\alpha\,\phi(\omega)}{(1-\alpha)^2\,\omega^2}$$

Change of controls:

$$c = \gamma\, u^{-1}(\omega), \qquad k = a\, u^{-1}(\omega)$$

which gives:

$$u(c) = \gamma^{1-\alpha}\,\omega \implies \omega - u(c) = (1-\gamma^{1-\alpha})\,\omega$$

$$u'(c) = \gamma^{-\alpha}\,\phi(\omega)^{-\alpha}$$

Static Optimisation Problem

Substituting into the HJB, all ω-dependence cancels. The problem reduces to:

$$\boxed{(\mu - \lambda)\, a^* = \min_{a,\gamma} \left\{ \gamma + \frac{\rho\, a^*}{1-\alpha}\left(1 - \gamma^{1-\alpha}\right) + \frac{\alpha\,(\rho\sigma)^2\, a^2\, \gamma^{-2\alpha}}{2}\, a^* - \lambda\, a \right\}}$$

This is a **static problem** in (a, γ):

- a^* appears *linearly* $\Rightarrow$ FOCs in a and γ pin down the optimal controls independently.
- The full equation then determines a^* as a **fixed point**.

First-Order Conditions for a and γ

FOC with respect to a:

$$\alpha\,(\rho\sigma)^2\,a\,\gamma^{-2\alpha}\,a^* = \lambda \implies a = \frac{\lambda}{\alpha\,(\rho\sigma)^2\,\gamma^{-2\alpha}\,a^*}$$

There is a unique λ^* such that $a = a^*$. Define $\gamma = \frac{\rho a^*}{y}$ and divide by ρa^* to get

$$\boxed{\frac{\mu}{\rho} = \inf_y \left\{ \frac{1}{y} + \frac{(\frac{\rho a^*}{y})^{1-\alpha} - 1}{\alpha - 1} + \frac{\alpha\sigma^2}{2\rho}(\rho a^*)^{2-2\alpha} y^{2\alpha} \right\}}.$$

Let us denote $I(a^*)$ the right-hand side. There is exactly one value a^* such that $I(a^*) = \frac{\mu}{\rho}$. Finally, y^* is the unique value of y for which the inf is reached for this particular value of a^*.

Social Planner Problem with common noise

The planner minimizes the expected discounted cost over a flow of random measures $\mathbb{P}_t$:

$$K(\mathbb{P}) = \min_{\{k,c,\beta,\beta^A\}} \mathbb{E}\left[\int_0^\infty e^{-\mu t}\left(\int_{\mathbb{R}} c_t(\omega)\, d\mathbb{P}_t(\omega)\right) dt\right] \tag{2}$$

where:

- $\mathbb{P}_t$ is the conditional law of ω_t given the common noise $\mathcal{F}_t^A$.
- $\mathbb{E}$ is the expectation over the realizations of the common noise Z^A.

Planner Constraints

- **Promise keeping**

$$d\omega_t = \rho\left[\omega_t - u(c_t)\right]dt + \sigma\beta_t dZ_t + \sigma^A\beta_t^A dZ_t^A$$

- **Incentive Compatibility:**

$$\beta_t \geq \rho k_t u'(c_t) \quad (3)$$

- **Stochastic Capital Allocation:**

$$K(\mathbb{P}_t) = \int_{\mathbb{R}} k_t(\omega)\, d\mathbb{P}_t(\omega) \quad (4)$$

- **Initial Condition:** $\mathbb{P}_0 = \mathbb{P}$.

The HJB Equation (Master Equation)

Without the capital allocation constraint, the value function $K(\mathbb{P})$ would satisfy the second-order HJB equation in the Wasserstein space:

$$\mu K(\mathbb{P}) = \inf_{\{c,\beta,\beta^A\}} \left\{ \int_{\mathbb{R}} c(\omega) d\mathbb{P}(\omega) + \int_{\mathbb{R}} \rho(\omega - u(c)) \partial_\omega \nabla K d\mathbb{P} \right.$$
$$+ \frac{1}{2} \int_{\mathbb{R}} (\sigma^2 \beta^2 + (\sigma^A)^2 (\beta^A)^2) \partial^2_{\omega\omega} \nabla K d\mathbb{P}$$
$$\left. + \frac{(\sigma^A)^2}{2} \int_{\mathbb{R}^2} \beta^A(\omega) \beta^A(\omega') \partial^2_{\omega\omega'} \nabla^2 K d\mathbb{P} d\mathbb{P}' \right\}$$

The Primal Problem

Assuming (IC) binding, we consider the following saddle-point problem over the conditional law $\mathbb{P}$ of ω given the common noise:

$$\mu K(\mathbb{P}) = \sup_{\lambda \in \mathbb{R}} \left\{ \inf_{c(\cdot), k(\cdot), \beta^A} \mathcal{L}(c, k, \beta^A, \lambda; \mathbb{P}) \right\}$$

Where the Lagrangian $\mathcal{L}$ is defined as:

$$\mathcal{L} = \int_{\mathbb{R}} \left[c(\omega) + \rho(\omega - u(c(\omega)))\partial_\omega \nabla K(\mathbb{P})(\omega) + \frac{(\sigma^2 + (\beta^A)^2)(\rho k(\omega) u'(c(\omega)))^2}{2} \partial^2_{\omega\omega} \nabla K(\mathbb{P})(\omega) + \lambda k(\omega) \right] d\mathbb{P}(\omega)$$
$$+ \frac{(\beta^A)^2}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial^2_{\omega\omega'} \nabla^2 K(\mathbb{P})(\omega, \omega') d\mathbb{P}(\omega) d\mathbb{P}(\omega') - \lambda K(\mathbb{P})$$

Ansatz

Assume $K(\mathbb{P})$ is a linear functional: $K(\mathbb{P}) = \int_{\mathbb{R}} g(\omega)\, d\mathbb{P}$.

The constraint $K(\mathbb{P}) = \int k(\omega)\, d\mathbb{P}$ must hold **path-by-path**, not just in expectation, which yields two fundamental conditions:

- Condition 1:
$$\int_{\mathbb{R}} (k(\omega) - g(\omega))\, d\mathbb{P}(\omega) = 0$$
- Condition 2:
$$\int_{\mathbb{R}} (\beta^A(\omega) g'(\omega) - g(\omega))\, d\mathbb{P}(\omega) = 0$$

Lagrange Multipliers and Point-wise HJB

Since the HJB must hold for any distribution $\mathbb{P}$, it reduces to a **point-wise ODE** for $g(\omega)$:

$$\mu g(\omega) = \sup_{\lambda_1,\lambda_2} \inf_{c,k,\beta^A} \Big\{ c + \rho(\omega - u(c))g'(\omega) + \frac{(\sigma\rho k u'(c))^2 + (\sigma^A\beta^A)^2}{2} g''(\omega) + \lambda_1(k - g(\omega)) + \lambda_2(\beta^A g'(\omega) - g(\omega)) \Big\}$$

The Case of Log Utilities

Assume log utility $u(c) = \log c$ and the functional guess $g(\omega) = a^* \exp(\omega)$. The solution is $g(\omega) = a^* \exp(\omega)$, where a^* is uniquely defined by:

$$\mu + \rho \log a^* - \frac{\sigma_A^2}{2} = \min_{\gamma} \left[\gamma - \rho \log \gamma + \frac{\sigma^2 \rho^2}{2\gamma^2} \right] \tag{5}$$

- **Consumption:** Agents consume a constant fraction γ^* of their capital: $c_t = \gamma^* k_t$.
- γ^* is the value that realizes the minimum in (5).

Aggregate Dynamics and Economic Interpretation

- **Aggregate Growth:** The economy grows at a random rate:

$$\frac{dK_t}{K_t} = (\mu - \gamma^*)dt + \sigma_A dZ_t^A$$

- **Individual Reallocation:** Capital evolves relative to the aggregate:

$$\frac{dk_t}{k_t} = \frac{dK_t}{K_t} + \sigma x^* dZ_t, \quad \text{where } x^* = \frac{\rho}{\gamma^*}$$

Conclusion

- **Theoretical Framework:** We established a Mean Field Control approach to characterize **Pareto optima** in economies with a continuum of heterogeneous agents.
- **Novelty:** Successfully integrated **value-dependent constraints** within a functional HJB equation on the **Wasserstein space**.
- Derived explicit solutions for the *AK growth model* by leveraging the properties of CRRA utility functions.
- **Future Work:** Extension to non-CRRA utilities.